

The inviscid limit of Navier-Stokes equations for vortex-wave data on \mathbb{R}^2

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February 21, 2019

*This paper is dedicated to Walter Strauss
on the occasion of his 80th birthday, as token of friendship and admiration.*

Abstract

We establish the inviscid limit of the incompressible Navier-Stokes equations on the whole plane \mathbb{R}^2 for initial data having vorticity as a superposition of point vortices and a regular component. In particular, this rigorously justifies the vortex-wave system from the physical Navier-Stokes flows in the vanishing viscosity limit, a model that was introduced by Marchioro and Pulvirenti in the early 90s to describe the dynamics of point vortices in a regular ambient vorticity background. The proof rests on the previous analysis of Gallay in his derivation of the vortex-point system.

1 Introduction

In this paper, we are interested in the vanishing viscosity limit of the incompressible Navier-Stokes equations on the plane \mathbb{R}^2 for irregular initial data; namely, we consider

$$\begin{aligned}\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu &= \nu \Delta u^\nu, \\ \nabla \cdot u^\nu &= 0,\end{aligned}\tag{1.1}$$

for fluid velocity $u^\nu(x, t) \in \mathbb{R}^2$ and pressure $p^\nu(x, t) \in \mathbb{R}$ at $x \in \mathbb{R}^2$ and $t \geq 0$. The interest is to understand the asymptotic behavior of solutions in the inviscid limit $\nu \rightarrow 0$.

It is straightforward to show that in the absence of spatial boundaries, regular solutions of the Navier-Stokes equations converge in strong Sobolev norms to the regular solutions of Euler equations as $\nu \rightarrow 0$ (e.g., [15, 31, 26]). The convergence (in L^2 for velocity fields) also holds for non-smooth solutions that include vortex patches [5, 6, 3, 26, 30]. The problem is largely open for less regular data [2, 4], or even for regular data in domains with a boundary (e.g., [28, 18, 27, 14] and the references therein).

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For initial data whose vorticity consists of a finite sum of point vortices (Dirac masses), Gallay [10] proved that the corresponding Navier-Stokes vorticity indeed converges weakly in the inviscid limit to the sum of point vortices whose centers evolve according to the Helmholtz-Kirchhoff point-vortex system. In this paper, we study the case when initial vorticity consists of one point vortex and a regular part. The case of finitely many point vortices can be treated similarly in combination of [10] where the vortex-point interaction is understood.

Let us now detail the problem. For velocity field $u^\nu = (u_1^\nu, u_2^\nu)$, let $\omega^\nu = \partial_{x_2} u_1^\nu - \partial_{x_1} u_2^\nu$ be the corresponding vorticity. Taking advantage of the divergence-free condition, we can recover the velocity from vorticity through the so-called Biot-Savart law

$$u^\nu = \nabla^\perp \Delta^{-1} \omega^\nu = K \star \omega^\nu, \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad (1.2)$$

where $K(x)$ denotes the Green kernel of $\nabla^\perp \Delta^{-1}$, the \star notation stands for the usual convolution in variable $x \in \mathbb{R}^2$, and $a^\perp = (a_2, -a_1)$ for vectors $a \in \mathbb{R}^2$. It follows from (1.1) that the vorticity solves

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu. \quad (1.3)$$

We solve the vorticity equation (1.3), together with (1.2), for initial data of the form

$$\omega^\nu|_{t=0} = \delta_{z_0}(x) + \omega_0^E(x), \quad (1.4)$$

where δ_{z_0} denotes the Dirac delta function centered at $x = z_0$ and ω_0^E is the regular component of vorticity that has compact support and vanishes in a neighborhood of z_0 . The existence and uniqueness for 2D Navier-Stokes equations with such initial data, or in fact more generally, with initial data of finite measures are known; see, for instance, [7, 12, 15, 9].

1.1 Vortex-wave system

In the inviscid limit, we do not expect the limiting solutions from (1.3)-(1.4) to satisfy Euler equations, even in a weak sense*, but rather the following so-called vortex-wave system coined by Marchioro and Pulvirenti [23, 25] in the early 90s:

$$\begin{aligned} \partial_t \omega^E + (v^E + H) \cdot \nabla \omega^E &= 0 \\ \dot{z}(t) &= v^E(t, z(t)), \\ \omega^E|_{t=0} &= \omega_0^E, \quad z(0) = z_0, \end{aligned} \quad (1.5)$$

in which $v^E = K \star \omega^E$ and $H = K(\cdot - z(t))$. That is, in the limit, the regular component of vorticity is transported by the full velocity, while the location of point vortex is propagated by the velocity v^E generated by the regular vorticity ω^E .

The global weak solutions of (1.5) in $L^1 \cap L^\infty$ were already obtained in [23, 25] (see also [17, 8] for an extension to L^p spaces), while their uniqueness is proved for Lipschitz or even bounded data [29, 16], provided the ambient velocity is constant in a neighborhood of the point vortex. In particular, let us recall the following theorem.

*In fact, it is not known whether weak solutions to Euler equations exist with point vortex data [23, 25].

Theorem 1.1 ([16]). *Consider initial data $z_0 \in \mathbb{R}$ and $\omega_0^E \in L^1 \cap L^\infty(\mathbb{R}^2)$. Assume that ω_0^E has compact support and is constant in a neighborhood of z_0 . Then, there are a unique global solution $(z(t), \omega^E(t))$ to (1.5) and a positive function $R(t)$ so that $\omega^E(t)$ remains constant in the ball centered at the point vortex $z(t)$ with radius $R(t)$ for all times $t \geq 0$. If we assume in addition that $\omega_0^E \in W^{k,p}$ for $kp > 2$ and $p > 1$, then for any $T \geq 0$, there holds*

$$\sup_{0 \leq t \leq T} \|\omega^E(t)\|_{W^{k,p}} \leq C_T \quad (1.6)$$

for some constant C_T .

Theorem 1.1 assures that $H = K(\cdot - z(t))$ remains regular in the support of $\nabla \omega^E(t)$. The stated regularity (1.6) thus follows from that of Euler equations on \mathbb{R}^2 ([19]).

The vortex-wave system (1.5) can be rigorously derived from Euler equations by replacing the initial Dirac mass δ_{z_0} by $\epsilon^{-2}\chi_\epsilon$, for χ_ϵ being the characteristic function of the ball $\{|x - z_0| \leq \epsilon\}$ and taking $\epsilon \rightarrow 0$. This was done in [24] (see also [1, 13]). It can also be derived from Navier-Stokes equations in the small viscosity limit, provided that $\nu \leq \epsilon^\alpha$ for $\alpha > 0$, as done similarly for the vortex-point system [20, 21, 22]. In this paper, we give a direct derivation of (1.5) as the inviscid limit of the Navier-Stokes flows (1.3) with data (1.4).

1.2 Main result

Consider the viscous problem (1.3) with initial data (1.4). Following [9, 10], we first decompose the vorticity into the so-called regular part $\omega^{E,\nu}$ and irregular part $\omega^{B,\nu}$, both of which are advected by the full velocity vector field $u^\nu = K \star \omega^\nu$. Precisely, we write

$$\omega^\nu = \omega^{E,\nu} + \omega^{B,\nu}, \quad (1.7)$$

where $\omega^{E,\nu}$ and $\omega^{B,\nu}$ solve

$$\begin{aligned} \partial_t \omega^{E,\nu} + u^\nu \cdot \nabla \omega^{E,\nu} &= \nu \Delta \omega^{E,\nu}, \\ \omega^{E,\nu}|_{t=0} &= \omega_0^E, \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} \partial_t \omega^{B,\nu} + u^\nu \cdot \nabla \omega^{B,\nu} &= \nu \Delta \omega^{B,\nu}, \\ \omega^{B,\nu}(t) &\rightharpoonup \delta_{z_0} \quad \text{as } t \rightarrow 0^+. \end{aligned} \quad (1.9)$$

Here and in what follows, the weak convergence for finite measures is understood in the following sense: $\mu_n \rightharpoonup \mu$ if and only if

$$\int_{\mathbb{R}^2} \phi d\mu_n \rightarrow \int_{\mathbb{R}^2} \phi d\mu,$$

for all the continuous functions ϕ that vanish at infinity. A direct computation shows that the decomposition preserves the mass:

$$\int_{\mathbb{R}^2} \omega^{E,\nu}(x, t) dx = \int_{\mathbb{R}^2} \omega_0^E(x) dx, \quad \int_{\mathbb{R}^2} \omega^{B,\nu}(x, t) dx = 1, \quad (1.10)$$

for all positive times. We shall prove that in the inviscid limit $\omega^{E,\nu} \rightarrow \omega^E$ and $\omega^{B,\nu}$ is concentrated near the point vortex $z(t)$, transported by v^E , yielding weak solutions to the vortex wave system with the same initial data (ω_0^E, z_0) . Precisely, our main theorem reads as follows.

Theorem 1.2. *Let $z_0 \in \mathbb{R}$ and $\omega_0^E \in W^{4,4}(\mathbb{R}^2)$ that has compact support and vanishes in a neighborhood of z_0 , and let $(z(t), \omega^E(t))$ and $\omega^\nu(t)$ be the unique solution to the vortex-wave system (1.5) and to the Navier-Stokes equation (1.3), respectively, with initial data $\omega_0 = \omega_0^E + \delta_{z_0}$. Then, there exists a time $T > 0$, independent of ν , such that the vorticity $\omega^\nu(t)$ can be written as*

$$\omega^\nu(x, t) = \omega^{E,\nu}(x, t) + \omega^{B,\nu}(x, t),$$

where $\omega^{E,\nu}(t), \omega^{B,\nu}(t)$ satisfy

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4 \cap L^{4/3}(\mathbb{R}^2)} &\leq C_T \nu, \\ \sup_{0 \leq t \leq T} t^{-1} \left\| \omega^{B,\nu}(t, x) - \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right\|_{L^1(\mathbb{R}^2)} &\leq C_T \nu, \end{aligned}$$

for some constant C_T independent of ν . In particular, $\omega^{E,\nu}(t) \rightarrow \omega^E(t)$ strongly in $L^4 \cap L^{4/3}$ and $\omega^{B,\nu}(t, \cdot) \rightharpoonup \delta_{z(t)}(\cdot)$ weakly in the sense of finite measures in the inviscid limit.

Theorem 1.2 derives the vortex-wave system (1.5) as an inviscid limit of Navier-Stokes flows on the whole plane, complementing the earlier derivation [24, 1, 13] from Euler equations. In addition, we obtain:

$$T \geq \min \left\{ T_*^-, \frac{1}{5 \|\nabla v^E\|_{L^\infty}} \right\}$$

for T_* being the smallest time when the point vortex $z(t)$ meets the support of $\omega^E(s)$ for some $s \in [0, t]$, recalling from Theorem 1.1 that $w(t)$ never meets the support of $\omega^E(t)$ for all times. See Proposition 2.1 and Remark 3.15.

Let us now discuss some difficulties in proving the theorem. First of all, the initial data containing a Dirac mass are too singular to perform a direct proof from the standard L^2 energy estimates. One then needs to construct a good approximation of solutions to treat the singular part, and control the remainder. The difficulty arises due to the presence of an vortex-wave interaction term of the form

$$v^{E,\nu}(t, x) \cdot \nabla_x \left(\frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right). \quad (1.11)$$

Formally, this term blows up when x is near the point vortex $z(t)$ and $\nu t \rightarrow 0$. To treat this singularity, we follow [10] to work in the vortex scaling variable, construct approximate solutions, and perform weighted energy estimates to control the remainder. However, the weighted energy estimates with the scaling variable $\xi = \frac{x-z(t)}{\sqrt{\nu t}}$ used in [10] are not enough to treat the interaction term (1.11), as it leaves a remainder of order one, but not smaller. To overcome this difficulty, we introduce an *approximate viscous* vortex wave system (Section 2), along with the new point vortex $\tilde{z}(t) = z(t) + O(\nu t)$ and the scaled variable $\xi = \frac{x-\tilde{z}(t)}{\sqrt{\nu t}}$ in order to close the estimate.

Lastly, we remark that we assume the initial vorticity to be $\delta_{z_0} + \omega_0^E$, where ω_0^E is smooth and compactly supported away from the point vortex z_0 . The regularity is needed in the construction of the high order approximation of solutions. It would be interesting to further combine our analysis with the viscous approximation near vortex-patch solutions constructed in [30] to treat the case when $\omega_0^E \in L^1 \cap L^\infty$.

1.3 Notations

We will denote $A \lesssim B$ to mean that $|A| \leq C_0|B|$ for some universal constant $C_0 > 0$ independent of the viscosity ν . We write $f = O(g)$ to mean that $f \lesssim g$, or simply $O(g)$ to mean that the term can be bounded by $C_0|g|$ for some constant $C_0 > 0$ independent of ν . We define the norm $\|\cdot\|_{L^4 \cap L^{4/3}}$ and $\|\cdot\|_{L^1 \cap L^\infty}$ of a function $\omega(x)$ in \mathbb{R}^2 to be

$$\|\omega\|_{L^4 \cap L^{4/3}} = \|\omega\|_{L^4} + \|\omega\|_{L^{4/3}}, \quad \|\omega\|_{L^1 \cap L^\infty} = \|\omega\|_{L^1} + \|\omega\|_{L^\infty}$$

We also denote by $\mathfrak{m}(\cdot)$ the Lebesgue measure on \mathbb{R}^2 .

Acknowledgement: The authors would like to thank Thierry Gallay and Christophe Lacave for their many insightful discussions on the subject. The research was supported by the NSF under grant DMS-1764119 and by an AMS Centennial Fellowship. Part of this work was done while the authors were visiting the Department of Mathematics and the Program in Applied and Computational Mathematics at Princeton University.

2 Approximate vortex wave system

Let $(z(t), \omega^E)$ be the global solution to the vortex-wave system (1.5) with initial data $\omega_0^E \in W^{4,4}$ that has compact support and vanishes in a neighborhood of z_0 . We introduce an *approximate viscous* vortex-wave system $(\tilde{z}(t), \tilde{\omega}^E)$, given by

$$\begin{aligned} \tilde{\omega}^E(x, t) &= \omega^E(x, t) + \nu w_{1,a}(x, t) \\ \partial_t \tilde{z} &= \tilde{v}^E(\tilde{z}(t), t) = K \star \tilde{\omega}^E(\tilde{z}(t), t), \quad \tilde{z}(0) = z_0, \end{aligned} \tag{2.1}$$

where the added vorticity component $w_{1,a}$ solves

$$\partial_t w_{1,a} + \left(v^E + \frac{1}{\sqrt{\nu t}} v^G \left(\frac{x - z(t)}{\sqrt{\nu t}} \right) \right) \cdot \nabla w_{1,a} + v_{1,a} \cdot \nabla \omega^E = \Delta \omega^E \tag{2.2}$$

with zero initial data. Here and in what follows, velocity and vorticity are defined through the Biot-Savart law (1.2). For instance, $v_{1,a} = K \star w_{1,a}$ and $v^G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2} (1 - e^{-|\xi|^2/4})$.

We obtain the following simple proposition.

Proposition 2.1. *Let T_* be defined by*

$$T_* = \inf_{t \geq 0} \left\{ t : z(t) \in \cup_{0 \leq s \leq t} \text{supp}(\omega^E(s)) \right\}, \tag{2.3}$$

with $T_* = \infty$ if $z(t)$ never meets the support of $\omega^E(s)$ for $s \in [0, t]$. Then, for any $T < T_*$, the unique smooth solution $w_{1,a}(t)$ of (2.2) exists on $[0, T]$, has compact support, vanishes in a neighborhood of $z(t)$, and satisfies

$$\mathbf{m}(\text{supp}(w_{1,a}(t))) + \|w_{1,a}(t)\|_{W^{2,4}(\mathbb{R}^2)} + \|\partial_t w_{1,a}(t)\|_{L^\infty(\mathbb{R}^2)} + \|v_{1,a}(t)\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C_T, \quad (2.4)$$

for $t \in [0, T]$ and for some constant C_T independent of ν . In addition, there holds

$$|\tilde{z}(t) - z(t)| \leq C_T \nu t \quad \text{for any } t \in [0, T]. \quad (2.5)$$

Here, \mathbf{m} denotes the Lebesgue measure on \mathbb{R}^2 .

Corollary 2.2. Let T_* be defined as in (2.3). For any $T < T_*$, $\tilde{\omega}^E(t)$ has compact support, vanishes in a neighborhood of $\tilde{z}(t)$, and satisfies

$$\mathbf{m}(\text{supp}(\tilde{\omega}^E(t))) + \|\tilde{\omega}^E(t)\|_{W^{2,4}(\mathbb{R}^2)} + \|\partial_t \tilde{\omega}^E(t)\|_{L^\infty(\mathbb{R}^2)} + \|\tilde{v}^E(t)\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C_T, \quad (2.6)$$

for $t \in [0, T]$ and for some constant C_T independent of ν .

Proof. The corollary is a direct consequence of Proposition 2.1 and Theorem 1.1. \square

Proof of Proposition 2.1. Recall from Theorem 1.1 that $\omega^E(t)$ has compact support and vanishes in a neighborhood of $z(t)$. This remains valid for $w_{1,a}(t)$ for small times, due to the transport structure of (2.2). Precisely, $w_{1,a}(t)$ is supported in $\cup_{0 \leq s \leq t} \text{supp}(\omega^E(s))$. Since $z(t) \notin \text{supp}(\omega^E(t))$ for all positive times, we have $T_* > 0$ by continuity. Thus, for any $T < T_*$, there is a positive distance d_T so that

$$|x - z(t)| \geq d_T > 0 \quad (2.7)$$

for all $x \in \text{supp}(w_{1,a}(t))$ and $0 \leq t \leq T$, which yields

$$\left| \frac{1}{\sqrt{\nu t}} v^G \left(\frac{x - z(t)}{\sqrt{\nu t}} \right) \right| = \frac{1}{2\pi|x - z(t)|} \left(1 - e^{-\frac{|x - z(t)|^2}{4\nu t}} \right) \leq \frac{1}{2\pi|x - z(t)|} \leq \frac{1}{2\pi d_T}.$$

Similar estimates hold for derivatives of $v^G(\cdot)$ for x away from $z(t)$. It follows from (2.2) that

$$\begin{aligned} \|w_{1,a}(t)\|_{L^4} &\leq \int_0^t (\|\Delta \omega^E(s)\|_{L^4} + \|v_{1,a}(s)\|_{L^\infty} \|\nabla \omega^E(s)\|_{L^4}) ds \\ &\lesssim \int_0^t (1 + \|v_{1,a}(s)\|_{L^\infty}) ds, \end{aligned}$$

which yields the estimate on $w_{1,a}$, upon using the elliptic estimate $\|v_{1,a}\|_{L^\infty} \lesssim \|w_{1,a}\|_{L^4 \cap L^{4/3}}$ and the fact that $w_{1,a}$ is compactly supported. The derivative estimates follow similarly.

Finally, let us prove the estimate on $\tilde{z}(t)$. By definition, we write

$$\begin{cases} \tilde{z}(t) &= z_0 + \int_0^t (v^E(\tilde{z}(s), s) + \nu v_{1,a}(\tilde{z}(s), s)) ds, \\ z(t) &= z_0 + \int_0^t v^E(z(s), s) ds, \end{cases} \quad (2.8)$$

which gives

$$\begin{aligned} |\tilde{z}(t) - z(t)| &\leq \int_0^t |(v^E(\tilde{z}(s), s) - v^E(z(s), s))| ds + \nu \int_0^t |v_{1,a}(\tilde{z}(s), s)| ds \\ &\leq \int_0^t \|\nabla v^E(s)\|_{L^\infty} |\tilde{z}(s) - z(s)| ds + \nu t \sup_{0 \leq s \leq t} \|v_{1,a}(s)\|_{L^\infty}. \end{aligned} \quad (2.9)$$

Applying the Gronwall's lemma gives (2.5). \square

3 Inviscid limit for the irregular part

In this section, we give estimates on the irregular part of vorticity $\omega^{B,\nu}$, solving (1.9). Let us recall the equation:

$$\begin{aligned} \partial_t \omega^{B,\nu} + u^\nu \cdot \nabla \omega^{B,\nu} &= \nu \Delta \omega^{B,\nu}, \\ \omega^{B,\nu}|_{t=0} &= \delta_{z_0}. \end{aligned} \quad (3.1)$$

Here $u^\nu = v^{E,\nu} + v^{B,\nu}$ is the velocity field for the full Navier-Stokes equations. Following [10], we introduce the change of variables

$$\xi = \frac{x - \tilde{z}(t)}{\sqrt{\nu t}}$$

and write

$$v^{B,\nu}(x, t) = \frac{1}{\sqrt{\nu t}} v_2(\xi, t), \quad \omega^{B,\nu}(x, t) = \frac{1}{\nu t} w_2(\xi, t). \quad (3.2)$$

Here, we recall that $\tilde{z}(t)$ to be the solution to the approximate vortex wave system, given in (2.1). Note that the change of variables is consistent with the Biot-Savart law: $v_2 = K \star_\xi w_2$. Putting the Ansatz into the equation (1.9) for $\omega^{B,\nu}$, we get the following equation

$$\Phi(w_2, v^{E,\nu}) := (t\partial_t - \mathcal{L}) w_2 + \sqrt{\frac{t}{\nu}} (v^{E,\nu}(\tilde{z}(t) + \xi\sqrt{\nu t}, t) - \partial_t \tilde{z}(t)) \cdot \nabla_\xi w_2 + \frac{1}{\nu} v_2 \cdot \nabla_\xi w_2 = 0, \quad (3.3)$$

where \mathcal{L} is defined by

$$\mathcal{L} w_2 := \Delta_\xi w_2 + \frac{1}{2} \xi \cdot \nabla_\xi w_2 + w_2.$$

In the vanishing viscosity limit, we expect that the viscous regular velocity remains close to the inviscid one: $v^{E,\nu} \rightarrow v^E$, and hence the irregular part should tend to the so-called Lamb-Oseen vortex, which is defined by

$$G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \quad v^G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2} \left(1 - e^{-|\xi|^2/4}\right).$$

It follows that $\mathcal{L}G = 0$ and $v^G \cdot \nabla_\xi G = 0$. Therefore, the pair $(G(\xi), v^{E,\nu})$ solves (3.3), up to the following error term $tR_1(\xi, t)$, with

$$R(\xi, t) := \frac{1}{\sqrt{\nu t}} \left(v^{E,\nu}(\tilde{z}(t) + \xi\sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t) \right) \cdot \nabla G \quad (3.4)$$

which does not vanish in the inviscid limit, upon recalling that $\partial_t \tilde{z}(t) = \tilde{v}^E(\tilde{z}(t), t)$. Roughly speaking, $R = \mathcal{O}(1)$ in the small viscosity limit.

We shall construct better approximate solutions to the equation (3.3). Here we stress that the equation (3.3) involves two unknown functions $w_2, v^{E,\nu}$ which are coupled through the full velocity u^ν . To leading order, let us take $v_{app}^{E,\nu} = \tilde{v}^E$ for \tilde{v}^E solving the approximate vortex-wave system (2.1) and

$$w_{2,app}(\xi, t) = G(\xi) + (\nu t)w_{2,a}(\xi, t) \quad (3.5)$$

where $w_{2,a}$ to be defined later. The pair $(w_{2,app}, v_{app}^{E,\nu})$ thus solves (3.3), leaving an error of the form

$$\begin{aligned} \Phi(w_{2,app}, v_{app}^{E,\nu}) &= t(\Lambda + \nu(1 - \mathcal{L}))w_{2,a} + \nu t^2 \partial_t w_{2,a} + \nu t^2 v_{2,a} \cdot \nabla w_{2,a} \\ &\quad + \sqrt{\nu} t^{3/2} (\tilde{v}^E(\tilde{z}(t) + \xi \sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t)) \cdot \nabla w_{2,a} + t R_1(\xi, t), \end{aligned} \quad (3.6)$$

where $R_1(\xi, t)$ is defined as in (3.4) with $v_{app}^{E,\nu} = \tilde{v}^E$, and

$$\Lambda w := v^G \cdot \nabla_\xi w + v \cdot \nabla_\xi G, \quad v = K \star w.$$

To treat the order one remainder $R_1(\xi, t)$, we first solve $(\Lambda + \nu(1 - \mathcal{L}))w_{2,a} = -R_1$ to leading order in ν . We recall the following proposition from [10], Lemma 5 and Remark 1.

Proposition 3.1. *Let $z = z(\xi)$ be a function of the form*

$$z(\xi) = a_1(r) \cos(2\theta) + a_2(r) \sin(2\theta) + a_3(r) \cos(3\theta) + a_4(r) \sin(3\theta)$$

for $\xi = re^{i\theta}$. Assume that the coefficients satisfy

$$\sum_{i=1}^4 (|a_i(r)| + |a_i'(r)|) \leq C_0 P(r) e^{-r^2/4} \quad \forall r > 0.$$

for some polynomial $P(r)$. Then for any $\nu > 0$, there exists a unique solution w^ν to the elliptic equation

$$\Lambda w^\nu + \nu(1 - \mathcal{L})w^\nu = z$$

such that

$$|w^\nu(\xi)| + |\nabla w^\nu(\xi)| \leq C_\gamma e^{-\gamma|\xi|^2/4}$$

for any $\gamma \in (0, 1)$ and for some constant C_γ that is independent of ν .

3.1 Vortex-wave reaction term

In this section, we show that the leading term in the reaction term in (3.4) satisfies the assumption of Proposition 3.1. Precisely, we introduce

$$R_1(\xi, t) = \frac{1}{\sqrt{\nu t}} (\tilde{v}^E(\tilde{z}(t) + \xi \sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t)) \cdot \nabla G. \quad (3.7)$$

We have the following lemma.

Lemma 3.2. For any $T > 0$, there is a constant C_T so that

$$|R_1(\xi, t) - A_0(\xi, t)| \leq C_T(\nu t)|\xi|^4 e^{-|\xi|^2/4},$$

where

$$\begin{aligned} A_0(\xi, t) &= \frac{1}{16\pi^2} |\xi|^2 e^{-|\xi|^2/4} \int_{\mathbb{R}^2} \frac{\sin(2\psi)}{|\tilde{z}(t) - y|^2} \tilde{\omega}^E(y, t) dy \\ &\quad - \frac{1}{16\pi^2} \sqrt{\nu t} |\xi|^3 e^{-|\xi|^2/4} \int_{\mathbb{R}^2} \frac{\sin(3\psi)}{|\tilde{z}(t) - y|^3} \tilde{\omega}^E(y, t) dy. \end{aligned} \quad (3.8)$$

Here, ψ denotes the angle between ξ and $\tilde{z}(t) - y$.

Proof. Recalling (3.7) and $G = \frac{1}{4\pi} e^{-|\xi|^2/4}$, and using the Biot-Savart law (1.2), we have

$$\begin{aligned} R_1(\xi, t) &= \frac{-1}{8\pi\sqrt{\nu t}} (\tilde{v}^E(\tilde{z}(t) + \xi\sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t)) \cdot \xi e^{-|\xi|^2/4} \\ &= \frac{-e^{-|\xi|^2/4}}{16\pi^2\sqrt{\nu t}} \int_{\mathbb{R}^2} \xi \cdot \left(\frac{(\tilde{z}(t) + \xi\sqrt{\nu t} - y)^\perp}{|\tilde{z}(t) + \xi\sqrt{\nu t} - y|^2} - \frac{(\tilde{z}(t) - y)^\perp}{|\tilde{z}(t) - y|^2} \right) \tilde{\omega}^E(y, t) dy \\ &= \frac{-e^{-|\xi|^2/4}}{16\pi^2\sqrt{\nu t}} \int_{\mathbb{R}^2} \xi \cdot (\tilde{z}(t) - y)^\perp \left(\frac{1}{|\tilde{z}(t) + \xi\sqrt{\nu t} - y|^2} - \frac{1}{|\tilde{z}(t) - y|^2} \right) \tilde{\omega}^E(y, t) dy \\ &=: A_1(\xi, t) + A_2(\xi, t), \end{aligned}$$

where $A_1(\xi, t), A_2(\xi, t)$ denote the integral over $\{|\xi|\sqrt{\nu t} \leq \frac{1}{2}|\tilde{z}(t) - y|\}$ and $\{|\xi|\sqrt{\nu t} \geq \frac{1}{2}|\tilde{z}(t) - y|\}$, respectively. Let us first treat $A_1(\xi, t)$. Applying Lemma A.2 for $|\xi|\sqrt{\nu t} \leq \frac{1}{2}|\tilde{z}(t) - y|$, we have

$$\frac{1}{|\tilde{z}(t) + \xi\sqrt{\nu t} - y|^2} - \frac{1}{|\tilde{z}(t) - y|^2} = \frac{1}{|\tilde{z}(t) - y|^2} \sum_{n=1}^{\infty} (-1)^n \frac{|\xi|^n \sqrt{\nu t}^n}{|\tilde{z}(t) - y|^n} \frac{\sin((n+1)\psi)}{\sin(\psi)}.$$

Here ψ is the angle between ξ and $\tilde{z}(t) - y$. Thus we get

$$\begin{aligned} &\xi \cdot (\tilde{z}(t) - y)^\perp \left(\frac{1}{|\tilde{z}(t) + \xi\sqrt{\nu t} - y|^2} - \frac{1}{|\tilde{z}(t) - y|^2} \right) \\ &= \sum_{n=2}^{\infty} (-1)^{n+1} (\nu t)^{\frac{n-1}{2}} \frac{|\xi|^n}{|\tilde{z}(t) - y|^n} \sin(n\psi) \\ &= -(\nu t)^{1/2} \frac{|\xi|^2}{|\tilde{z}(t) - y|^2} \sin(2\psi) + (\nu t) \frac{|\xi|^3}{|\tilde{z}(t) - y|^3} \sin(3\psi) + \frac{1}{\sqrt{\nu t}} \sum_{n \geq 4} (-1)^{n+1} \frac{(|\xi|\sqrt{\nu t})^n}{|\tilde{z}(t) - y|^n} \sin(n\psi), \end{aligned}$$

in which we can estimate

$$\left| \frac{1}{\sqrt{\nu t}} \sum_{n \geq 4} (-1)^{n+1} \frac{(|\xi|\sqrt{\nu t})^n}{|\tilde{z}(t) - y|^n} \sin(n\psi) \right| \leq 2 \frac{(\nu t)^{3/2} |\xi|^4}{|\tilde{z}(t) - y|^4},$$

since $|\xi|\sqrt{\nu t} \leq \frac{1}{2}|\tilde{z}(t) - y|$. Hence, we have

$$\begin{aligned} A_1(\xi, t) &= \frac{|\xi|^2 e^{-|\xi|^2/4}}{16\pi^2} \int_{|\xi|\sqrt{\nu t} \leq \frac{1}{2}|\tilde{z}(t)-y|} \frac{1}{|\tilde{z}(t) - y|^2} \sin(2\psi) \tilde{\omega}^E(y, t) dy \\ &\quad - \frac{\sqrt{\nu t} |\xi|^3 e^{-|\xi|^2/4}}{16\pi^2} \int_{|\xi|\sqrt{\nu t} \leq \frac{1}{2}|\tilde{z}(t)-y|} \frac{1}{|\tilde{z}(t) - y|^3} \sin(3\psi) \tilde{\omega}^E(y, t) dy \\ &\quad + \mathcal{O}(\nu t |\xi|^4 e^{-|\xi|^2/4}) \int_{|\xi|\sqrt{\nu t} \leq \frac{1}{2}|\tilde{z}(t)-y|} \frac{1}{|\tilde{z}(t) - y|^4} \sin(4\psi) \tilde{\omega}^E(y, t) dy. \end{aligned}$$

We note that all the integrals above are bounded by $\|\tilde{\omega}^E(t)\|_{L^1}$, since $\tilde{z}(t)$ is bounded away from the support of $\tilde{\omega}^E(t)$ by Corollary 2.2. Therefore, defining $A_0(\xi, t)$ as in (3.8), we can write

$$\begin{aligned} A_1(\xi, t) &= A_0(\xi, t) - \frac{|\xi|^2 e^{-|\xi|^2/4}}{16\pi^2} \int_{|\xi|\sqrt{\nu t} \geq \frac{1}{2}|\tilde{z}(t)-y|} \frac{1}{|\tilde{z}(t) - y|^2} \sin(2\psi) \tilde{\omega}^E(y, t) dy \\ &\quad + \frac{\sqrt{\nu t} |\xi|^3 e^{-|\xi|^2/4}}{16\pi^2} \int_{|\xi|\sqrt{\nu t} \geq \frac{1}{2}|\tilde{z}(t)-y|} \frac{1}{|\tilde{z}(t) - y|^3} \sin(3\psi) \tilde{\omega}^E(y, t) dy + \mathcal{O}(\nu t |\xi|^4 e^{-|\xi|^2/4}). \end{aligned}$$

It remains to treat the integral over the domain $\{|\xi|\sqrt{\nu t} > \frac{1}{2}|\tilde{z}(t) - y|\}$. Since $\tilde{z}(t)$ is bounded away from the support of $\tilde{\omega}^E(t)$, the above (explicitly written) integrals vanish for $|\xi|\sqrt{\nu t} \leq c_T$ for all $t \in [0, T]$, for some constant c_T . On the other hand, for $|\xi|\sqrt{\nu t} \geq c_T$, we have

$$\left| \frac{|\xi|^2 e^{-|\xi|^2/4}}{16\pi^2} \int_{|\xi|\sqrt{\nu t} \geq \frac{1}{2}|\tilde{z}(t)-y|} \frac{1}{|\tilde{z}(t) - y|^2} \sin(2\psi) \tilde{\omega}^E(y, t) dy \right| \leq C_T \nu t |\xi|^4 e^{-|\xi|^2/4} \|\tilde{\omega}^E(t)\|_{L^1},$$

for some constant C_T . Similarly, we also have $A_2(\xi, t) = 0$ for $|\xi|\sqrt{\nu t} \leq c_T$ for all $t \in [0, T]$, for some constant c_T , while for $|\xi|\sqrt{\nu t} \geq c_T$, we have

$$\begin{aligned} |A_2(\xi, t)| &\leq |A_1(\xi, t)| + |A(\xi, t)| \\ &\leq C_T |\xi|^2 (1 + \nu t |\xi|^2) e^{-|\xi|^2/4} \|\tilde{\omega}^E(t)\|_{L^1} + C_T (\nu t)^{-1/2} |\xi| e^{-|\xi|^2/4} \|\tilde{v}^E\|_{L^\infty} \\ &\leq C_T (\nu t) |\xi|^4 e^{-|\xi|^2/4}, \end{aligned}$$

upon using Corollary 2.2 to bound \tilde{v}^E and $\tilde{\omega}^E$. The lemma follows. \square

3.2 Construction of an approximation solution

We now construct $w_{2,a}$ that solves the following elliptic equation

$$\Lambda w_{2,a} + \nu(1 - \mathcal{L})w_{2,a} = -A_0(\xi, t) \tag{3.9}$$

with $A_0(\xi, t)$ defined as in (3.8). We have the following.

Lemma 3.3. *There exists a solution $w_{2,a}$ to (3.9) so that, for any $\gamma \in (0, 1)$, there holds*

$$|w_{2,a}(t, \xi)| + |\nabla w_{2,a}(\xi, t)| \leq C_\gamma e^{-\gamma|\xi|^2/4}$$

uniformly in $\nu > 0$. In particular, we have

$$\|v_{2,a}(t)\|_{L^\infty} + \int_{\mathbb{R}^2} |w_{2,a}(\xi, t)|^2 e^{|\xi|^2/4} d\xi + \int_{\mathbb{R}^2} |\nabla w_{2,a}(\xi, t)|^2 e^{|\xi|^2/4} d\xi \lesssim 1. \quad (3.10)$$

Proof. For each $y \in \mathbb{R}^2$, we introduce

$$B_0(\xi, y, t) = \frac{-1}{16\pi^2} |\xi|^2 e^{-|\xi|^2/4} \frac{\sin(2\psi)}{|\tilde{z}(t) - y|^2} \tilde{\omega}^E(y, t) + \frac{1}{16\pi^2} \sqrt{\nu t} |\xi|^3 e^{-|\xi|^2/4} \frac{\sin(3\psi)}{|\tilde{z}(t) - y|^3} \tilde{\omega}^E(y, t), \quad (3.11)$$

recalling ψ the angle between ξ and $\tilde{z}(t) - y$. It follows from (3.8) that $A_0(\xi, t) = \int_{\mathbb{R}^2} B_0(\xi, y, t) dy$. It is clear that for each y , $B_0(\xi, y, t)$ satisfies the assumption of Proposition 3.1 and hence we can define

$$W_{2,a}(\xi, y, t) := \left(\Lambda + \nu(1 - \mathcal{L}) \right)^{-1} B_0(\xi, y, t),$$

stressing that $y \in \mathbb{R}^2$ and $t \geq 0$ play a role as independent parameters. The solution $w_{2,a}$ is thus defined by the average of $W_{2,a}(\xi, y, t)$ with respect to y . The pointwise estimates follow directly from Proposition 3.1 and the estimates on $\tilde{\omega}^E$. Taking $\gamma > 1/2$ and using the elliptic estimate $\|v_{2,a}\|_{L^\infty} \lesssim \|w_{2,a}\|_{L^1 \cap L^\infty}$, we obtain the estimates (3.10). \square

3.3 Estimating the error term

Construct $w_{2,a}$ as in Lemma 3.3. Then, $w_{2,\text{app}} = G(\xi) + \nu t w_{2,a}$ and $v_{\text{app}}^{E,\nu} = \tilde{v}^E$ approximately solves (3.3) in the following sense.

Proposition 3.4. *For any $\gamma \in (0, 1)$, there holds*

$$|\Phi(w_{2,\text{app}}, v_{\text{app}}^{E,\nu})(\xi, t)| \leq C_\gamma \nu t^{3/2} e^{-\gamma|\xi|^2/4} \quad (3.12)$$

for some constant C_γ .

Proof. Fix a $\gamma \in (0, 1)$. Using (3.9) into (3.6), we write

$$\begin{aligned} \Phi(w_{2,\text{app}}, v_{\text{app}}^{E,\nu})(\xi, t) &= \nu t^2 v_{2,a} \cdot \nabla w_{2,a} + \sqrt{\nu t} t^{3/2} (\tilde{v}^E(\tilde{z}(t) + \xi \sqrt{\nu t}, t) - \tilde{v}^E(\tilde{z}(t), t)) \cdot \nabla w_{2,a} \\ &\quad + \nu t^2 \partial_t w_{2,a} + t(R_1(\xi, t) - A_0(\xi, t)) \\ &=: \sum_{i=1}^4 \Phi_i(\xi, t). \end{aligned}$$

Let us estimate each term on the right. Using Proposition 2.1 and Lemma 3.3, we get

$$|\Phi_1(\xi, t)| \leq \nu t^2 \|v_{2,a}(t)\|_{L^\infty} |\nabla w_{2,a}(\xi, t)| \lesssim \nu t^2 e^{-\gamma|\xi|^2/4}.$$

Similarly, using Corollary 2.2, we bound

$$|\tilde{v}^E(\xi\sqrt{\nu t} + \tilde{z}(t), t) - \tilde{v}^E(\tilde{z}(t), t)| \lesssim |\xi|\sqrt{\nu t}\|\nabla\tilde{v}^E\|_{L^\infty}$$

and hence

$$\begin{aligned} |\Phi_2(\xi, t)| &\leq \sqrt{\nu t}^{3/2} |\tilde{v}^E(\xi\sqrt{\nu t} + \tilde{z}(t), t) - \tilde{v}^E(\tilde{z}(t), t)| \|\nabla w_{2,a}(\xi, t)\| \\ &\lesssim \nu t^2 |\xi| e^{-\gamma'|\xi|^2/4} \\ &\lesssim \nu t^2 e^{-\gamma|\xi|^2/4}, \end{aligned}$$

upon taking γ' from Lemma 3.3 so that $\gamma' > \gamma$.

Next, we treat $\Phi_3(\xi, t) = \nu t^2 \partial_t w_{2,a}$. Since $\sqrt{t} \partial_t$ commutes with Λ and \mathcal{L} , the equation (3.9) gives

$$(\nu(1 - \mathcal{L}) + \Lambda)(\sqrt{t} \partial_t w_{2,a}) = -\sqrt{t} \partial_t A_0(\xi, t).$$

To apply Proposition 3.1, it suffices to prove that

$$\sqrt{t} |\partial_t A_0(\xi, t)| \lesssim |\xi|^2 (1 + |\xi|) e^{-|\xi|^2/4}. \quad (3.13)$$

Indeed, we recall from (3.11) that

$$\begin{cases} A_0(\xi, t) &= \int_{\mathbb{R}^2} B_0(\xi, y, t) dy \\ B_0(\xi, y, t) &= \frac{-1}{16\pi^2} |\xi|^2 e^{-|\xi|^2/4} \frac{\sin(2\psi)}{|\tilde{z}(t) - y|^2} \tilde{\omega}^E(y, t) + \frac{1}{16\pi^2} \sqrt{\nu t} |\xi|^3 e^{-|\xi|^2/4} \frac{\sin(3\psi)}{|\tilde{z}(t) - y|^3} \tilde{\omega}^E(y, t), \end{cases} \quad (3.14)$$

where ψ is the angle between ξ and $\tilde{z}(t) - y$. By Corollary 2.2, $\tilde{\omega}^E(t)$ and $\partial_t \tilde{\omega}^E(t)$ are both bounded, compactly supported, and vanishing in a neighborhood of $\tilde{z}(t)$. In particular, $|\tilde{z}(t) - y|$ is bounded below away from zero for y in the support of $\tilde{\omega}^E(t)$. The estimate (3.13) thus follows, upon recalling that $\partial_t \tilde{z}(t) = \tilde{v}^E(\tilde{z}(t), t)$ and \tilde{v}^E is bounded (Corollary 2.2). Arguing similarly as in Lemma 3.3, we obtain

$$|\sqrt{t} \partial_t w_{2,a}(\xi, t)| \leq C_\gamma e^{-\gamma|\xi|^2/4}.$$

Finally, the last term $\Phi_4(\xi, t) = t(R_1(\xi, t) - A_0(\xi, t))$ is already treated in Lemma 3.2. This concludes the proof. \square

3.4 Equations for the remainder

Having introduced the approximate solutions $w_{2,app}$ and $v_{app}^{E,\nu}$, let us now study the remainder. Precisely, we search for solutions of (3.3) in the following form

$$\begin{cases} w_2 &= G(\xi) + (\nu t)w_{2,a} + (\nu t)\bar{w}_2 \\ v^{E,\nu} &= \tilde{v}^E + \nu^{3/2}\bar{v}_1, \end{cases} \quad (3.15)$$

in which \tilde{v}^E and $w_{2,a}$ are constructed in the previous sections. Putting this Ansatz into (3.3), we have

$$\begin{aligned} (t\partial_t - \mathcal{L} + 1)\bar{w}_2 + \frac{1}{\nu}\Lambda\bar{w}_2 + \sqrt{\frac{t}{\nu}}(\tilde{v}^E - \dot{\tilde{z}}) \cdot \nabla\bar{w}_2 + t(\bar{v}_2 \cdot \nabla w_{2,a} + v_{2,a} \cdot \nabla\bar{w}_2) \\ + \frac{1}{\sqrt{t}}(\bar{v}_1 \cdot \nabla G) + \nu\sqrt{t}(\bar{v}_1 \cdot \nabla w_{2,a}) + t(\bar{v}_2 \cdot \nabla\bar{w}_2) + \nu\sqrt{t}(\bar{v}_1 \cdot \nabla\bar{w}_2) + \frac{1}{\nu t}\Phi(w_{2,app}, v_{app}^{E,\nu}) = 0, \end{aligned} \quad (3.16)$$

in which we stress that \tilde{v}^E and \bar{v}_1 are functions of (x, t) , while G , $w_{2,a}$, and \bar{w}_2 are functions of ξ, t . Again, velocity and vorticity are defined through the Biot-Savart law in their respective variables.

Our goal is to derive estimates for the remainder solution (\bar{w}_2, \bar{v}_1) in suitable function spaces. Precisely, we shall work with the following weighted L^2 norm

$$\|\omega\|_{L_p^2}^2 := \int_{\mathbb{R}^2} |\omega(\xi)|^2 p(\xi) d\xi, \quad p(\xi) = e^{|\xi|^2/4}.$$

The weight function is natural in view of the following lemma.

Lemma 3.5. *The operator \mathcal{L} is self-adjoint in L_p^2 , while Λ is skew-symmetric in L_p^2 . In particular, $\mathcal{L} \leq 0$, we have*

$$\langle \Lambda \omega, \omega \rangle_{L_p^2} = 0$$

for any $\omega(\xi)$ in the domain of Λ .

Proof. The lemma follows from a direct calculation; see [11, Lemma 4.8]. \square

Lemma 3.6 (Elliptic estimates). *Let $\bar{v}_2 = K \star_\xi \bar{w}_2$ be the velocity obtained from \bar{w}_2 by the Biot-Savart law. There holds*

$$\|\bar{v}_2\|_{L^\infty} \lesssim \|\bar{w}_2\|_{L_p^2} + \|\bar{w}_2\|_{L_p^2}^{1/2} \|\nabla \bar{w}_2\|_{L_p^2}^{1/2}.$$

Proof. By Hölder inequality and Sobolev embeddings, we have

$$\begin{aligned} \|\bar{v}_2\|_{L^\infty} &\lesssim \|\bar{w}_2\|_{L^{4/3}}^{1/2} \|\bar{w}_2\|_{L^4}^{1/2} \lesssim \|\bar{w}_2\|_{L_p^2}^{1/2} \left(\|\bar{w}_2\|_{L_p^2} + \|\nabla \bar{w}_2\|_{L_p^2} \right)^{1/2} \\ &\lesssim \|\bar{w}_2\|_{L_p^2}^{1/2} \left(\|\bar{w}_2\|_{L_p^2}^{1/2} + \|\nabla \bar{w}_2\|_{L_p^2}^{1/2} \right) \\ &= \|\bar{w}_2\|_{L_p^2} + \|\bar{w}_2\|_{L_p^2}^{1/2} \|\nabla \bar{w}_2\|_{L_p^2}^{1/2}. \end{aligned}$$

The proof is complete. \square

3.5 Estimates for the remainder

This section is devoted to prove the following proposition.

Proposition 3.7. *There are a positive constant κ and a positive time T so that*

$$\begin{aligned} t \frac{d}{dt} \|\bar{w}_2(t)\|_{L_p^2}^2 + \kappa (\|(1 + |\xi|)\bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2) \\ \lesssim t \|\bar{w}_2(t)\|_{L_p^2}^5 + \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + t^{-1} \|\bar{v}_1(t)\|_{L^\infty}^2 \end{aligned} \tag{3.17}$$

uniformly in ν and in $t \in [0, T]$.

The proposition follows from weighted energy estimates. To proceed, using the equation (3.16) for $t\partial_t\bar{w}_2$, we compute

$$t\frac{d}{dt}\|\bar{w}_2(t)\|_{L_p^2}^2 = \int_{\mathbb{R}^2} (t\partial_t\bar{w}_2(\xi, t))\bar{w}_2(\xi, t)p(\xi)d\xi = \sum_{i=1}^9 \mathcal{E}_i(t), \quad (3.18)$$

where

$$\begin{cases} \mathcal{E}_1(t) &= \int_{\mathbb{R}^2} p(\xi)(\mathcal{L}\bar{w}_2 - \bar{w}_2)(\xi, t)d\xi, \\ \mathcal{E}_2(t) &= -\frac{1}{\nu} \int_{\mathbb{R}^2} \Lambda\bar{w}_2(\xi, t)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_3(t) &= -\sqrt{\frac{t}{\nu}} \int_{\mathbb{R}^2} ((\tilde{v}^E - \tilde{z}) \cdot \nabla\bar{w}_2)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_4(t) &= -t \int_{\mathbb{R}^2} (\bar{v}_2 \cdot \nabla w_{2,a} + v_{2,a} \cdot \nabla\bar{w}_2)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_5(t) &= -t \int_{\mathbb{R}^2} (\bar{v}_2 \cdot \nabla\bar{w}_2)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_6(t) &= -\nu\sqrt{t} \int_{\mathbb{R}^2} (\bar{v}_1 \cdot \nabla\bar{w}_2)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_7(t) &= -\frac{1}{\nu t} \int_{\mathbb{R}^2} \Phi_{\text{app}}(\xi, t)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_8(t) &= -\frac{1}{\sqrt{t}} \int_{\mathbb{R}^2} (\bar{v}_1 \cdot \nabla G)\bar{w}_2(\xi, t)p(\xi)d\xi, \\ \mathcal{E}_9(t) &= -\nu\sqrt{t} \int_{\mathbb{R}^2} (\bar{v}_1 \cdot \nabla w_{2,a})\bar{w}_2(\xi, t)p(\xi)d\xi. \end{cases}$$

Let us estimate each term \mathcal{E}_i . Thanks to Lemma 3.5, we have $\mathcal{E}_2(t) = 0$, while $\mathcal{E}_1(t) \leq -\|\bar{w}_2(t)\|_{L_p^2}^2$. In fact, the following lemma gives a better coercive estimate for $\mathcal{E}_1(t)$.

Lemma 3.8 (Diffusive term). *There holds*

$$\mathcal{E}_1(t) \leq -\frac{1}{24} \left(\|\nabla\bar{w}_2(t)\|_{L_p^2}^2 + \|(1 + |\xi|)\bar{w}_2(t)\|_{L_p^2}^2 \right).$$

Proof. Recalling $\mathcal{L} = 1 + \frac{1}{2}\xi \cdot \nabla + \Delta$ and integrating by parts, we compute

$$\begin{aligned} & \int_{\mathbb{R}^2} (\mathcal{L}\bar{w}_2 - \bar{w}_2)(\xi, t)p(\xi)\bar{w}_2(\xi, t)d\xi \\ &= \int_{\mathbb{R}^2} \left(\Delta\bar{w}_2 + \frac{1}{2}\xi \cdot \nabla\bar{w}_2 \right) \bar{w}_2(\xi, t)p(\xi)d\xi, \\ &= - \int_{\mathbb{R}^2} |\nabla\bar{w}_2|^2 p(\xi)d\xi - \int_{\mathbb{R}^2} \bar{w}_2(\nabla p \cdot \nabla\bar{w}_2)d\xi + \frac{1}{4} \int_{\mathbb{R}^2} (\xi \cdot \nabla(|\bar{w}_2|^2)) p(\xi, t)d\xi \\ &= - \int_{\mathbb{R}^2} |\nabla\bar{w}_2|^2 p(\xi, t)d\xi - \int_{\mathbb{R}^2} \bar{w}_2(\nabla p \cdot \nabla\bar{w}_2)d\xi - \frac{1}{2} \int_{\mathbb{R}^2} |\bar{w}_2|^2 p(\xi, t)d\xi - \frac{1}{4} \int_{\mathbb{R}^2} |\bar{w}_2|^2 (\xi \cdot \nabla p)d\xi. \end{aligned}$$

The second integral is treated by

$$- \int_{\mathbb{R}^2} \bar{w}_2(\nabla p \cdot \nabla\bar{w}_2)d\xi \leq \frac{3}{4} \int_{\mathbb{R}^2} |\nabla\bar{w}_2|^2 p(\xi, t) + \frac{1}{3} \int_{\mathbb{R}^2} \frac{|\nabla p|^2}{p^2} |\bar{w}_2|^2 p(\xi)d\xi.$$

Recalling now the weight function $p(\xi) = e^{|\xi|^2/4}$, we obtain the lemma at once. \square

Lemma 3.9. *There holds*

$$\mathcal{E}_3(t) \lesssim t \|\xi \bar{w}_2(t)\|_{L_p^2}^2.$$

Proof. Integrating by parts and using the fact that $\tilde{v}^E - \dot{\tilde{z}}$ is divergence free, we have

$$\begin{aligned} \mathcal{E}_3(t) &= -\sqrt{\frac{t}{\nu}} \int_{\mathbb{R}^2} \left((\tilde{v}^E - \dot{\tilde{z}}) \cdot \nabla \bar{w}_2 \right) \bar{w}_2(\xi, t) p(\xi) d\xi \\ &= \frac{1}{2} \sqrt{\frac{t}{\nu}} \int_{\mathbb{R}^2} (\tilde{v}^E - \dot{\tilde{z}}) \cdot \nabla p(\xi) |\bar{w}_2(\xi, t)|^2 d\xi. \end{aligned}$$

Recalling $\dot{\tilde{z}} = \tilde{v}^E(\tilde{z}(t), t)$ and using Corollary 2.2, we estimate

$$|\tilde{v}^E(\xi \sqrt{\nu t} + \tilde{z}(t), t) - \dot{\tilde{z}}(t)| = |\tilde{v}^E(\xi \sqrt{\nu t} + \tilde{z}(t), t) - \tilde{v}^E(\tilde{z}(t), t)| \lesssim \sqrt{\nu t} |\xi|.$$

The lemma follows, upon using $\nabla p = \frac{1}{2} \xi p(\xi)$. □

Lemma 3.10. *There holds*

$$\mathcal{E}_4(t) \lesssim t \left(\|\bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 \right).$$

Proof. We write $\mathcal{E}_4(t) = -t(\mathcal{E}_{41}(t) + \mathcal{E}_{42}(t))$, where

$$\begin{cases} \mathcal{E}_{41}(t) &= \int_{\mathbb{R}^2} (\bar{v}_2 \cdot \nabla w_{2,a}) \bar{w}_2(\xi, t) p(\xi) d\xi, \\ \mathcal{E}_{42}(t) &= \int_{\mathbb{R}^2} (v_{2,a} \cdot \nabla \bar{w}_2) \bar{w}_2(\xi, t) p(\xi) d\xi. \end{cases}$$

Using Hölder's inequality, we estimate

$$|\mathcal{E}_{41}(t)| \leq \|\bar{v}_2(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2} \left(\int_{\mathbb{R}^2} |\nabla w_{2,a}(\xi, t)|^2 p(\xi) d\xi \right)^{1/2},$$

in which the integral is bounded by Lemma 3.3. As for $\|\bar{v}_2(t)\|_{L^\infty}$, we use the elliptic estimate and Sobolev embedding, giving

$$\|\bar{v}_2\|_{L^\infty}^2 \lesssim \|\bar{w}_2\|_{L^{4/3}} \|\bar{w}_2\|_{L^4} \lesssim \|\bar{w}_2\|_{L^{4/3}} \|\bar{w}_2\|_{L^2}^{1/2} (\|\bar{w}_2\|_{L^2} + \|\nabla \bar{w}_2\|_{L^2})^{1/2}.$$

Recalling the weight function $p = e^{|\xi|^2/4}$, we have $\|\bar{w}_2\|_{L^{4/3}} \lesssim \|\bar{w}_2\|_{L_p^2}$. Thus, we get

$$\|\bar{v}_2\|_{L^\infty}^2 \lesssim \|\bar{w}_2\|_{L_p^2}^{3/2} (\|\bar{w}_2\|_{L_p^2} + \|\nabla \bar{w}_2\|_{L_p^2})^{1/2} \lesssim \|\bar{w}_2\|_{L_p^2}^2 + \|\nabla \bar{w}_2\|_{L_p^2}^2, \quad (3.19)$$

and so

$$|\mathcal{E}_{41}(t)| \lesssim \|\bar{w}_2(t)\|_{L_p^2} (\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2}) \lesssim \|\bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2.$$

On the other hand, the estimate on $\mathcal{E}_{42}(t)$ is direct, since $v_{2,a}$ is bounded. The lemma follows. □

Lemma 3.11. *There holds*

$$\mathcal{E}_5(t) \lesssim t \left(\|\bar{w}_2(t)\|_{L_p^2}^2 + \|\bar{w}_2(t)\|_{L_p^2}^5 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 \right).$$

Proof. By Hölder's inequality and (3.19), we get

$$\begin{aligned} |\mathcal{E}_5(t)| &= t \left| \int_{\mathbb{R}^2} (\bar{v}_2 \cdot \nabla \bar{w}_2) \bar{w}_2(\xi, t) p(\xi) d\xi \right| \\ &\leq t \|\bar{v}_2(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2} \|\nabla \bar{w}_2(t)\|_{L_p^2} \\ &\lesssim t \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right)^{1/4} \|\bar{w}_2(t)\|_{L_p^2}^{7/4} \|\nabla \bar{w}_2(t)\|_{L_p^2}. \end{aligned}$$

The lemma follows upon using Young's inequality. \square

Lemma 3.12. *There holds*

$$\mathcal{E}_6(t) \lesssim \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + \nu t \|\bar{w}_2(t)\|_{L_p^2}^4 + \nu \|\nabla \bar{w}_2(t)\|_{L_p^2}^2.$$

Proof. Again by Hölder inequality, we get

$$\begin{aligned} |\mathcal{E}_6(t)| &= \nu \sqrt{t} \left| \int_{\mathbb{R}^2} (\bar{v}_1 \cdot \nabla \bar{w}_2) \bar{w}_2(\xi, t) p(\xi) d\xi \right| \\ &\lesssim \nu t^{1/2} \|\bar{v}_1(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2} \|\nabla \bar{w}_2(t)\|_{L_p^2}, \end{aligned}$$

which yields the lemma upon using Young's inequality. \square

Lemma 3.13. *There holds*

$$\mathcal{E}_7(t) \lesssim t^{1/2} \|\bar{w}_2(t)\|_{L_p^2}.$$

Proof. Using the estimates from (3.12) for a fixed $\gamma \in (\frac{1}{2}, 1)$ and Hölder inequality, we get

$$\begin{aligned} |\mathcal{E}_7(t)| &\leq (\nu t)^{-1} \int_{\mathbb{R}^2} |\Phi_{\text{app}}(\xi, t)| |\bar{w}_2(\xi, t)| p(\xi) d\xi \\ &\leq (\nu t)^{-1} \int_{\mathbb{R}^2} (\nu t^{3/2}) C_\gamma e^{-\gamma|\xi|^2/4} |\bar{w}_2(\xi, t)| p(\xi) d\xi \\ &\leq C_\gamma t^{1/2} \left(\int_{\mathbb{R}^2} e^{-2\gamma|\xi|^2/4} p(\xi) d\xi \right)^{1/2} \left(\int_{\mathbb{R}^2} |\bar{w}_2(\xi, t)|^2 p(\xi) d\xi \right)^{1/2} \\ &\lesssim t^{1/2} \|\bar{w}_2(t)\|_{L_p^2}, \end{aligned}$$

where we used $\gamma > 1/2$. This concludes the proof. \square

Lemma 3.14. *There hold*

$$\mathcal{E}_8(t) \lesssim t^{-1/2} \|\bar{v}_1(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2}, \quad \mathcal{E}_9(t) \lesssim \nu t^{1/2} \|\bar{v}_1(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2}.$$

Proof. We recall that

$$\mathcal{E}_8(t) = -\frac{1}{\sqrt{t}} \int_{\mathbb{R}^2} (\bar{v}_1(\xi, t) \cdot \nabla G(\xi)) \bar{w}_2(\xi, t) p(\xi) d\xi$$

where $G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}$ and $p(\xi) = e^{|\xi|^2/4}$. We have

$$|\mathcal{E}_8(t)| \lesssim t^{-1/2} \|\bar{v}_1(t)\|_{L^\infty} \int_{\mathbb{R}^2} |\xi| |\bar{w}_2(\xi, t)| d\xi \lesssim t^{-1/2} \|\bar{v}_1(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2}.$$

The proof for $\mathcal{E}_9(t)$ is identical, upon recalling the pointwise bound on $\nabla w_{2,a}$ from Lemma 3.3. \square

Proof of Proposition 3.7. We are now ready to prove Proposition 3.7. Collecting and combining all the estimates from the previous lemmas, we get

$$\begin{aligned} & t \frac{d}{dt} \|\bar{w}_2(t)\|_{L_p^2}^2 + \kappa (\|(1 + |\xi|) \bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2) \\ & \lesssim t \left(\|(1 + |\xi|) \bar{w}_2(t)\|_{L_p^2}^2 + \|\bar{w}_2(t)\|_{L_p^2}^5 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 \right) + t^{1/2} \|\bar{w}_2(t)\|_{L_p^2} \\ & \quad + \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + \nu t \|\bar{w}_2(t)\|_{L_p^2}^4 + \nu \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 + t^{-1/2} \|\bar{v}_1(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L_p^2}, \end{aligned} \quad (3.20)$$

for $\kappa = 1/24$. Taking t and ν sufficiently small and using Young's inequality, we obtain

$$\begin{aligned} & t \frac{d}{dt} \|\bar{w}_2(t)\|_{L_p^2}^2 + \frac{\kappa}{2} (\|(1 + |\xi|) \bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2) \\ & \lesssim t \|\bar{w}_2(t)\|_{L_p^2}^5 + \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + t^{-1} \|\bar{v}_1(t)\|_{L^\infty}^2. \end{aligned} \quad (3.21)$$

This completes the proof of the proposition. \square

Remark 3.15. *The constraint on the smallness of times T is precisely due to the term $\mathcal{E}_3(t)$ treated in Lemma 3.9. The remaining terms are treated using the standard Young's inequality. Hence, we in fact obtain*

$$\begin{aligned} & t \frac{d}{dt} \|\bar{w}_2(t)\|_{L_p^2}^2 + \kappa \left(\|\bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 + (1 - 5t \|\nabla v^E(t)\|_{L^\infty}) \|\xi \bar{w}_2(t)\|_{L_p^2}^2 \right) \\ & \lesssim t (\|\bar{w}_2(t)\|_{L_p^2}^2 + \|\bar{w}_2(t)\|_{L_p^2}^5) + \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + t^{-1} \|\bar{v}_1(t)\|_{L^\infty}^2 \end{aligned} \quad (3.22)$$

for all positive times, as long as the estimates from Proposition 2.1 and Corollary 2.2 on the approximate vortex-wave solutions are valid. This yields a lower bound on the smallness of T so that $\sup_{0 \leq t \leq T} 5t \|\nabla v^E(t)\|_{L^\infty} \leq 1$.

Remark 3.16. *One may try to improve the time interval by introducing a new weight function, as done similarly in [10], $p_{\text{new}}(\xi) = p(\xi)(1 + \nu t q(\xi, t))$, where $q(\xi, t)$ solves*

$$v^G(\xi) \cdot \nabla_\xi q = \frac{1}{\sqrt{\nu t}} \left(v^E(z(t) + \xi \sqrt{\nu t}, t) - v^E(z(t), t) \right) \cdot \xi,$$

whose solution is however unclear for large $\xi \sqrt{\nu t}$.

4 Inviscid limit for the regular part

In the previous section, we have proved the apriori estimate for $\omega^{B,\nu}$ and $v^{E,\nu}$ in the weighted energy space with the re-scaled variable $\xi = \frac{x-\tilde{z}(t)}{\sqrt{\nu t}}$. In this section, we derive estimates on the regular vorticity component $\omega^{E,\nu}$, which solves

$$\partial_t \omega^{E,\nu} + u^\nu \cdot \nabla \omega^{E,\nu} = \nu \Delta \omega^{E,\nu} \quad (4.1)$$

with the initial data ω_0^E . We write:

$$\begin{cases} \omega^{E,\nu}(t, x) &= \tilde{\omega}^E(t, x) + \nu^{3/2} \bar{w}_1(t, x), \\ v^{E,\nu}(t, x) &= \tilde{v}^E(t, x) + \nu^{3/2} \bar{v}_1(t, x), \\ v^{B,\nu}(t, x) &= \frac{1}{\sqrt{\nu t}} v^G \left(\frac{x-\tilde{z}(t)}{\sqrt{\nu t}} \right) + \sqrt{\nu t} (v_{2,a} + \bar{v}_2) \left(\frac{x-\tilde{z}(t)}{\sqrt{\nu t}}, t \right), \\ u^\nu(t, x) &= v^{E,\nu}(t, x) + v^{B,\nu}(t, x), \end{cases} \quad (4.2)$$

where $(\tilde{z}(t), \tilde{\omega}^E)$ is the solution to the viscous vortex-wave system introduced in Section 2, while v^G and $v_{2,a}$ are constructed in Section 3. Here, we note that the form of the common velocity $u^\nu(t, x)$ is compatible with the form in (3.15) and (3.2) in the scaled variable ξ . The velocity \bar{v}_2 is kept the same as in the previous section, with ξ is replaced by $\frac{x-\tilde{z}(t)}{\sqrt{\nu t}}$ and $\bar{v}_2 = K \star_\xi \bar{w}_2$. It is natural to work in the original variables (x, t) instead of (ξ, t) , since $\omega^{E,\nu}(t)$ solves (4.1) with regular initial data ω_0^E . Hence one does not expect $\omega^{E,\nu}$ to have the localized behavior near the point vortex. Roughly speaking, we want to get an apriori bound on $\|\bar{v}_1(t)\|_{L^\infty}$ (in terms of $\bar{w}_2(t)$) on a time interval independent of ν . Precisely, we shall prove the following proposition.

Proposition 4.1. *Let \bar{w}_1 solve the equations (4.1) and (4.2). There exists a positive time T , independent of $\nu > 0$, such that*

$$\|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} \lesssim \int_0^t s^{3/2} (\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2}) ds + \nu^{1/2} t$$

for $t \in [0, T]$.

4.1 Equations for the remainder

In this subsection, we derive the equations for the remainder \bar{w}_1 as well as \bar{v}_2 appearing in (4.1) and (4.2). Putting the Ansatz (4.2) into equation (4.1) and using equation (2.2), we obtain the following transport-diffusion equation for \bar{w}_1 :

$$\partial_t \bar{w}_1 + u^\nu \cdot \nabla \bar{w}_1 - \nu \Delta \bar{w}_1 = f(x, t),$$

where $f(x, t)$ are given by

$$\begin{aligned} f(x, t) &= -\frac{1}{\nu \sqrt{t}} \left(v^G \left(\frac{x-\tilde{z}(t)}{\sqrt{\nu t}} \right) - v^G \left(\frac{x-z(t)}{\sqrt{\nu t}} \right) \right) \cdot \nabla w_{1,a} - \bar{v}_1 \cdot \nabla \tilde{\omega}^E - \frac{\sqrt{t}}{\nu} \bar{v}_2 \cdot \nabla \tilde{\omega}^E \\ &\quad - \sqrt{\nu} (v_{1,a} \cdot \nabla w_{1,a}) + \frac{1}{2\pi \nu^{3/2}} \frac{(x-z(t))^\perp}{|x-z(t)|^2} e^{-\frac{|x-z(t)|^2}{4\nu t}} \cdot \nabla \omega^E + \sqrt{\nu} \Delta w_{1,a}. \end{aligned} \quad (4.3)$$

4.2 Estimating the forcing term $f(x, t)$

In this subsection, we prove the following proposition

Proposition 4.2. *Let $f(x, t)$ be defined as in (4.3). There holds*

$$\|f(t)\|_{L^4 \cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} + t^{3/2} \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right) + \sqrt{\nu}.$$

We will give a proof at the end of this subsection, after proving some useful lemmas. First, let us write f as:

$$f(x, t) = f_1(x, t) + f_2(x, t) + f_3(x, t)$$

where

$$\begin{cases} f_1(x, t) &= -\frac{1}{\nu\sqrt{t}} \left(v^G \left(\frac{x - \tilde{z}(t)}{\sqrt{\nu t}} \right) - v^G \left(\frac{x - z(t)}{\sqrt{\nu t}} \right) \right) \cdot \nabla w_{1,a} - \sqrt{\nu} (v_{1,a} \cdot \nabla w_{1,a}) \\ &\quad + \frac{1}{2\pi\nu^{3/2}} \frac{(x - z(t))^\perp}{|x - z(t)|^2} e^{-\frac{|x - z(t)|^2}{4\nu t}} \cdot \nabla \omega^E + \sqrt{\nu} \Delta w_{1,a}, \\ f_2(x, t) &= -\bar{v}_1 \cdot \nabla \tilde{\omega}^E, \\ f_3(x, t) &= -\frac{\sqrt{t}}{\nu} \bar{v}_2 \cdot \nabla \tilde{\omega}^E, \end{cases}$$

In what follows, we bound $\|f_i(t)\|_{L^4 \cap L^{4/3}}$ for each $i \in \{1, 2, 3\}$.

Lemma 4.3. *There holds*

$$\|f_1(t)\|_{L^4 \cap L^{4/3}} \lesssim \sqrt{\nu}$$

uniformly in $\nu > 0$.

Proof. First we see that

$$\left\| -\sqrt{\nu} (v_{1,a} \cdot \nabla w_{1,a}) - \frac{1}{2\pi\nu^{3/2}} \frac{(x - z(t))^\perp}{|x - z(t)|^2} e^{-\frac{|x - z(t)|^2}{4\nu t}} \cdot \nabla \omega^E + \sqrt{\nu} \Delta w_{1,a} \right\|_{L^4 \cap L^{4/3}} \lesssim \sqrt{\nu}$$

thanks to the fact that ω^E is supported away from $z(t)$ and $\tilde{z}(t)$, and $w_{1,a}$ is bounded in $W^{2,4}$, by Proposition 2.1. Now for the first term in f_1 , it suffices to prove that

$$\frac{1}{\sqrt{\nu t}} \left| v^G \left(\frac{x - \tilde{z}(t)}{\sqrt{\nu t}} \right) - v^G \left(\frac{x - z(t)}{\sqrt{\nu t}} \right) \right| \lesssim \nu t \quad \text{for all } x \in \text{supp}(w_{1,a}). \quad (4.4)$$

As long as the above claim is proved, we would get

$$\begin{aligned} & \left\| \frac{1}{\nu\sqrt{t}} \left(v^G \left(\frac{x - \tilde{z}(t)}{\sqrt{\nu t}} \right) - v^G \left(\frac{x - z(t)}{\sqrt{\nu t}} \right) \right) \cdot \nabla w_{1,a} \right\|_{L^4 \cap L^{4/3}} \\ & \lesssim \sqrt{\nu} \|\nabla w_{1,a}(t)\|_{L^4 \cap L^{4/3}(\text{supp}(w_{1,a}))} \lesssim \sqrt{\nu} \end{aligned}$$

by Proposition 2.1.

Now we shall prove the inequality (4.4). To this end, let us denote

$$\eta_1 = x - \tilde{z}(t), \quad \text{and} \quad \eta_2 = x - z(t) \quad (4.5)$$

The left hand side of (4.4) can be re-written as:

$$\frac{1}{\sqrt{\nu t}} \left(v^G \left(\frac{\eta_1}{\sqrt{\nu t}} \right) - v^G \left(\frac{\eta_2}{\sqrt{\nu t}} \right) \right) = \frac{1}{2\pi} (V_1(\eta_1, \eta_2) + V_2(\eta_1, \eta_2)) \quad (4.6)$$

where

$$\begin{cases} V_1(\eta_1, \eta_2) &= \left(\frac{\eta_1^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_2|^2} \right), \\ V_2(\eta_1, \eta_2) &= \left(\frac{\eta_2^\perp}{|\eta_2|^2} e^{-|\eta_2|^2/4\nu t} - \frac{\eta_1^\perp}{|\eta_1|^2} e^{-|\eta_1|^2/4\nu t} \right). \end{cases}$$

When $x \in \text{supp}(\tilde{\omega}^E(t))$, by the properties established in Section 2, we have a positive constant c_T , independent of ν , such that

$$|x - z(t)| \geq c_T \quad \text{and} \quad |x - \tilde{z}(t)| \geq c_T \quad \forall t \in [0, T]. \quad (4.7)$$

This implies that $|\eta_1| \geq c_T$ and $|\eta_2| \geq c_T$, upon recalling the notations (4.5). Thus, we get

$$\begin{aligned} |V_1(\eta_1, \eta_2)| &= \left| \frac{\eta_1^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_2|^2} \right| \leq \left| \frac{\eta_1^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_1|^2} \right| + \left| \frac{\eta_2^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_2|^2} \right| \\ &\leq \frac{|\eta_1 - \eta_2|}{|\eta_1|^2} + |\eta_2| \frac{||\eta_2|^2 - |\eta_1|^2|}{|\eta_1|^2 |\eta_2|^2} \\ &\leq c_T^{-2} |\eta_1 - \eta_2| + \frac{1}{|\eta_1|^2 |\eta_2|} \left| |\eta_2|^2 - |\eta_1|^2 \right| (|\eta_1| + |\eta_2|) \lesssim |\eta_1 - \eta_2| \\ &= |(x - \tilde{z}(t)) - (x - z(t))| = |\tilde{z}(t) - z(t)| \lesssim \nu t \quad (\text{by the estimate (2.5)}). \end{aligned}$$

Hence

$$|V_1(\eta_1, \eta_2)| \lesssim \nu t. \quad (4.8)$$

Now for $V_2(\eta_1, \eta_2)$, note that we shall only consider $x \in \text{supp}(\tilde{\omega}^E(t))$, in which we get (4.7). In this case we get

$$|V_2(\eta_1, \eta_2)| \leq |\eta_2|^{-1} e^{-|\eta_2|^2/4\nu t} + |\eta_1|^{-1} e^{-|\eta_1|^2/4\nu t} \leq 2c_T^{-1} e^{-c_T^2/4\nu t} \lesssim \nu t \quad (4.9)$$

Combining (4.6), (4.8) and (4.9), we get the desired inequality (4.4). The bound for the first term is complete. This concludes the proof. \square

Lemma 4.4. *There holds*

$$\|f_2(t)\|_{L^4 \cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}}$$

Proof. We have

$$\|f_2(t)\|_{L^4 \cap L^{4/3}} = \|\bar{v}_1(t) \cdot \nabla \tilde{\omega}^E(t)\|_{L^4 \cap L^{4/3}} \leq \|\bar{v}_1(t)\|_{L^\infty} \|\nabla \tilde{\omega}^E(t)\|_{L^4 \cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}}$$

by Corollary 2.2 and Lemma A.1. The proof is complete. \square

Lemma 4.5. *There holds*

$$\|f_3(t)\|_{L^4 \cap L^{4/3}} \lesssim t^{3/2} \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right)$$

Proof. We recall that

$$f_3(x, t) = \frac{\sqrt{t}}{\nu} \bar{v}_2(\xi, t) \cdot \nabla \tilde{\omega}^E(t, x), \quad \xi = \frac{x - \tilde{z}(t)}{\sqrt{\nu t}}.$$

We shall only consider $x \in \text{supp}(\tilde{\omega}^E(t))$. Since $\tilde{\omega}^E(t)$ is supported away from $\tilde{z}(t)$, there exists $d_T > 0$ such that

$$|x - \tilde{z}(t)| \geq d_T \quad \text{for } x \in \text{supp}(\tilde{\omega}^E(t)). \quad (4.10)$$

Since $\int_{\mathbb{R}^2} \bar{w}_2(\xi, t) d\xi = 0$, by Lemma (A.1), we get

$$\begin{aligned} \|(1 + |\xi|^2) \bar{v}_2(t)\|_{L^\infty} &\lesssim \|(1 + |\xi|^2) \bar{w}_2(t)\|_{L^4} + \|(1 + |\xi|^2) \bar{w}_2(t)\|_{L^{4/3}} \\ &\lesssim \|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2}. \end{aligned}$$

This implies that, for x in the support of $\tilde{\omega}^E(t)$, we get

$$|\bar{v}_2(t, \xi)| \lesssim \frac{1}{1 + |\xi|^2} \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right) \lesssim (\nu t) \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right).$$

Thus we get

$$\|f_3(t)\|_{L^4 \cap L^{4/3}} \lesssim \frac{\sqrt{t}}{\nu} \|\bar{v}_2(\xi, t) \cdot \nabla \tilde{\omega}^E(t, x)\|_{L^4 \cap L^{4/3}} \lesssim t^{3/2} \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right).$$

The proof is complete. \square

We conclude this subsection by proving the Proposition 4.2

Proof of Proposition 4.2. The proof follows as a direct consequence of the previous lemmas for f_i , $i \in \{1, 2, 3\}$ in this subsection. \square

4.3 Apriori estimates for the remainder

In this section, we give a proof for our main Theorem 4.1, stated at the beginning of this subsection. We recall from Section 4.1 that \bar{w}_1 solves the heat transport equation

$$\partial_t \bar{w}_1 + u^\nu \cdot \nabla \bar{w}_1 - \nu \Delta \bar{w}_1 = f(x, t).$$

A standard $L^4 \cap L^{4/3}$ estimate for the heat transport equation yields

$$\begin{aligned} \frac{d}{dt} (\|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}}) &\lesssim \|f(t)\|_{L^4 \cap L^{4/3}} \\ &\lesssim \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} + t^{3/2} \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\nabla \bar{w}_2(t)\|_{L_p^2} \right) + \sqrt{\nu}, \end{aligned}$$

using Proposition 4.2. Now applying Gronwall lemma for the above inequality, we have

$$\begin{aligned} \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} &\lesssim \int_0^t \left(s^{3/2} (\|\bar{w}_2(s)\|_{L_p^2} + \|\nabla \bar{w}_2(s)\|_{L_p^2}) + \sqrt{\nu} \right) ds \\ &\lesssim \int_0^t s^{3/2} (\|\bar{w}_2(s)\|_{L_p^2} + \|\nabla \bar{w}_2(s)\|_{L_p^2}) ds + \nu^{1/2} t. \end{aligned} \quad (4.11)$$

The proof is complete.

5 Proof of inviscid limit

In this section, we conclude the proof for inviscid limit, using the apriori estimates obtained from the previous sections. Let us first prove the following proposition, before proving our main theorem, stated in the first part of this paper.

Proposition 5.1. *There exists a time $T > 0$, independent of the viscosity ν , such that*

$$\sup_{0 \leq t \leq T} \left(\|\bar{w}_2(t)\|_{L_p^2} + \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} \right) \lesssim 1,$$

uniformly in ν .

Proof. First, we recall the following estimates for $\|\bar{w}_2(t)\|_{L_p^2}$ and $\|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}}$ proven in Propositions 3.7 and 4.1.

$$\begin{aligned} \frac{d}{dt} \|\bar{w}_2(t)\|_{L_p^2}^2 + \frac{\kappa}{t} \left(\|(1 + |\xi|)\bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 \right) &\lesssim \|\bar{w}_2(t)\|_{L_p^2}^5 + \nu \|\bar{v}_1(t)\|_{L^\infty}^4 + t^{-2} \|\bar{v}_1(t)\|_{L^\infty}^2 \\ \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} &\lesssim \int_0^t s^{3/2} \left(\|\bar{w}_2(s)\|_{L_p^2} + \|\nabla \bar{w}_2(s)\|_{L_p^2} \right) ds + \nu^{1/2} t. \end{aligned} \tag{5.1}$$

Let

$$\mathcal{G}(t) = \|\bar{w}_2(t)\|_{L_p^2}^2 + \int_0^t s^{-1} \left(\|\bar{w}_2(s)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(s)\|_{L_p^2}^2 \right) ds.$$

From the inequality (5.1), it is straight-forward that

$$\|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} \lesssim t^{5/2} \mathcal{G}(t)^{1/2} + \nu^{1/2} t. \tag{5.2}$$

Thus, we have

$$\begin{aligned} \mathcal{G}'(t) &= \frac{d}{dt} \|\bar{w}_2(t)\|_{L_p^2}^2 + t^{-1} \left(\|\bar{w}_2(t)\|_{L_p^2}^2 + \|\nabla \bar{w}_2(t)\|_{L_p^2}^2 \right) \\ &\lesssim \|\bar{w}_2(t)\|_{L_p^2}^5 + \nu \|\bar{v}_1(t)\|_{L^\infty}^4 + t^{-2} \|\bar{v}_1(t)\|_{L^\infty}^2 \quad (\text{by (5.1)}) \\ &\lesssim \mathcal{G}(t)^{5/2} + \nu \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}}^4 + t^{-2} \|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}}^2 \\ &\lesssim \mathcal{G}(t)^{5/2} + \nu \left(t^{5/2} \mathcal{G}(t)^{1/2} + \nu^{1/2} t \right)^4 + t^{-2} \left(t^{5/2} \mathcal{G}(t)^{1/2} + \nu^{1/2} t \right)^2 \quad (\text{by (5.2)}) \\ &\lesssim \mathcal{G}(t)^{5/2} + \nu t^{10} \mathcal{G}(t)^2 + \nu^3 t^4 + t^3 \mathcal{G}(t) + \nu. \end{aligned}$$

By standard ODE theory, we have a time $T > 0$, which is independent of $\nu > 0$, such that $\mathcal{G}(t)$ is uniformly bounded for $t \in [0, T]$. Since $\mathcal{G}(t) \geq \|\bar{w}_2(t)\|_{L_p^2}^2$, the proof for $\|\bar{w}_2(t)\|_{L_p^2}$ is complete. The bound $\|\bar{w}_1(t)\|_{L^4 \cap L^{4/3}} \lesssim 1$ follows from the inequality (5.2). \square

We conclude this section by proving our main theorem, stated in the first part of this paper.

Proof of theorem 1.2. We have proved that $\|\bar{w}_2(t)\|_{L^2_p}$ is uniformly bounded in ν . We recall from Section 3 that

$$\omega^{B,\nu}(t, x) = \frac{1}{\nu t} w_2(\xi, t) = \frac{1}{\nu t} (G(\xi) + (\nu t)w_{2,a} + (\nu t)\bar{w}_2) = \frac{1}{\nu t} G(\xi) + w_{2,a} + \bar{w}_2,$$

where $G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}$ and $\xi = (x - \tilde{z}(t))/\sqrt{\nu t}$. We compute

$$\begin{aligned} \left\| \omega^{B,\nu}(t, x) - \frac{1}{4\pi\nu t} e^{-\frac{|x-\tilde{z}(t)|^2}{4\nu t}} \right\|_{L^1_x} &= \|w_{2,a}(\xi, t) + \bar{w}_2(\xi, t)\|_{L^1_\xi} \\ &= \nu t \|w_{2,a}(t) + \bar{w}_2(t)\|_{L^1_\xi} \lesssim (\nu t) \left(\|w_{2,a}(t)\|_{L^2_p} + \|\bar{w}_2(t)\|_{L^2_p} \right) \\ &\lesssim (\nu t). \end{aligned} \quad (5.3)$$

For simplicity of notations, we denote by $G_{\tilde{z}(t)}(x)$ and $G_{z(t)}(x)$ the Gaussians $\frac{1}{4\pi\nu t} e^{-\frac{|x-\tilde{z}(t)|^2}{4\nu t}}$ and $\frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}}$, respectively. Our goal now is to compare the two Gaussians in L^1 norm. To this end, let us denote $A = \frac{|x-\tilde{z}(t)|^2}{4\nu t}$ and $B = \frac{|x-z(t)|^2}{4\nu t}$. We have

$$G_{\tilde{z}(t)}(x) - G_{z(t)}(x) = e^{-A} - e^{-B} = e^{-B} (e^{B-A} - 1).$$

We have

$$\begin{aligned} B - A &= (4\nu t)^{-1} (|x - z(t)|^2 - |x - \tilde{z}(t)|^2) = (4\nu t)^{-1} (2x \cdot (\tilde{z}(t) - z(t)) + |z(t)|^2 - |\tilde{z}(t)|^2) \\ &\lesssim (4\nu t)^{-1} (|x| |\tilde{z}(t) - z(t)| + |\tilde{z}(t) - z(t)|) \\ &\lesssim |x| + 1 \quad (\text{since } |\tilde{z}(t) - z(t)| \lesssim \nu t) \\ &\leq |x - z(t)| + |z(t)| + 1 \lesssim \frac{|x - z(t)|}{\sqrt{\nu t}} + 1. \end{aligned}$$

Here we used the standard fact of the vortex-wave system that $|z(t)| \lesssim 1$ for any fixed interval of time. For, one can see that $|z(t)| \leq |z_0| + \int_0^t |v^E(z(s), s)| ds \leq |z_0| + t \|v^E\|_{L^\infty}$. Hence we get

$$|G_{\tilde{z}(t)}(x) - G_{z(t)}(x)| \lesssim e^{-\frac{|x-z(t)|^2}{4\nu t} + M_T \frac{|x-z(t)|}{\sqrt{\nu t}}} \quad \text{for some } M_T > 0. \quad (5.4)$$

Integrating both sides of the inequality (5.4) in $x \in \mathbb{R}^2$, we have

$$\|G_{z(t)} - G_{\tilde{z}(t)}\|_{L^1_x} \lesssim \int_{\mathbb{R}^2} e^{-\frac{|x-z(t)|^2}{4\nu t} + M_T \frac{|x-z(t)|}{\sqrt{\nu t}}} dx.$$

Making the change of variables $y = \frac{x-z(t)}{\sqrt{\nu t}}$ in the above integral, we thus obtain

$$\|G_{z(t)} - G_{\tilde{z}(t)}\|_{L^1_x} \lesssim \nu t. \quad (5.5)$$

Combining the inequalities (5.3) and (5.5), we get

$$\left\| \omega^{B,\nu}(t, x) - \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right\|_{L^1_x} \lesssim \nu t.$$

The inequality $\|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4 \cap L^{4/3}} \lesssim \nu$ follows directly from the expansion (4.2), the inequality (5.2) and the uniform bound of $\mathcal{G}(t)$. The proof is complete. \square

A Appendix

In this section, we collect several useful lemmas used in this paper.

Lemma A.1 (Elliptic estimates). *Let $v = K \star \omega$ be the velocity vector field obtained from the vorticity ω on \mathbb{R}^2 . Define the norm $\|\cdot\|_{L^4 \cap L^{4/3}} = \|\cdot\|_{L^4} + \|\cdot\|_{L^{4/3}}$. There hold the following inequalities*

$$\|v\|_{L^\infty} \lesssim \|\omega\|_{L^4 \cap L^{4/3}}, \quad \|v\|_{L^\infty} \lesssim \|\omega\|_{L^1 \cap L^\infty}.$$

Moreover, if $\int_{\mathbb{R}^2} \omega(x) dx = 0$, then

$$\|(1 + |x|^2)v\|_{L^\infty} \lesssim \|(1 + |x|^2)\omega\|_{L^4 \cap L^{4/3}}.$$

Proof. From the Biot-Savart law (1.2), we estimate

$$\begin{aligned} |v(x)| &\lesssim \int_{\mathbb{R}^2} \frac{|\omega(y)|}{|x-y|} dy = \left(\int_{|x-y| \leq R} + \int_{|x-y| \geq R} \right) \frac{|\omega(y)|}{|x-y|} dy \\ &\lesssim \left(\int_{|x-y| \leq R} |x-y|^{-4/3} dy \right)^{3/4} \|\omega\|_{L^4} + \left(\int_{|x-y| \geq R} |x-y|^{-4} dy \right)^{1/4} \|\omega\|_{L^{4/3}} \\ &\lesssim R^{1/2} \|\omega\|_{L^4} + R^{-1/2} \|\omega\|_{L^{4/3}}. \end{aligned} \tag{A.1}$$

Thus choosing $R = \frac{\|\omega\|_{L^{4/3}}}{\|\omega\|_{L^4}}$, we have $\|v\|_{L^\infty} \lesssim \|\omega\|_{L^{4/3}}^{1/2} \|\omega\|_{L^4}^{1/2}$, which gives the first inequality. As for the second, we use $\|\omega\|_{L^p} \leq \|\omega\|_{L^1}^{1/p} \|\omega\|_{L^\infty}^{1-1/p}$.

It remains to check the last inequality. We shall check it for v_2 , the second component of v . The estimate on v_1 is similar. First, we check

$$|x| |v_2(x)| \lesssim \int_{\mathbb{R}^2} \frac{1}{|x-y|} |y| |\omega(y)| dy. \tag{A.2}$$

By Biot-Savart law and $\int_{\mathbb{R}^2} \omega(y) dy = 0$, we have

$$|v_2(x)| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x-y|^2} \omega(y) dy \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \left| \frac{x_1 - y_1}{|x-y|^2} - \frac{x_1}{|x|^2} \right| |\omega(y)| dy.$$

Now we have

$$\frac{x_1 - y_1}{|x-y|^2} - \frac{x_1}{|x|^2} = \frac{1}{|x|^2 |x-y|^2} (|x|^2(x_1 - y_1) - x_1|x-y|^2).$$

It follows that $|x|^2(x_1 - y_1) - x_1|x-y|^2 \leq 4|x||y||x-y|$. Hence,

$$|x| \left[\frac{x_1 - y_1}{|x-y|^2} - \frac{x_1}{|x|^2} \right] \leq \frac{4|y|}{|x-y|},$$

which gives (A.2). Now multiplying both sides of (A.2) by $|x|$, we have

$$\begin{aligned} |x|^2 |v_2(x)| &\lesssim \int_{\mathbb{R}^2} \frac{|x||y|}{|x-y|} |\omega(y)| dy \leq \int_{\mathbb{R}^2} \frac{|y|+|x-y|}{|x-y|} |y| |\omega(y)| dy \\ &= \int_{\mathbb{R}^2} \frac{1}{|x-y|} |y|^2 |\omega(y)| dy + \int_{\mathbb{R}^2} |y| |\omega(y)| dy \end{aligned}$$

Let us first treat the first term in the above. Repeating the argument of (A.1) for $\omega = |y|^2 |\omega(y)|$, we have

$$\int_{\mathbb{R}^2} \frac{1}{|x-y|} |y|^2 |\omega(y)| dy \lesssim \|(1+|y|^2)\omega(y)\|_{L^4 \cap L^{4/3}}.$$

For the second term, using Hölder inequality, we get

$$\int_{\mathbb{R}^2} |y| |\omega(y)| dy = \int_{\mathbb{R}^2} \frac{|y|}{1+|y|^2} (1+|y|^2) |\omega(y)| dy \lesssim \|(1+|y|^2)\omega(y)\|_{L^{4/3}}.$$

Thus

$$|x|^2 |v_2(x)| \lesssim \|(1+|x|^2)\omega\|_{L^4 \cap L^{4/3}}.$$

The lemma follows. \square

Lemma A.2. *Let $z_1, z_2 \in \mathbb{C}$ and ψ be the angle between z_1 and z_2 . Assuming that $|z_1| < |z_2|$ and $\sin(\psi) \neq 0$, there holds*

$$\frac{1}{|z_1 + z_2|^2} - \frac{1}{|z_2|^2} = \frac{1}{|z_2|^2} \sum_{n=1}^{\infty} (-1)^n \frac{|z_1|^n}{|z_2|^n} \frac{\sin((n+1)\psi)}{\sin(\psi)}.$$

Proof. Let $\frac{z_1}{z_2} = z = re^{i\psi}$. We have

$$\frac{1}{|z_1 + z_2|^2} - \frac{1}{|z_2|^2} = \frac{1}{|z_2|^2} \left(\frac{1}{|1+z|^2} - 1 \right).$$

Now for $|z| < 1$, we have

$$\begin{aligned} \frac{1}{|1+z|^2} &= \frac{1}{(1+z)(1+\bar{z})} = (1-z+z^2-\dots)(1-\bar{z}+\bar{z}^2+\dots) \\ &= 1 - (z+\bar{z}) + (z^2+z\bar{z}+\bar{z}^2) - (z^3+z^2\bar{z}+z\bar{z}^2+\bar{z}^3) + \dots \end{aligned}$$

Now for each n , we have

$$z^n + z^{n-1}\bar{z} + \dots + z\bar{z}^{n-1} + \bar{z}^n = \frac{z^{n+1} - \bar{z}^{n+1}}{z - \bar{z}} = r^n \frac{\sin((n+1)\psi)}{\sin \psi}.$$

This concludes the proof. \square

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