Kinetic theory of gases

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Fall 2017, I teach a graduate topics course: same topics!
**Figure**: 4 states of matter, plus Bose-Einstein condensate!
James Clerk Maxwell, in 1859, gave birth to kinetic theory: use the statistical approach to describe the dynamics of a (rarefied) gas.
Figure: Kinetic theory of gases: phase space!
Gas molecules are identical.

One-particle phase space: position $x \in \Omega \subset \mathbb{R}^d$, momentum $p \in \mathbb{R}^d$
Gas molecules are identical.

One-particle phase space: position $x \in \Omega \subset \mathbb{R}^d$, momentum $p \in \mathbb{R}^d$

$f(t, x, p)$ the probability density distribution. Mass: $f(t, x, p) \, dx \, dp$.

Interest: the dynamics of $f(t, x, p)$? Non-equilibrium theory.
I. Collisionless Kinetic Theory
Hamilton 1830’s classical mechanics\textsuperscript{1}: one-particle Hamiltonian $H(x, p)$ (induced from Lagrangian):

$$\dot{x} = \nabla_p H, \quad \dot{p} = -\nabla_x H$$

\textsuperscript{1}In truth, it should be an N-particle Hamiltonian dynamics, then mean field limit or BBGKY hierarchy to kinetic theory. \textbf{S1}: Presentation on its formal derivation?
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Examples:

- Free particles:
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Examples:

- Free particles:
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  H(x, p) = \frac{1}{2m} |p|^2
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- Particles in a potential:
  \[
  H(x, p) = \frac{1}{2m} |p|^2 + V_{pot}(x)
  \]

---

\(^1\)In truth, it should be an \(N\)-particle Hamiltonian dynamics, then mean field limit or BBGKY hierarchy to kinetic theory. \textbf{S1:} Presentation on its formal derivation?
• Also for plasmas: charged particles in electromagnetic fields

\[ H(x, p) = \frac{1}{2m} |p - qA(x)|^2 + \phi(x) \]

with electric and magnetic potentials \( \phi, A \).
• Also for plasmas: charged particles in electromagnetic fields

\[ H(x, p) = \frac{1}{2m} |p - qA(x)|^2 + \phi(x) \]

with electric and magnetic potentials \( \phi, A \).

• Force acting on particles:

\[ F = -\nabla_x H = E + \frac{q}{m} (p \times B) \]

which is the Lorentz force, with \( E = -\nabla_x \phi \) and \( B = \nabla_x \times A \).
Figure: Liouville's Theorem: volume in phase space remains constant (Exercise: prove this).
Collisionless kinetic theory

Liouville's Theorem: volume in phase space remains constant (Exercise: prove this).

This yields the kinetic equation: \( \partial_t f = \{H, f\} \) (Poisson bracket), explicitly

\[
0 = \frac{d}{dt} f(t, x(t), p(t)) = \partial_t f + \nabla_p H \cdot \nabla_x f - \nabla_x H \cdot \nabla_p f
\]

\[
= \partial_t f + \mathbf{v} \cdot \nabla_x f + F_{\text{force}} \cdot \nabla_v f
\]
Three examples of kinetic equations (remain very active):

Vlasov dynamics (transport in phase space):
\[
\frac{\partial}{\partial t} f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0
\]

Vlasov-Poisson:
\[
F = -\nabla_x \phi, \quad -\Delta_x \phi = \sigma \int_{\mathbb{R}^3} f(t,x,v) \, dv,
\]

with \( \sigma = 1 \) (plasma physics: charged particles) or \( \sigma = -1 \) (gravitational case: stars in a galaxy).

Vlasov-Maxwell (plasma): Lorentz force
\[
F = E + v \times B
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with the electromagnetic fields \( E \), \( B \) solving Maxwell (generated by charge and current density).
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- **Vlasov-Maxwell** (plasma): Lorentz force \( F = E + v \times B \) with the electromagnetic fields \( E, B \) solving **Maxwell** (generated by charge and current density).
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Then, Vlasov \( f(t, x, v) = f_0(x_{-t}, v_{-t}) \).
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Collisionless kinetic theory
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• Vlasov-Maxwell: Outstanding open problem! Glassey Strauss’s theorem.
Several mathematical issues: $\partial_t f = \{H, f\}$ (focus on previous examples).
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- Infinitely many conserved quantities, and infinitely many equilibria:

\[
\frac{D}{dt} \varphi(f) = 0, \quad f^* = \varphi(H(x, p))
\]

for arbitrary $\varphi(\cdot)$. Which one is dynamically stable? and shouldn’t it be just Gaussian? Maxwell-Boltzmann distribution (next section)?
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for arbitrary \( \varphi(\cdot) \). Which one is dynamically stable? and shouldn’t it be just Gaussian? Maxwell-Boltzmann distribution (next section)?

- Hence, large time dynamics and stability of equilibria are very delicate!
Collisionless kinetic theory

VP: \[ \partial_t f + \bf{v} \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0, \quad -\Delta \phi = \rho - 1. \]

- For instance, homogenous equilibria

\[ \mu = \mu(e), \quad e := \frac{1}{2}|\bf{v}|^2, \quad \phi_0 = 0. \]
Collisionless kinetic theory

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- Arnold: Look at casimir’s invariant

\[ \mathcal{A}[f] = \int \int \left[ \frac{1}{2} |v|^2 f + \varphi(f) \right] dvdx + \frac{1}{2} \int |\nabla \phi|^2 dx. \]
Collisionless kinetic theory

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- \( \mu \) is a critical point iff \( \varphi'(\mu) = -\frac{1}{2}|v|^2 \) or \( \varphi' = -\mu^{-1} \). Convexity of \( \mathcal{A}[f] \) requires

\[ \mu_e < 0. \]
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- \( \mu_e < 0 \) implies Arnold’s nonlinear stability. But, what’s long-time dynamics?
Collisionless kinetic theory

\[ \mu(v) \]

\[ v_{\text{min}} \]

\[ v \]

**Figure:** Stable vs unstable equilibria. Beautiful Penrose's iff criteria:

Linearly unstable iff 
\[ v_{\text{min}} \text{ exists and } \int_{\mathbb{R}} \mu'(v) \, dv > 0. \]
Collisionless kinetic theory

Figure: Stable vs unstable equilibria. Beautiful Penrose’s iff criteria: linearly unstable iff $v_{\text{min}}$ exists and

$$\text{P.V. } \int_{\mathbb{R}} \frac{\mu'(v) dv}{v - v_{\text{min}}} > 0.$$  

S2: Presentation on Penrose’s criteria?

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• Mouhot-Villani: \textbf{nonlinear} Landau damping of Penrose stable homogenous equilibria of VP (earning Villani a Fields medal).
• \textbf{S3}: presentation on linearized case?

\[ f^* = \varphi(H(x,p)). \]
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• \textbf{Open:} Are there stable inhomogenous equilibria (VP)? \( f_* = \varphi(H(x, p)). \)
• Mouhot-Villani: **nonlinear** Landau damping of Penrose stable homogenous equilibria of VP (earning Villani a Fields medal).

• **S3**: presentation on linearized case?

• **Open**: Are there stable inhomogenous equilibria (VP)? $f_* = \varphi(H(x, p))$.

• Some of my works (available on my homepage):
  - Stability of inhomogenous equilibria of VM (w/ Strauss).
  - Various asymptotic limits of Vlasov (w/ Han-Kwan).
  - Non-relativistic limits of VM (w/ Han-Kwan and Rousset).
II. Collisional Kinetic Theory
Collisional kinetic theory

\[ v' \ast v = 0 \]

\[ v' + v = v' + v' \ast \]

\[ |v|^2 + |v'\ast|^2 = |v'|^2 + |v'\ast|^2 \]

**Figure:** Elastic collisions: momentum and energy are exchanged between particles, however conserved, after collision:
Collisional kinetic theory

\[ v_{\ast} = 0 \]

\[ v + v_{\ast} = v' + v'_{\ast} \]

\[ |v|^2 + |v_{\ast}|^2 = |v'|^2 + |v'_{\ast}|^2 \]

Sphere \( S_{vv_{\ast}} \):
One has the **Boltzmann equation:**

\[
\partial_t f - \{ H, f \} = Q[f]
\]

in which collision operator \( Q[f] \):

\[
Q[f](x, v) = \int \int \int \omega(v, v') |v' - v| dv^* dv'' dv''^*
\]

with \( f_2(x, v, v^*) \) the probability distribution of finding two particles with respective momenta.

- **Molecular chaos assumption:**
  
  \( f_2(x, v, v^*) = f(x, v) f(x, v^*) \)

  asserting velocities are uncorrelated before the collision (unlike Hamiltonian, here it introduces a notion of time arrow!).
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Q[f](x, v) = \int \int \int_{\mathbb{R}^9} \omega(v, v_* | v', v'_*) \left[ f_2(x, v', v'_*) - f_2(x, v, v_*) \right] dv_* dv' dv'_*
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Collisional kinetic theory

- Conservation laws: $v', v' \in S_{vv^*}$ and hence

$$dv' dv^* = d\sigma(S_{vv^*}) = |v - v^*| d\sigma(n), \quad n \in S_2.$$

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• (symmetric) **Scattering**: Elastic hard-sphere collisions:

\[
\omega(v, v_*|v', v'_*) = \frac{|n \cdot (v - v_*)|}{|v - v_*|} \quad \text{(angle of collision)}
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• The Boltzmann equation (1872) for hard-sphere gases:

\[
\partial_t f - \{ H, f \} = \int_{\mathbb{R}^3} \int_{S_2} |n \cdot (v - v^*_*)| \left[ f' f^*_* - ff^*_* \right] dndv^*_*.
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Boltzmann’s H-Theorem: Look at the entropy:

\[ \mathcal{H}(t) = \int \int_{\mathbb{R}^6} f \log f \, dx \, dv. \]
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• One has

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\]

\[
= \frac{1}{4} \int \int_{\mathbb{R}^3} \int_{S^2} |n \cdot (v - v_*)| \left[ f' f'_* - ff_* \right] \log \frac{ff_*}{f'f'_*} \, dndxdvdv_*
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\leq 0.
\]

• Entropy is decreasing ! (unlike Hamiltonian, there is a time arrow!).
• **Boltzmann’s H-Theorem:** There is more: equilibria iff

\[ ff_\ast = f' f'_\ast, \quad v', v'_\ast \in S_{vv'} \]

and so iff (presentation?):
- **Boltzmann’s H-Theorem**: There is more: equilibria iff

\[ ff_* = f' f'_*, \quad v', v'_* \in S_{vv_*} \]

and so iff (presentation?): Maxwellian distribution

\[
f(x, v) = C e^{-\beta |v-u|^2} = \frac{\rho(x)}{(2\pi \theta(x))^{3/2}} e^{-\frac{|v-u(x)|^2}{2\theta(x)}}
\]

in which \((\rho, u, \theta)\) denotes macroscopic density, velocity, and temperature of gases.
• Hydrodynamics limits (if time permits).
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• Mathematical difficulty of the Cauchy problem and convergence to equilibrium.

• Some of my works (available on my homepage):
  • On justification of Maxwell-Boltzmann distribution (w/ Bardos, Golse, Sentis)
  • on Quantum Boltzmann, interaction between fermions, bosons, and Bose-Einstein condenstate (w/ Tran)