Math Review

Lines:

We use these all the time in finance and economics.
A linear relationship between two variables X and Y is defined by the function

$$\mathbf{Y = mX + b}$$

Here **m** is the slope of the line and **b** is the intercept. The intercept represents where the line crosses the Y-axis and slope defines the direction in which the line tilts (slope=rise over run). Positive slope implies that the line tilts up and a negative slope means the line slopes down.

As an example, the demand curve is often assumed to be a linear relationship between quantity demanded (Q) and price (P). That is, quantity demanded *depends* on the price of a good. In the most general terms we say quantity demanded is a function of price and we represent that general statement with the mathematical statement Q(P), read Q is a function of P. If we assume a linear relationship between quantity demanded and price, then we know the form of the demand function, it is a line. Let the slope of the demand curve, m, be -2 and the intercept, b, be 4. The *demand function* is then

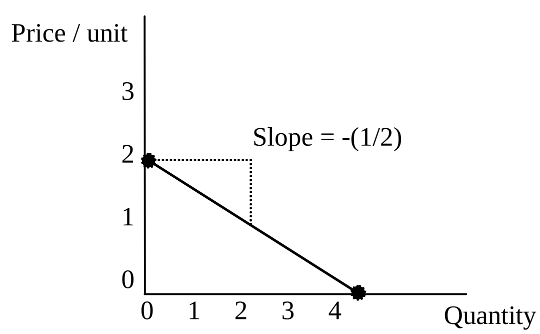
$$\mathbf{Q = 4 - 2P.}$$

Rearranging, the *inverse demand function* is then

$$\mathbf{P = 2 - (1/2)Q.}$$

(The inverse demand function is usually what you will see graphed, this is just the way economists do things. There is a one-to-one transformation between the demand and inverse demand functions so it doesn't really matter which way you write it.) Note that the negative slope indicates that as the price increases the quantity demanded falls. [This should be a familiar relationship – the *First Law of Demand*.]

Graphing lines is easy; just find one of the variables when the other is zero. For instance, if Q equals zero, then P equals 2. I got that from the inverse demand function. From the demand function we can find that Q equals 4 when P equals zero. To get the graph, plot these two intercepts and connect the dots.



This trick works for upward sloping lines as well. Try graphing the peculiar demand function Q = 4 + 2P. In this case the inverse demand function is P = -2 + (1/2)Q. Once again, first solve for Q when P = 0 (Q=4) and then solve for P when Q is 0 (P = -2). Here we must extend the price axis down into the negative quadrant to plot our line.

The Calculus:

The Calculus is probably the most useful of all the mathematics you will ever learn. We will be using it all the time in finance. You will also see many applications in your business school studies especially in the areas of Economics, Statistics, and Inventory Control. The really neat thing about calculus is how much easier problems become when you can use calculus rather than rely solely on algebra to find a solution.

We will be using calculus primarily to find minimums and maximums of functions. All you have to remember is a few rules and a couple of key concepts and you will be able to do all the calculus you will ever need in the real world; not to mention, the significant improvement your grade will experience if you know this stuff. Let's get on with it.

Remember that the slope of any function is the *first derivative* of that function. Remember that the slope of a line is the 'rise over the run' which is equivalent to 'the change in Y over the change in X'. Take the inverse demand function

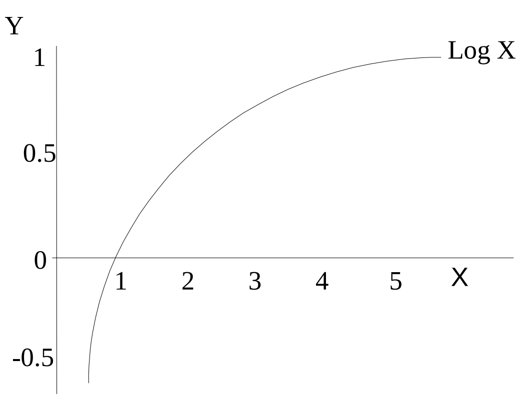
$$P = 2 - (1/2)Q.$$

Here **dP/dQ,** (read as the first derivative of price with respect to quantity) is **-(1/2)**. Notice that the slope of a *linear* function is always going to be a constant like **-(1/2)**.

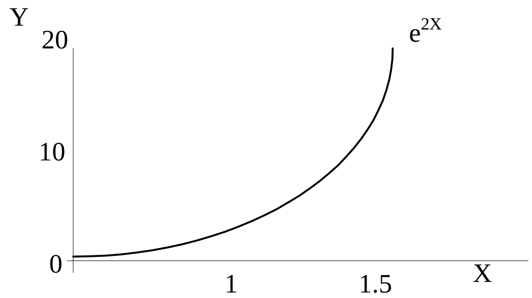
What if we are dealing with a function that is non-linear such as the following?

$$Y = \log X.$$

What is the slope of this function? Taking the first derivative with respect to X we get **dY/dX = 1/X**. Notice that the slope is <u>not</u> a constant. This should make sense since we have a curve instead of a line. The slope will be different at all points along X, i.e. when X =1 the slope is 1, when X = 2 the slope is ½.



Note that as we increase X the slope of the function log X is always positive but shrinking. Said differently, the function, log X, increases at a decreasing rate. If we plotted an exponential function such as $Y = e^{2X}$ we would have a function with a positive slope ($dY/dX = 2e^{2x}$) but that increased at an increasing rate.



A cool trick that allows you to know how a function behaves is to take the second derivative (the derivative of the derivative). This will tell you the rate at which the function is increasing or decreasing.

The first derivative of $Y = \log X$ is $dY/dX = 1/X$ and so the second derivative (the derivative of the derivative) is $d^2Y/dX^2 = -1/(X^2)$. Since the first derivative is positive we know that the function is increasing AND since the second derivative is negative we know that it is increasing at a decreasing rate. For $Y = e^{2X}$ we have the first derivative as $dY/dX = 2e^{2x}$ and so $d^2Y/dX^2 = 4e^{2x}$. Here the positive first and second derivatives indicate that the function is increasing at an increasing rate.

I know at this point you head is about to explode, but just hold on. Let me show you how to use the calculus to make money!

Optimization:

Financial economists are always trying to optimize stuff. Often we will be concerned with ideas like the maximum amount of profit a firm can earn, or the maximum amount of utility a consumer can achieve, or the minimum amount of cost to produce a particular level of output, or the minimum amount or risk in a portfolio.

Let's suppose we want to find the maximum point of the following function.

$$\mathbf{y = f(x) = - 4x^2 + 2x + 9}$$

Just looking at it we know that it is not linear (it is a quadratic function). We want to know what x will give us the biggest y. How do we find the maximum point of this function? Well, without calculus we would have to draw the function and find the highest point on the graph and call that the maximum. Using calculus we have it much easier.

Here is the recipe for optimizing any function:
Step 1) find the first derivative and set it equal to zero. (this is called the *necessary first order condition*)

Step 2) find the second derivative and figure out whether it is positive or negative. (this is called the *sufficient second order condition*)

$$\mathbf{y = f(x) = - 4x^2 + 2x + 9}$$

$$\frac{dy}{dx} = f'(x) = -8x + 2 = 0$$
$$\Rightarrow -8x = -2$$
$$\Rightarrow x = \frac{1}{4}$$

The first order condition says that our quadratic will be optimized when **x = 1/4**.

We have to check the second order condition to figure out if we have a maximum or minimum at this point. So, we just take the derivative of the function that is left after we took the first derivative.

$$\frac{dy}{dx} = f'(x) = -8x + 2$$
$$\frac{d^2y}{dx^2} = f''(x) = -8$$

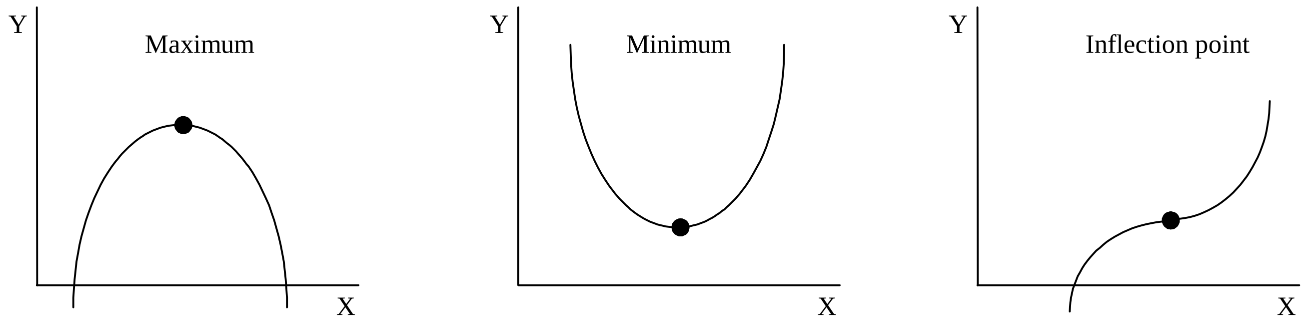
Since the second derivative is negative, we know we have a maximum.

Now think about this for a second. If we are at a point where the slope is zero (we set the first derivative equal to zero), the second derivative tells us that if we move a little bit to the right of that point that the slope of the line will decrease. Since a decrease in the slope from zero means the slope would be negative then we would be heading downhill and we must be at a maximum! If the second derivative was positive, then we know that a small movement away from zero would increase the slope and hence we must be at the bottom of the hill or at the minimum. Fancy huh?

Here is a little table to help you remember how to tell whether the first order condition represents a maximum, minimum, or inflection point.

if f''(x) > 0	**then f'(x) is a minimum**
if f''(x) < 0	**then f'(x) is a maximum**
if f''(x) = 0	**then f'(x) is a inflection point**

Maybe some pictures will help to clear up the difference between minimums, maximums, and inflection points.



Looking at the pictures it should be pretty intuitive why the necessary first order condition is to set the first derivative equal to zero. A line with a slope of zero is a horizontal line. Since the first derivative is the slope of the function, when the slope is zero we are either at the top or the bottom of the function.

Areas under curves:

Sometimes we may be interested in finding the area under a curve. To do this algebraically would be a real pain, but using calculus it is a snap. Suppose we wanted to find the area under our quadratic between the two points x = 1 and x = 5. To do this we just need to integrate the function over the range {1,5}.

$$\int_1^5 -4x^2 + 2x + 9 \text{ dx}$$

$$= -\frac{4}{3}x^3 + x^2 + 9\text{x} \Big|_1^5$$

$$= \left(-\frac{4}{3}5^3 + 5^2 + 9(5)\right) - \left(-\frac{4}{3}1^3 + 1^2 + 9(1)\right)$$

$$= (-96.67) - (8.67)$$

$$= -105.34$$

The steps for finding the area under a curve are as follows.

Step 1) Find the integral

Step 2) Plug the number on top of the integral sign (this will be the bigger number) into the integral for x. Subtract off of that the integral evaluated at the number from the bottom of the integral sign.

Step 3) Do the necessary arithmetic.

Matrices:

Working with matrices is probably new to most of you so let's start with the basics. A matrix is any set of numbers or variables grouped together in a rectangle. There are two dimensions to every matrix, the number of rows and the number of columns. The dimensions are always given as the number of rows and then the number of columns. (Say that last sentence out loud a few times until you memorize it.)

For example here is the simplest matrix possible known as a scalar matrix

$$[1]$$

This is a 1x1 matrix, one row and one column. Here are some matrices with the following dimensions: a 2x1, a 1x2, a 3x3 and a 4x2.

$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a & b \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 4 & 3 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ a & c \\ a & d \\ a & f \end{bmatrix}$$

A matrix with just one row is called a row vector, such as the 1x2 above and a matrix with just one column is called a column vector such as the 2x1 matrix above. Sometimes people just say vector rather than row or column vector. When talking about particular elements of a matrix you use the row and column position. So the variable *b* is the (2,1) element in the first matrix above and is the (1,2) element of the last matrix.

Define a matrix *A* as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 4 & 3 & 1 \end{bmatrix}.$$

The transpose of a matrix is denoted as *A'* or A^T and is simply a rearrangement of the matrix so that the first row becomes the first column, the second row becomes the second column and so on. The transpose of *A* is

$$A' = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 3 \\ 3 & 4 & 1 \end{bmatrix}.$$

The transpose of a row vector is a column vector and vice versa. There are some other neat properties of transposes that we might get to later.

You can do arithmetic with matrices as long as you follow some simple rules.

1) **To add or subtract two matrices they must have the same dimensions and the adding or subtracting is done element by element.** So to add a matrix *B* to *A'* above, *B* must me a 3x3.

If $B = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 3 \\ 5 & 3 & 1 \end{bmatrix}$ then $A' + B = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 3 \\ 3 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 9 \\ 6 & 6 & 6 \\ 8 & 7 & 2 \end{bmatrix}$

2) To multiply two matrices

a) **they must have the same inner dimensions**
b) **the dimensions of the result will be the outer dimensions**
c) **the multiplication is done row by column.**

When you look at the dimensions of the two matrices the number of columns of the first matrix must be the same as the number rows in the second matrix. Hence, you can multiply a 3x2 by a 2x3 to get a 3x3 but you cannot multiply a 3x2 by another 3x2. For example let's multiply *A'* by a couple of different matrices.

Define *C* as $C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then $A' * C = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 3 \\ 3 & 4 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 14 \end{bmatrix}$.

We can do this since *A* is a 3x3 and *C* is a 3x1. That means that the answer of *A***C* will be a 3x1. Here is how you get the answer. Multiply the elements of the first row of *A* by the elements of the first column of *C* and add up the results.

1*1 + 1*2 + 4*3 = 15
2*1 + 3*2 + 3*3 = 17
3*1 + 4*2 + 1*3 = 14

Now let's multiply *A* and *B*.

$$A * B = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 3 \\ 3 & 4 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 3 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 26 & 19 & 12 \\ 31 & 26 & 22 \\ 27 & 27 & 28 \end{bmatrix}$$

The first column of the result comes from

1*2 + 1*4 + 4*5 = 26
2*2 + 3*4 + 3*5 = 31
3*2 + 4*4 + 1*5 = 27

The second column of the result comes from

1*4 + 1*3 + 4*3 = 26

2*4 + 3*3 + 4*3 = 31
3*4 + 4*3 + 1*3 = 27

and so on. Notice that to get the first column of the result we used all the rows of *A* and the first column of *B*. To get the second column of the result we used all the rows of *A* and the second column of *B* and so on.

You might be freaking out at this point but just wait. Once you see how convenient it is to work with matrices in finance you will wonder how you ever made it without them. I will also show you how to do the arithmetic in Excel so that you don’t have to do this all by hand.

Cheat sheet

Exponents: Here are a couple of rules for exponents that may come in handy.

$$(X^n)(X^m) = X^{n+m} \qquad (XY)^m = X^m Y^m$$
$$(X^n)^m = X^{nm} \qquad (X/Y)^m = X^m / Y^m$$

Logarithms: Here are some rules for logarithms that *will* come in handy.

$$\log U + \log V = \log UV$$
$$n(\log U) = \log U^n$$
$$\log(U/V) = \log U - \log V$$

Derivatives: Here are the basic rules for taking derivatives. In the table c and n are constants, x is a variable, u and v are functions of x.

In General		Example	
F(x)	f'(x)	f(x)	f'(x)
C	0	4	0
x^n	nx^{n-1}	x^3	$3x^2$
uv	u'v + v'u	xlog(x)	$1(\log x) + \left(\frac{1}{x}\right)x$
$\frac{u}{v}$	$\frac{u'v - v'u}{v^2}$	$\frac{x}{\log x}$	$\frac{1(\log x) - \left(\frac{1}{x}\right)x}{(\log x)^2}$
Log(c + nx)	$\frac{n}{c+nx}$	log(2 + 3x)	$\frac{3}{2+3x}$
	or	log(x)	1 / x
e^{cx}	ce^{cx}	e^x	e^x

Integrals: Here are the basic rules for taking anti-derivatives. In the following table c and n are constants, x is a variable. Let k be the constant of integration in these examples. If you have an indefinite integral you MUST include that constant.

In General		Example	
F(x)	∫f(x) dx	f(x)	∫f(x) dx
C	cx + k	4	4x + k
x^n	$\frac{1}{n+1}x^{n+1} + k$	x^3	$\frac{1}{4}x^4 + k$
1/x	logx + k		
e^{cx}	$\frac{1}{c}e^{cx} + k$	e^x	$e^x + k$
	or	$e^{(3x + 2)}$	$\frac{1}{3}e^{(3x+2)} + k$