1. **Risk and Risk Aversion**
   A. ***Risk***: The possibility of more than one outcome.
   B. ***Risk premium***: incremental expected gain from taking a risk.

Example: a simple prospect with only two outcomes

Consider investing an initial amount of wealth, W = $100, in an investment with two known outcomes. The investor will receive W_1 = $200 with probability p = 0.8 or W_2 = $50 with probability (1-p) = 0.2.

What is the terminal wealth and expected profit of this gamble?

$E(W) = pW_1 + (1-p)W_2$
E(W) = .8(200) + .2(50) = **$170**
E(profit) = E(π) = W - E(W) = $70

What is the variance and standard deviation of the payoff?

$\sigma^2 = p[W_1 - E(W)]^2 + (1-p)[W_2 - E(W)]^2$
$\sigma^2 = .8(200 - 170)^2 + .2(50 - 170)^2 = 3600$
σ **= $60**.

Now, suppose there is a *risk-free* investment paying 12% over the life of the investment. In this scenario investing all of initial wealth in the risk-free investment pays out $12.

To find the *risk premium*: find the incremental expected gain from taking the risk.
RP = 70 - 12 = $58

Risk premium: **RP = E(π) - rf** Think of the risk premium as compensation (over the risk-free investment) for the standard deviation of the investment payoffs.

   C. Speculation vs. Gambling.
      i. Speculation is the taking on of risk to make money.
      ii. Gambling is taking on of risk for the fun of taking on the risk.
         a) Speculation involves a risk-premium while gambling does not.
         b) Another way to differentiate between the two is to think of a gamble as having an expected payoff of $0.

Example: Consider two parties in a betting transaction.
Think of the writer of the call versus the holder of the call. The parties have differing probabilities assigned to whether the stock price will increase or decrease (*heterogeneous expectations*), hence they are both speculating. If they had homogeneous beliefs about the probabilities and the expected payoff was $0, then they would be gambling.

2. **Risk Types**

Risk lovers: people who engage in fair games and gambles. **Risk neutral investors**: people who only care about expected return without regard for risk. **Risk averse investors**: people who *must* have a risk-premium - avoid fair games or worse investments.

3. **Utility and a little "Theory of the Consumer"**
   A. **<u>Rational behavior</u>**: Behavior consistent with the behavior postulates and preference axioms.

      i. **Behavior Postulates**: (a postulate is a statement made without proof - usually something that is obvious or well known.)

a) Each person desires many goods and has many goals.
b) For each person, some goods are scarce.
c) Each person is willing to forsake some of a good to get more of other goods.
d) The more of a good one has the larger the total value, but the lower the marginal value of a unit.
e) Not all people have identical tastes and preferences.
f) People are innovative but consistent.

ii. **Preference axioms**: describe how people deal with preferences between different goods or bundles of goods. Let **R** stand for "at least as good as." i.e. if **X^a R X^b** then bundle A is at least as good as bundle Let **P** stand for preferred and **I** stand for indifferent.

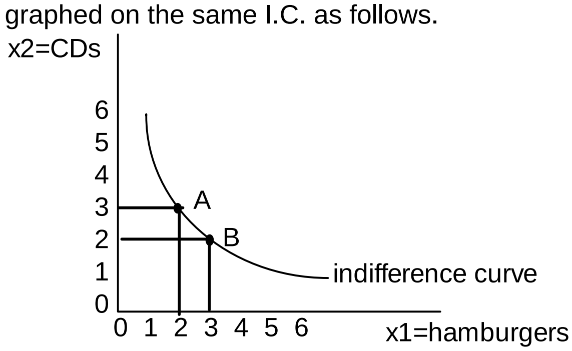
(1) <u>preferences are *reflexive*</u>: **X^a R X^a**
(2) <u>*preferences are complete*</u>: **X^a R X^b** or **X^b R X^a** (all bundles are comparable)
(3) <u>*preferences are transitive*</u>: if **X^a R X^b** and **X^b R X^c** => **X^a R X^c**
(4) transitivity also implies the following
  (a) if X^a P X^b and X^b P X^c => X^a P X^c
  (b) if X^a I X^b and X^b I X^c => X^a I X^c
  (c) if X^a P X^b and X^b I X^c => X^a P X^c
  (d) if X^a I X^b and X^b P X^c => X^a P X^c
(5) <u>*preferences are monotonic*</u>: more is better

B. Utility: A ranking of preferences or a measure of welfare/happiness. More preferred goods or bundles of goods have higher utility.
i. Utility is *only* a ranking,
ii. We can not add utilities,
iii. We can not compare utilities across people,
iv. We cannot say utility has anything to do with usefulness.

C. **Utility Maximization**: If someone is maximizing utility they are simply trying to get to the most preferred bundle of goods they can afford. ***<u>We always assume that people are rational utility maximizers.</u>***

D. **Indifference Curves (I.C.'s)**: lines connecting bundles of goods that we prefer by the same amount.

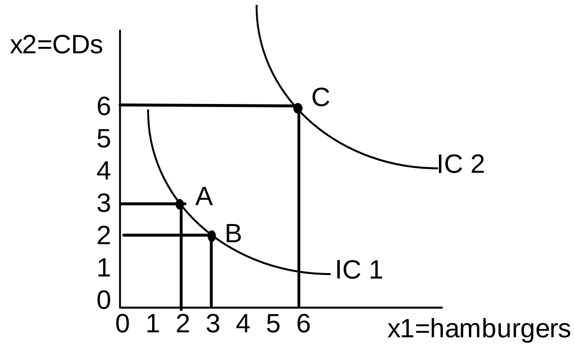
i. <u>Example:</u> If A=(2,3) and B=(3,2) and we would be willing to pay the same amount for either bundle (we value the two bundles equally), then we are *indifferent* between them. These two bundles would be graphed on the same I.C. as follows.



4.

i. In fact, any bundle which we value the same as A and B will show up on this I.C..

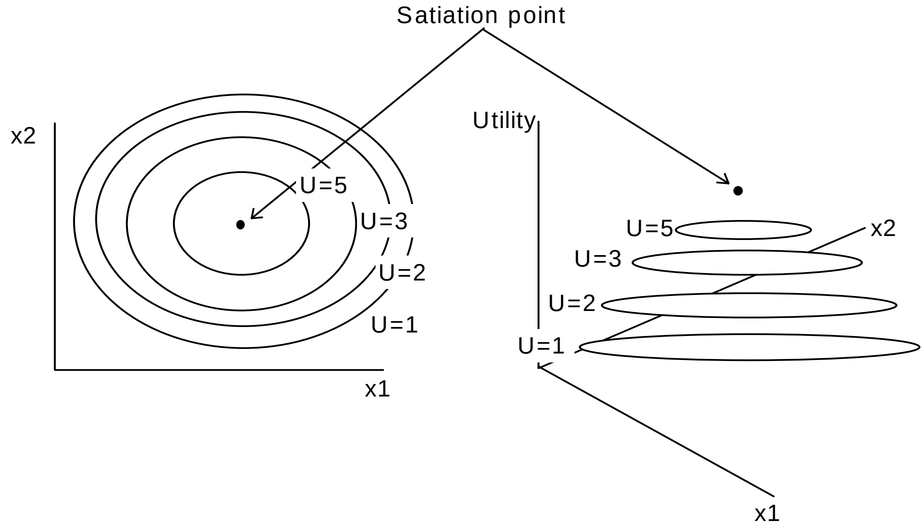
ii. What if we value a bundle C = (6,6) more than A and B. (Is this rational?) This new bundle of hamburger's and CD's will appear on a higher indifference curve.



iii. Properties of Indifference Curves:

a) Higher indifference curves imply higher utility.
b) I.C.'s give a visual representation of utility and of bundles that have the same utility since the utility along any indifference curve is constant.
c) I.C.'s almost always have the convex to the origin shape
d) I.C.'s never intersect: that would violate transitivity
e) I.C.'s are infinite in number.

iv. Why do I.C.'s usually have that shape? Consider the two graphs below.



a) The set of I.C.'s is what is called a *mapping*.
b) The graph on the left is what is a ***contour map***. Think of the maps of you have seen of mountainous regions were different elevations are labeled with different lines. Each circle is an indifference curve at different levels of utility.
c) Previously, we only viewed a portion of the map. Here we have the same picture, but we have backed up to look at the whole map.
d) To see this, start at the origin and move towards the satiation point. The curves are just the I.C.'s and we are moving to higher and higher levels of utility until we reach the satiation point.
e) If you keep moving out past the satiation point, utility starts to fall.
f) The graph on the right is the same picture, but now you can see the hill more clearly. This is a 3-D picture showing utility as elevation and the I.C.'s as describing particular levels of that elevation. *All this is shows that if an I.C. is bowed out from the origin we are really on the back side of the hill and the bundle of goods has become a bundle of* ***bads***.

B. What the *%!#$ does all that have to do with Investments?!
i. Consider two aspects of any investment: risk and return.

a) Risk is the variance of the payoff and is a bad.
b) Expected return is the good part of the investment.

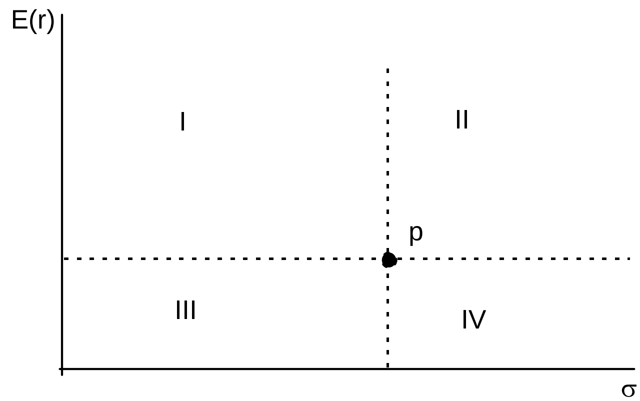
ii. A reasonable utility function for investments (there are many different types of utility functions as we will see) is

(1) $U = E(r) - .005A\sigma^2$
(2) where A is a measure of risk aversion.

b) More risk averse investors penalize risky investments more severely and will have a higher value of A.
c) What the equation says is that the utility from an investment is larger for higher expected rates of return but lower for higher variance.

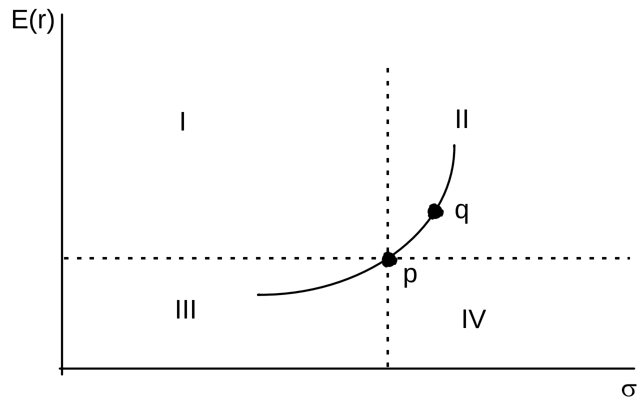
Example. a portfolio has E(r) = 20% and σ = 20%. T-Bills offer a sure rate of return of 7%. What investment will be taken if A = 4, if A = 8?

iii. Instead of graphing two *goods* and looking at utility, graph a *good and a bad*, E(r) and σ.



a) If p is a portfolio and we are dealing with a risk averse investor, then:
b) Any portfolio in quadrant IV is never preferred to p.
c) All portfolios in quadrant I are preferred to p.
d) a) and b) above form the *"mean-variance criterion"* which states that any asset A dominates B if
   (1) $E(r_a) \geq E(r_b)$ and
   (2) $\sigma_a \leq \sigma_b$
   (3) With one equality being strict.
   (4) Some of the portfolios in quadrants II and III may be preferred to p and others may not. It depends on the amount of risk aversion, A.

C. Indifference curves in E(r) - σ space
i. Some of the portfolios in quadrants II and III will have expected returns and standard deviations, which will yield the same level of utility as portfolio p.
ii. Consider portfolio q. It has a higher E(r) and higher σ. If given the level of risk aversion A, this portfolio has the same utility as portfolio p, then the risk averse investor should be indifferent between the two portfolios



iii. If we connect all the portfolios that yield the same level of utility as portfolio p, we have an indifference curve. This will be very useful in developing a model of investor behavior concerning investments.

D. Other Utility Functions and a more Formal derivation of Risk Aversion

i. There are many different utility functions to work with. Choice of any particular one is usually made on the basis of mathematical ease or particular constraints that may be associated with a given problem.

Log utility:	$U(W) = \ln(W)$
Power utility:	$U(W) = -W^{-1}$
Quadratic utility:	$U(W) = aW - bW^2$

5. **Tool Box: Portfolio Mathematics**

A. Four rules for dealing with risky assets and portfolios

$E(r) = \Sigma Pr(s)r(s)$: Where Pr(s) = probability of scenario s
$\sigma^2 = \Sigma Pr(s)[r(s) - E(r)]^2$

$E(r_p) = w_1E(r_1) + w_2E(r_2)$:
Where w_i is the weight or amount of the portfolio invested in asset i.

$\sigma_p^2 = w_1^2\,\sigma_1^2 + w_2^2\,\sigma_2^2 + 2w_1\,w_2\sigma_{1,2}$
Where $\sigma_{1,2} = Cov(r_1, r_2) = E\{[r_1 - E(r_1)][r_2 - E(r_2)]\}$
$= \Sigma Pr(s)[r_1 - E(r_1)][r_2 - E(r_2)]$
$= \rho_{1,2}\sigma_1\sigma_2$
and $\rho_{1,2}$ is the correlation coefficient.

Example: Consider an assets return in three possible states

	States of the World		
	Good	Fair	Crisis
Probability	.5	.3	.2
Return	20%	10%	-5%

What is the expected return on this asset?

$E(r) = \Sigma Pr(s)r(s) = Pr(Good)(20) + Pr(Fair)(10) + Pr(Crisis)(-5)$
$= .5*20 + .3*10 + .2*(-5)$
$= 12\%$

What is the variance of the asset return?

$\sigma^2 = \Sigma Pr(s)[r(s) - E(r)]^2$
$= Pr(Good)(20 - E(r))^2 + Pr(Fair)(10- E(r))^2 + Pr(Crisis)(-5- E(r))^2$

= 91

What is the standard deviation of the asset return?

$= \sqrt{\sigma^2}$ = 9.539%

Build a portfolio with 20% of the portfolio being the asset above, asset 1, and 80% of the portfolio being the risk free asset, (asset 2) earning 8%.

What is the portfolio expected return?

$E(r_p) = w_1E(r_1) + w_2E(r_2)$
$= (.2)E(r_1) + (.8)E(r_2)$
= 8.8%

What is the portfolio variance and standard deviation?

$\sigma_p^2 = w_1^2\, \sigma_1^2 + w_2^2\, \sigma_2^2 + 2w_1\, w_2\sigma_{1,2}$
= 3.64

Note how the risk-free asset makes this calculation easy

$\sigma_p^2 = w_1^2\, \sigma_1^2$

B. What happens when we combine two risky assets into a portfolio?

	States of the World		
	Good	Fair	Crisis
Probability	.25	.5	.25
Return (1)	18%	16%	10%
Return (2)	6%	15%	24%

$\mathbf{E(r_1)}$: .25 * 18% + .5 * 16% + .25 * 10% = **15%**
$\mathbf{E(r_2)}$: .25 * 6% + .5 * 15% + .25 * 24% = **15%**

$\sigma^2_1 = .25(18\text{-}15)^2 + .5(16\text{-}15)^2 + .25(10\text{-}15)^2 = \mathbf{9}$
$\sigma_1 = (9)^{1/2} = \mathbf{3\%}$
$\sigma^2_2 = .25(6\text{-}15)^2 + .5(15\text{-}15)^2 + .25(24\text{-}15)^2 = \mathbf{40.5}$
$\Rightarrow \sigma_2 = (40.5)^{1/2} = \mathbf{6.4\%}$

Which stock would a risk averse investor choose?
What is the expected return and standard deviation of an equally weighted portfolio?

$\mathbf{E(R_p)}$ = .5 * 15% + .5 *15% = **15%**
$\sigma^2_p = (.5)^2 (9) + .5^2(40.5) + 2\,(.5)(.5)\sigma_{AB}$
σ_{AB} **=** .25(18 -15)(6 -15) +.5(16 -15)(15 - 15) + .25(10 - 15)(24 - 15)
= .25(3)(-9) + .5(1)(0) + .25(-5)(9) = **-18**
σ^2_p = 2.25 + 10.125 - 9 = **3.375**
σ_p **= 1.83**

C. We have reduced risk but *not* eliminated it. Why?
i. Total risk = Market risk + Idiosyncratic risk
ii. What assets have a low correlation with stocks that might yield this kind of powerful diversification?

6. Two Asset Model
A. Can we *reduce* risk further using the above portfolio?
i. Consider the portfolio as an asset. That is, fix the weights on the individual securities so that we have a new security, call it ***P***, with $E(r_p)$ = 15% and σ_p = 1.83
ii. Now, form a new portfolio by combining the new security *P* with the risk-free asset, call it *F*, earning 8%.

iii. DETOUR: what do we mean by "risk-free asset"?

a) Typically T-bills OR

b) money market mutual funds consisting of T-bills, CD's, and commercial paper

c) Are T-bills really risk-free?

(1) Duration

(2) *Interest rate risk*

B. Back to the risk reduction problem. Let *y* be the proportion of an investor's portfolio invested in the risky asset *P* and (1-*y*) be how much is put in the risk-free asset *F*.

i. Define r_c as the return on the "complete" portfolio, r_p as the return on the risky portfolio, and r_f as the return on the risk-free asset. Then we have:

$r_c = yr_p + (1-y)r_f$

and

$E(r_c) = yE(r_p) + (1-y)r_f$

$= r_f + y(E(r_p) - r_f)$

$= 8 + y(15 - 8)$

ii. In a word that means the expected return on the complete portfolio equals the risk-free rate plus a risk premium weighted by the level of exposure to the risky part as measured by *y*.

iii. *(What other formula does that look like?)*

iv. What is the variance of the complete portfolio?

$\sigma_c^2 =$

v. Therefore, our complete portfolio has an $E(r_c) = 8 + 7y$ and a standard deviation of $\sigma_c = y\sigma_p$.

C. I know that in and of itself this is pretty darn exciting to know, but check this out. Follow these next steps and watch what happens!

i. Solve out for *y* from the standard deviation formula to get

$y = \sigma_c/\sigma_p$

ii. Substitute that into the $E(r_c)$ formula to get

$E(r_c) = r_f + (\sigma_c/\sigma_p)(E(r_p) - r_f) = r_f + ([E(r_p) - r_f]/\sigma_p)\ \sigma_c$

iii. for us we have

$E(r_c) = 8 + \sigma_c[7/1.83]$

$= 8 + [3.83]\sigma_c$

iv. But what kind of function is that last equation?

v. The intercept is $r_f = 8$

vi. The slope is $S = [E(r_p) - r_f]/\ \sigma_p = 3.83$

a) This line is called the Capital Allocation Line (CAL)

b) The slope, S, is called the reward-to-variability ratio

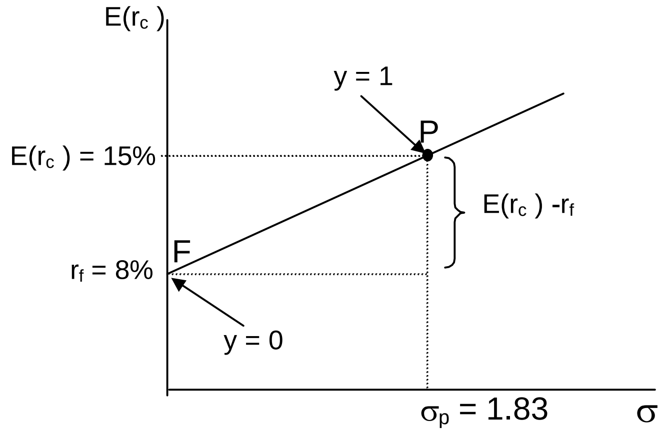
D. But take a look at the formula $\sigma_c = y\sigma_{p.}$ We have a fixed value for $\sigma_p = 1.83$, so as we vary the amount of our portfolio we invest in the risky asset *P*, we change the standard deviation of our complete portfolio. AND since the expected value of the complete portfolio only depends on σ_c, we also vary the expected return as we vary *y*.

If $y = 0 \Rightarrow E(r_c) = r_f = 8\%$

If $y = 1 \Rightarrow E(r_c) = r_p = 15\%$

If $0 < y < 1 \Rightarrow 8\% < E(r_c) < 15\%$

i. Check out the graph



ii. So, if our investor wants to reduce risk all she has to do is put some of her portfolio into the risk-free asset.

Example:

Let $y = 0.5$.
Then E(r) = 8 + .5(15 - 8) = 11.5
and σ = .5(1.83) = 0.915
Where does the portfolio with $y = 0.5$ fall on the line?

E. If our investor wants to increase risk and return using only these two assets, what can she do?
i. Simple, she should just set $y > 1$.
ii. This just means that she will be borrowing at the risk-free rate rather than lending.

Example:

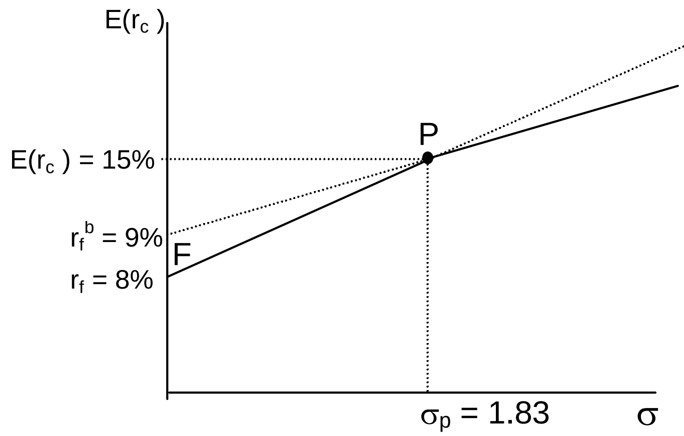
Let $y = 1.5$.
Then E(r) = 8 + 1.5(15 - 8) = 18.5
and σ = 1.5(1.83) = 2.745
Where does the portfolio with $y = 1.5$ fall on the line?

Does the reward-to-variability ratio change in this situation?
$S = [E(r_p) - r_f]/ \sigma_p =$

F. It is more realistic to assume that investors can lend at the risk-free rate r_f but must borrow at a higher rate, r_f^b.
i. What does this do to the model?
Let $r_f = 8\%$ as before and let $r_f^b = 9\%$.
a) If the investor is lending then
$S = [E(r_p) - r_f]/ \sigma_p = 3.83$

b) If the investor is borrowing then
$S = [E(r_p) - r_f^b]/ \sigma_p = 3.28$

ii. Hence, the slope of the CAL is different for the two positions (leveraged positions lay on the flatter portion). See the graph.



Remark: The CAL defines the *set of possible investments* for our investor.

7. The problem of an investor.
   A. If our investor is restricted to invest in the two assets F and P above, then her problem comes down to finding the *y* that will make her the happiest.
   B. Mathematically we write the problem down as

      Max U = $E(r_c) - .005A\sigma_c^2$
      But $E(r_c) = r_f + y(E(r_p) - r_f)$
      and $\sigma_c^2 = y^2\sigma_p^2$.
      so
      Max U = $r_f + y(E(r_p) - r_f) - .005Ay^2\sigma_p^2$.

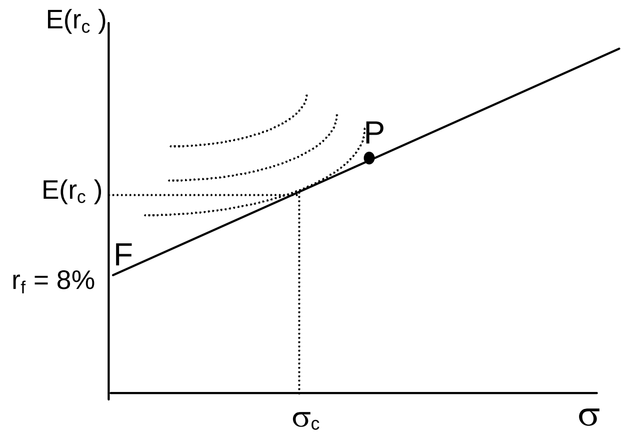
   C. Now, how do we find the *y* that maximizes utility?
      i. Simple…set the first derivative of the utility function with respect to *y* equal to zero and solve for *y*.

$$\frac{\partial U}{\partial y} = E(r_p) - r_f - .01yA\sigma_p^2 = 0$$

$$\Rightarrow y^* = \frac{E(r_p) - r_f}{0.01A\sigma_p^2}$$

      ii. In words, this *y** is the optimal amount for our investor who has the given utility function and level of risk aversion A to invest in the risky portfolio.
      iii. Remark: Notice that *y** is inversely proportional to the level of risk aversion AND the level of risk, but directly proportional to the risk premium offered on the risky asset.

   D. Graphically, we just found the point along the CAL that is tangent to the highest indifference curve. This gives the *y** that yields the utility maximizing "optimal" portfolio for this investor



In our example: y^* = [15 - 8]/(0.1*A*(1.83)^2)
Let A = 2, then
y^* =

$E(r_c)$ =

σ_c =

8. Passive investment strategy
   A. If we let the portfolio P be a market index such as the S&P 500 or an index mutual fund and we combine that with a risk-free asset such as a mutual fund of T-bills the CAL becomes what is known as the Capital Market Line (CML).
      i. A passive strategy is one that does not involve any security analysis.
      ii. Holding a portfolio of a market index and the risk-free asset is one type of passive strategy.
      iii. Advantages of a passive strategy
      iv. Disadvantages of a passive strategy

9. Attributes and implications of diversification:
   A. Diversification reduces risk:
      i. Why?
      ii. Correlation
      iii. Most stocks are positively correlated so we will never completely eliminate risk

Investors should not care about the risk of individual securities but they should be concerned about how the risk of that security affects the risk of their portfolio.

      iv. We don't have to care about the idiosyncratic risk of any particular asset, only about how it affects the risk of our portfolio
      v. The contribution of an individual asset to portfolio risk depends on
         a) the standard deviation of the asset's returns,
         b) the correlation of the asset's returns with returns of the other assets in the portfolio, and
         c) the proportion of the portfolio represented by that asset.

   B. We had a two stock portfolio and it reduced risk; what happens when we add more assets?
      i. We only need to invest in about 10 securities to almost completely eliminate any *unique risk*.
      ii. That does NOT imply that it is possible to eliminate risk altogether.

   C. Total risk = Market risk + Unique risk

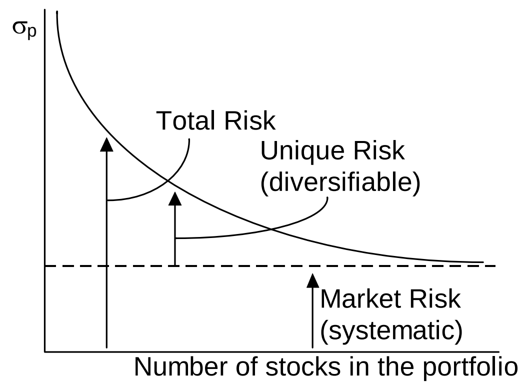
Market or **systematic risk,** which all stocks experience and to a greater or lesser extent, cannot be diversified away.

      i. Includes: unexpected changes in GDP, interest rates, oil price, and business cycles.

Unique risk is **diversifiable** or **unsystematic risk**. Represents risks that are unique to a particular firm, industry, location, country, and asset group.

ii. Includes: changing tastes, labor strikes, new product developments, marketing, raw materials price shocks.

D. What we have been doing with diversification is reducing the unsystematic risk of the given stocks.



E. How does the level of correlation between assets affect the mean and standard deviation of a portfolio?

i. For any portfolio of two assets we have

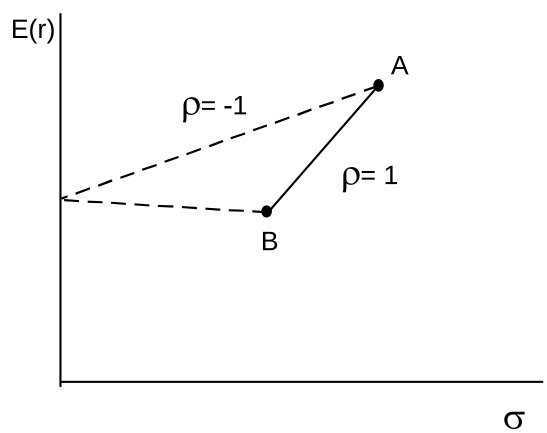
$E(r_p) = w_1E(r_1) + w_2E(r_2)$

$\sigma_p^2 = w_1^2\,\sigma_1^2 + w_2^2\,\sigma_2^2 + 2w_1\,w_2\rho_{1,2}\sigma_1\sigma_2$

ii. The expected return depends on the weights.

iii. The variance depends on the weights and on the correlation structure between the assets.

iv. To see how the correlation affects the portfolio opportunity set, fix the correlation and graph all possible combinations of the weights.



The remarkable aspect of this is that portfolios of less than perfectly correlated assets always offer better risk-return opportunities than the individual component securities on their own.

F. Is it possible to define the weights of a two-asset portfolio so that we always have the "minimum variance" portfolio along the opportunity set?

i. Since $w_2 = (1 - w_1)$ we can substitute that into the formula for the variance, take the first derivative, set it equal to zero and solve for w_1.

ii. The solution to that minimization problem is

$w_{1min} = [(\sigma_2^2 - cov(r_1, r_2)] / [\sigma_1^2 + \sigma_2^2 - 2cov(r_1, r_2)]$
and
$w_{2min} = 1 - w_1$.

10. The investors' problem reformed.
   A. Given two (or more) risky assets and a risk-free asset, how does the investor go about deciding which portfolio of assets to invest in to maximize utility?

      i. Recipe:
         a) Specify the return characteristics of all securities.
         b) Calculate the "optimal" risky portfolio
         c) Calculate the weight *y** to invest in the risky portfolio
         d) Calculate the weights to all the assets in the complete portfolio

Example:
Specify the return characteristics of all securities

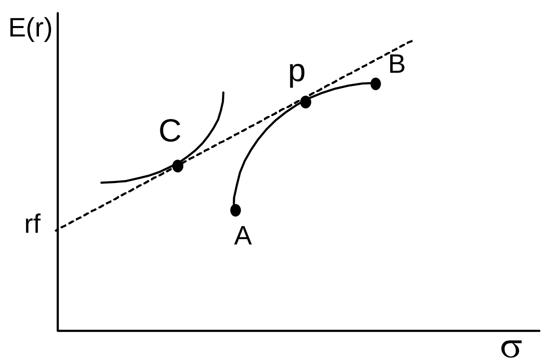
This is just calculating the expected returns, variances and covariance's. We may do this either by forecasting out probabilities or by using historical data.

Suppose these are given as

Asset	E(r)	Std. Dev
A	10	20
B	30	60
Risk-free	5	0
Corr(A,B)	-0.2	

Calculate the optimal risky portfolio.

This will be the portfolio that is the point on the curve that is tangent to a line from the risk-free asset.



In this step we are looking for the weights that give us the portfolio P. This portfolio has the maximum Sharpe ratio. All we have to do then is to solve the maximization problem

$$\underset{W_i}{Max}\, S = \frac{E(r_p) - rf}{\sigma_P}$$
$$s.t. \sum_i w_i = 1$$

This is straightforward calculus, but it can be extremely tedious if there are more than two assets in the risky portfolio.

Using solver, I got $w_a = .6818$ and $w_b = .3281$.

Now we need to calculate *y**, the amount to invest in the optimal risky portfolio P.

Remember that this is really just a utility maximization problem. We are trying to find the weights of the portfolio, C, that will be tangent to the highest indifference curve.

That is we want to

Max U = $r_f + y(E(r_p) - r_f) - .005Ay^2\sigma_p^2$
$y^* = [E(r_p) - r_f] / [.01A\sigma_p^2]$

We need the risky portfolio expected return, variance, and a level of risk aversion for this investor.

The expected return of portfolio P is
$E(r_p) =$

= 16.26

The variance of portfolio P is
$\sigma_p^2 =$

= 21.13

Let A = 5, then
y* =

= .5089

Hence 50.89% of the complete portfolio should be in the optimal risky portfolio P and the rest should be in the risk-free asset.

The only remaining question is how much of stock A to buy and how much of stock B to buy.

Simply multiply the weights in the optimal risky portfolio by .5089.