1. Review Notes 1

Go over the materials from Review Notes 1: http://www.personal.psu.edu/ttn12/files/review1.pdf

2. Review Notes 2

Go over the materials from Review Notes 2: http://www.personal.psu.edu/ttn12/files/review2.pdf

3. Harmonic functions

Harmonic functions are solutions to the Laplace equation:
\[ \Delta u = 0, \quad \text{in a domain } D, \]
together with a boundary condition. \( D \) is a domain in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Typical boundary conditions are one of the following types:

- Dirichlet \( u = h(x) \), for \( x \in \partial D \).
- Neumann \( \nabla u \cdot n = h(x) \), for \( x \in \partial D \).
- Robin \( \nabla u \cdot n + a(x)u = h(x) \), for \( x \in \partial D \).


**Theorem 3.1.** Let \( D \) be a connected and bounded domain in \( \mathbb{R}^n \), with \( n = 2 \) or \( 3 \). Then the maximum and minimum of a Harmonic function are attained on the boundary of \( D \).

**Proof.** Indeed, take arbitrary small \( \varepsilon > 0 \) and study the function \( v(x) = u(x) + \epsilon |x|^2 \). Calculation shows
\[ \Delta v = \Delta u + 2n\epsilon = 2n\epsilon, \]
since \( \Delta u = 0 \) (Harmonic function). Now, the maximum of \( v \) can't be attained in the interior, since otherwise at that point \( \Delta v \leq 0 \), which can't be equal to a positive number \( 2n\epsilon \). (by changing the sign of \( \epsilon \) to be negative, you can show that minimum of \( v \) can't be attained in the interior). This proves the maximum of \( v \) is attained on the boundary:
\[ v(x) \leq \max_{x \in \partial D} v(x). \]

Writing \( v = u + \epsilon |x|^2 \), the above inequality is equivalent to
\[ u(x) \leq v(x) \leq \max_{x \in \partial D} (u(x) + \epsilon |x|^2) \leq \max_{x \in \partial D} u(x) + \epsilon C_0, \]
for some large constant \( C_0 \), since the domain \( D \) is bounded. Taking the limit of \( \epsilon \to 0 \) in the above inequality yields
\[ u(x) \leq \max_{x \in \partial D} u(x). \]
This is the weak Maximum Principle. Later on, we shall prove a stronger version of MP. \( \square \)

**Theorem 3.2.** If \( u \) is the sub-Harmonic function, that is \( \Delta u \geq 0 \), then the same proof of MP as above yields that
\[ u(x) \leq \max_{x \in \partial D} u(x). \]

**Theorem 3.3** (Comparison Principle). If \( \Delta u \geq \Delta v \) in \( D \) and if \( u \leq v \) on the boundary \( \partial D \), then
\[ u \leq v \quad \text{in} \quad D. \]
Theorem 3.4. Let $B$ be the unit ball in the plane, and let $u(x, y)$ be the solution to the Poisson problem:

\[
\begin{align*}
-\Delta u &= f(x, y) & \text{in } & B \\
u &= 0 & \text{on } & \partial B
\end{align*}
\]

Prove that at all points $(x, y) \in B$, we have

\[
\frac{1}{4} \max_{B} |f| \leq u(x, y) \leq \frac{1}{4} \max_{B} |f|.
\]

Proof. Introduce $v = u + \frac{x^2 + y^2}{4} \max_{B} |f|$. It turns out that $v$ is a sub-harmonic function, and hence the MP can be applied on $v$. \qed

Theorem 3.5 (Uniqueness of the Dirichlet problem). There is at most one solution to the following Poisson problem:

\[
\Delta u = f(x) \quad \text{in } D, \quad u = h(x) \quad \text{on } \partial D,
\]

for any given functions $f(x), h(x)$.

Proof. Indeed, assuming there are two such solutions $u_1, u_2$. Look at their difference: $w = u_1 - u_2$. It solves the Laplace equation, with zero Dirichlet boundary condition. Hence, by the Maximum Principle, it must be zero. This proves the uniqueness of the solution to the Poisson problem. \qed

3.2. Mean-value property. Suppose that $u \in C^2$. If $u$ is a harmonic function, then $u$ satisfies the mean-value property:

\[
u(x) = \int_{\partial B(x, r)} u(y) \, dS(y) = \int_{B(x, r)} u(y) \, dy
\]

for all ball $B(x, r)$, centered at $x$ and of radius $r$, as long as the ball is contained inside $D$. The converse is also true. The proof goes as follows: set

\[
\phi(r) = \int_{\partial B(x, r)} u(y) \, dS(y).
\]

Then, we calculate

\[
\begin{align*}
\phi'(r) &= \frac{d}{dr} \int_{\partial B(0, 1)} u(x + rz) \, dS(z) = \int_{\partial B(0, 1)} z \cdot \nabla u(x + rz) \, dS(z) \\
&= \int_{\partial B(x, r)} n \cdot \nabla u(y) \, dS(y) = \frac{1}{|\partial B(x, r)|} \int_{B(x, r)} \Delta u \, dy.
\end{align*}
\]

It is now clear that $u$ is harmonic iff $\Delta u = 0$ iff $\phi(r)$ is constant for all $r > 0$. Since $\phi(r)$ is constant in $r$, we have

\[
\phi(r) = \lim_{r \to 0} \phi(r_*). = \lim_{r_* \to 0} \int_{\partial B(x, r_*)} u(y) \, dS(y) = u(x),
\]

proving the Mean Value Property.

3.3. Strong Maximum Principle.

Theorem 3.6. Let $D$ be a connected and bounded domain in $\mathbb{R}^n$, with $n = 2$ or 3. Then, if the maximum or minimum of a Harmonic function is attained in the interior of the domain $D$, the Harmonic function must be a constant.

Proof. The proof uses the mean-value property of harmonic functions:

\[
u(x) = \int_{B(x, r)} u(y) \, dy
\]

for any $r > 0$, as long as the ball is contained inside $D$. Now, if there is a point $x_0$ so that

\[
\max_{D} u = u(x_0) = \int_{B(x_0, r)} u(y) \, dy \leq \max_{D} u \int_{B(x_0, r)} dS(y) = \max_{D} u.
\]
This proves that $u$ must be a constant (equal to $\max_D u$) in the ball $B(x_0, r)$. Now, for any other points $x_1$ in the domain $D$, we want to prove that $u(x_1) = u(x_0)$, and hence $u$ is a constant. Indeed, since $D$ is connected, we can always construct a finite ball covering the path from $x_0$ to $x_1$. Within each ball, $u$ is the same constant, and hence $u(x_1)$ must be the same as $u(x_0)$.

This proves that $u$ must be a constant, if its maximum is attained inside of $D$. \qed

\textbf{Remark 3.7.} In the other words, the strong MP asserts that either the Harmonic function $u$ is constant or $u$ satisfies

$$\min_{\partial D} u < u(x) < \max_{\partial D} u, \quad \forall \ x \in D.$$ 

4. \textsc{Laplace equations on some special geometries}

The Laplace equation can be solved via the method of separation of variables in certain symmetric domains. Below are a few examples.

4.1. \textbf{In a rectangle} $(x, y) \in [0, a] \times [0, b]$. Write $u = X(x)Y(y)$ and deduce from $\Delta u = 0$ the following identity:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0. \quad (4.1)$$

This shows that the ratios $\frac{X''}{X}$ and $\frac{Y''}{Y}$ must be constants and have opposite signs. Depending on the imposed boundary conditions on the edges of the rectangle, we can find these constants: \textit{this is an eigenvalue problem}!

For instance, we impose the zero Dirichlet boundary conditions on the top and bottom of the rectangle, that is, at $y = 0, b$. It follows that

$$\frac{Y''}{Y} = -\lambda, \quad Y(0) = Y(b) = 0.$$ 

This eigenvalue problem yields a set of solutions

$$\lambda_n = \frac{n^2 \pi^2}{b^2}, \quad Y_n(y) = \sin \frac{n\pi y}{b}, \quad n = 1, 2, \cdots$$

Next, from (4.1), we get

$$\frac{X''}{X} = \lambda_n,$$

which yields the fundamental solutions:

$$e^{\sqrt{\lambda_n}x}, \quad e^{-\sqrt{\lambda_n}x}$$

and hence, $X_n(x)$ is a linear combination of these fundamental solutions. Use the boundary conditions on $u$ on two side edges (that is, at $x = 0, a$) to determine the coefficients in the Fourier series of the solution:

$$u(x, y) = \sum_{n \geq 1} X_n(x)Y_n(y) = \sum_{n \geq 1} \left(A_ne^{\frac{n\pi x}{a}} + B_ne^{-\frac{n\pi x}{a}}\right) \sin \frac{n\pi y}{b}.$$ 

\textbf{NOTE: occasionally, it is more convenient to take a different set of fundamental solutions to determine $X_n$.} For instance, if we assume that $X_n(a) = 0$, we should then take $X_n(x)$ to be of the form:

$$X_n(x) = A_n \sinh \left(\frac{n\pi (x - a)}{b}\right) + B_n \cosh \left(\frac{n\pi (x - a)}{b}\right)$$

Using the zero condition for $X_n(a) = 0$, we conclude that $B_n$ must be zero, and hence

$$X_n(x) = A_n \sinh \left(\frac{n\pi (x - a)}{b}\right).$$
4.2. **In a box** $[0, a] \times [0, b] \times [0, c]$. Separation of variables $u = X(x)Y(y)Z(z)$ yields from the Laplace equation

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0,$$

from which all the three ratios must be constants (and they add up to zero). Again, depending on zero boundary conditions on each sides of the box, we can find these constants via the corresponding eigenvalue problems. Very similar steps as above (or as in our previous steps to solve the diffusion problem) lead to the solution in the box.

4.3. **In a disk**: $\{x^2 + y^2 \leq a^2\}$. Due to the nature of the disk, we first write the Laplace equation in the polar coordinates $u = u(r, \theta)$, which is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0.$$

Luckily, the method of separation of variables works again! Write $u = R(r)\Theta(\theta)$. The Laplace equation reduces to

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} = 0.$$

Varying $r$, we note that the ratio $\frac{\Theta''}{\Theta}$ must be constants. **First, find these constants!**

Indeed, we study the problem in the disk, and so, we must require that $\Theta(\theta)$ is $2\pi$-periodic: $\Theta(\theta + 2\pi) = \Theta(\theta)$. The eigenvalue problem $\Theta'' = -\lambda \Theta$ with periodic boundary conditions gives a set of solutions

$$\lambda_n = n^2, \quad \Theta_n(\theta) = A_n \sin(n\theta) + B_n \cos(n\theta), \quad n = 0, 1, 2, \ldots$$

With each of these $\lambda = n^2$, we find that

$$R_n = C_n r^n + D_n r^{-n}$$

are solutions to the equation (4.3), with the ratio $\frac{\Theta''}{\Theta} = -\lambda_n$. We want bounded solutions, so we take $D_n = 0$ in the solution for $R_n$.

The general solution in the disk is then

$$u(r, \theta) = \sum_{n \geq 0} r^n \left( A_n \sin(n\theta) + B_n \cos(n\theta) \right).$$

Use the Fourier theory to determine the coefficients!

4.4. **In the exterior of a disk**: $\{x^2 + y^2 \geq a^2\}$. Same calculations as above up to the computation (4.4) for $R_n$. Since it is the exterior of the disk, that is, $r$ can be large, we are forced to take $C_n = 0$, yielding $R_n = r^{-n}$, and hence the general solution is

$$u(r, \theta) = \sum_{n \geq 0} r^{-n} \left( A_n \sin(n\theta) + B_n \cos(n\theta) \right).$$

Use the Fourier theory to determine the coefficients!

4.5. **In a wedge**: $(r, \theta) \in [0, a] \times [\alpha, \beta]$. Same calculations work, with an exception that $\Theta(\theta)$ is now no longer periodic, but satisfy an imposed boundary conditions on the edge of the wedge: $\theta = \alpha, \beta$.

4.6. **Spherical coordinates**. It is also possible to solve the Laplace equation via the method of separation of variables in the spherical coordinates!
5. GREEN FUNCTION TO THE LAPLACE OPERATOR

Let $G(x, x_0)$ be the Green function to the Laplace operator in domain $D$ with Dirichlet boundary conditions: namely,

- $\Delta G(x, x_0) = \delta_{x=x_0}$.
- $G(x, x_0) = 0$ for $x \in \partial D$.

Then, consider the Poisson problem

$$\Delta u = f$$

in $D$ with Dirichlet boundary condition $u = h(x)$ on $\partial D$. Then, you can check easily that the unique solution can be expressed in term of the Green function as follows:

$$u(x) = \int_D G(y, x) f(y) \, dy + \int_{\partial D} \frac{\partial G(y, x)}{\partial n} h(y) \, dS(y).$$

Here are examples of the Green function in some special cases:

- Green function on the whole space $\mathbb{R}^d$:

$$G(x, x_0) = \begin{cases} \log |x - x_0|, & d = 2 \\ \frac{1}{|x - x_0|^{d-2}}, & d \geq 3 \end{cases}$$

- Green function on the half-space $\mathbb{R}^d_+ = \{x_d > 0\}$ with Dirichlet boundary conditions:

$$H(x, x_0) = G(x, x_0) - G(x, x_0^*)$$

where $x_0^*$ is the reflection of $x_0$ across the boundary $x_d = 0$: precisely, if $x_0 = (x', x_d)$ then $x_0^* = (x', -x_d)$.

- Green function on the ball $\{|x| < a\}$ with Dirichlet boundary conditions:

$$H(x, x_0) = G(x, x_0) - \frac{a}{|x_0|} G(x, x_0^*)$$

where $x_0^*$ is the reflection of $x_0$ across the boundary surface $|x| = a$: precisely, $x_0^* = \frac{a^2 x_0}{|x_0|^2}$.