Kinetic theory of gases

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Fall 2017, I teach a graduate topics course: same topics !

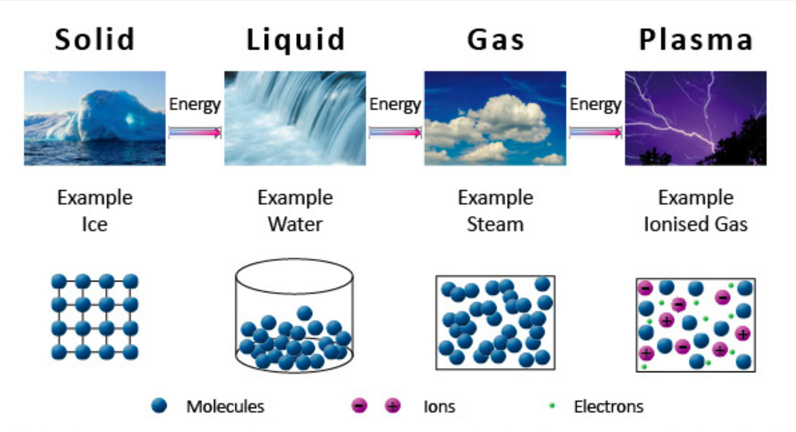
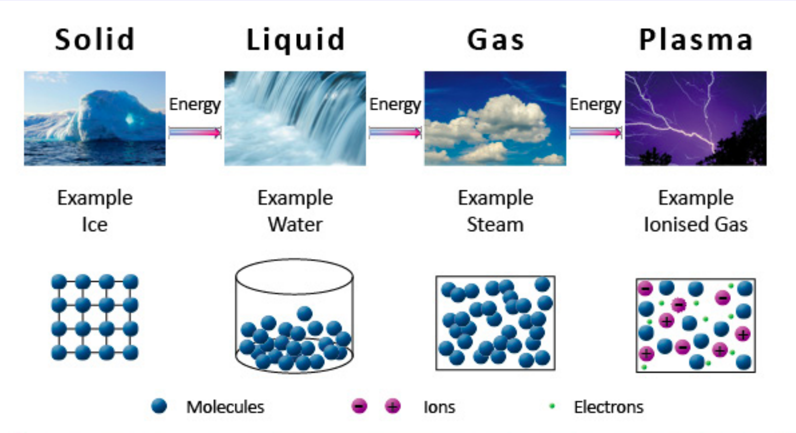


Figure : 4 states of matter, plus Bose-Einstein condensate!



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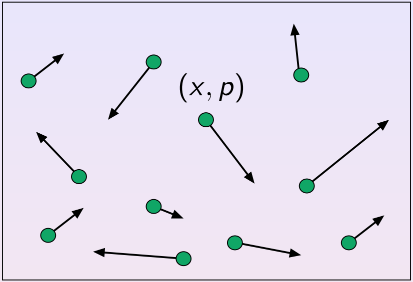
- James Clerk Maxwell, in 1859, gave birth to kinetic theory: use the statistical approach to describe the dynamics of a (rarefied) gas.



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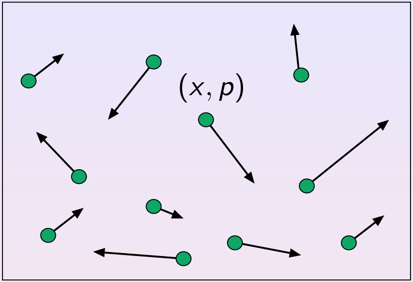

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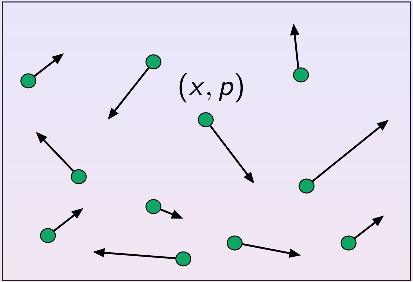

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- Gas molecules are identical.
- One-particle phase space: position $x \in \Omega \subset \mathbb{R}^d$, momentum $p \in \mathbb{R}^d$
- $f(t, x, p)$ the probability density distribution. Mass: $f(t, x, p)\, \mathrm{d}x\mathrm{d}p$.
- Interest: the dynamics of $f(t, x, p)$? Non-equilibrium theory.

I. Collisionless Kinetic Theory

Hamilton 1830's classical mechanics[1]: one-particle Hamiltonian $H(x,p)$ (induced from Lagrangian):

$$\dot{x} = \nabla_p H, \qquad \dot{p} = -\nabla_x H$$

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- Particles in a potential:

$$H(x,p) = \frac{1}{2m}|p|^2 + V_{pot}(x)$$

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- Force acting on particles:

$$F = -\nabla_x H = E + \frac{q}{m}(p \times B)$$

which is the Lorentz force, with $E = -\nabla_x \phi$ and $B = \nabla_x \times A$.

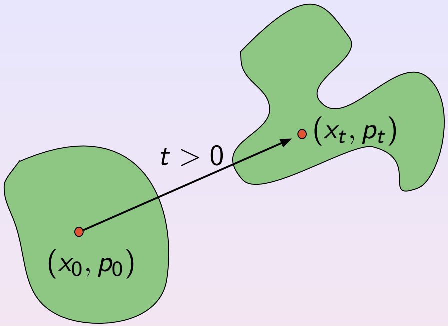
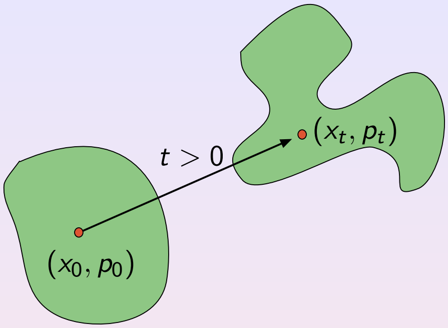


Figure : Liouville's Theorem: volume in phase space remains constant (Exercise: prove this).



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This yields the kinetic equation: $\partial_t f = \{H, f\}$ (Poisson bracket), explicitly

$$0 = \frac{d}{dt} f(t, x(t), p(t)) = \partial_t f + \nabla_p H \cdot \nabla_x f - \nabla_x H \cdot \nabla_p f$$
$$= \partial_t f + v \cdot \nabla_x f + F_{\rm force} \cdot \nabla_v f$$

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$$F = -\nabla_x \phi, \qquad -\Delta_x \phi = \sigma \int_{\mathbb{R}^3} f(t,x,v)\, dv,$$

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- Vlasov-Maxwell (plasma): Lorentz force $F = E + v \times B$ with the electromagnetic fields E, B solving Maxwell (generated by charge and current density).

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- Vlasov-Maxwell: Outstanding open problem! Glassey Strauss's theorem.

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$$\frac{D}{dt}\varphi(f) = 0, \qquad f_* = \varphi(H(x,p))$$

for arbitrary $\varphi(\cdot)$. Which one is dynamically stable ? and shouldn't it be just Gaussian ? Maxwell-Boltzmann distribution (next section) ?

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- Hence, large time dynamics and stability of equilibria are very delicate!

VP: $$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0, \qquad -\Delta \phi = \rho - 1.$$

- For instance, homogenous equilibria

$$\mu = \mu(e), \qquad e := \frac{1}{2}|v|^2, \qquad \phi_0 = 0.$$

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- Arnold: Look at casimir's invariant

$$\mathcal{A}[f] = \iint \Big[\frac{1}{2}|v|^2 f + \varphi(f)\Big]\, dvdx + \frac{1}{2}\int |\nabla \phi|^2\, dx.$$

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- $\mu_e < 0$ implies Arnold's nonlinear stability. But, what's long-time dynamics?

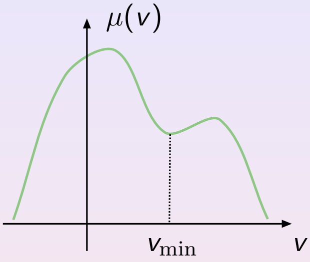

Figure : Stable vs unstable equilibria. Beautiful Penrose's iff criteria:

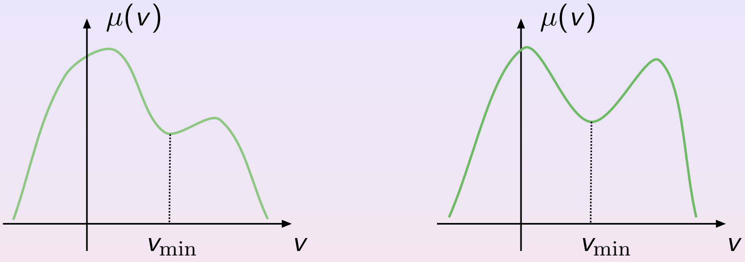

Figure : Stable vs unstable equilibria. Beautiful Penrose's iff criteria: linearly unstable iff $v_{\min}$ exists and

$$\mathrm{P.V.} \int_{\mathbb{R}} \frac{\mu'(v)dv}{v - v_{\min}} > 0.$$

S2: Presentation on Penrose's criteria?

• Mouhot-Villani: nonlinear Landau damping of Penrose stable homogenous equilibria of VP (earning Villani a Fields medal).
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• Some of my works (available on my homepage):

- Stability of inhomogenous equilibria of VM (w/ Strauss).
- Various asymptotic limits of Vlasov (w/ Han-Kwan).
- Non-relativistic limits of VM (w/ Han-Kwan and Rousset).

II. Collisional Kinetic Theory

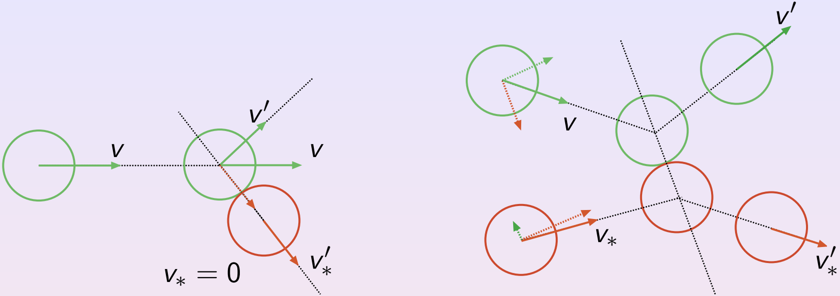


Figure : Elastic collisions: momentum and energy are exchanged between particles, however conserved, after collision:

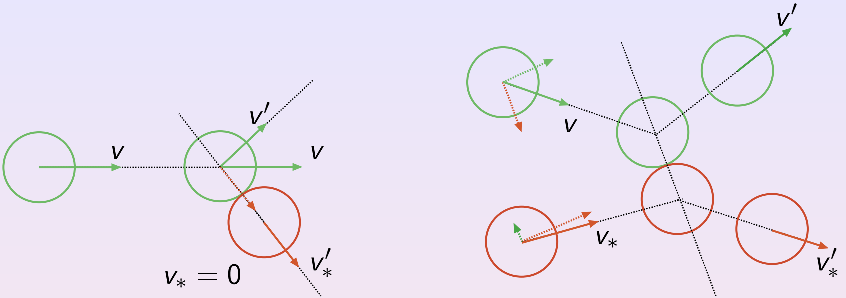
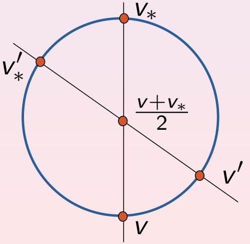


Figure : Elastic collisions: momentum and energy are exchanged between particles, however conserved, after collision:

- $v + v_* = v' + v'_*$
- $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$

Sphere S_{vv_*} :



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$$Q[f](x,v) = \iiint_{\mathbb{R}^9} \underbrace{\omega(v, v_*|v', v'_*)}_{Scattering} \Big[\underbrace{f_2(x, v', v'_*)}_{Gain} - \underbrace{f_2(x, v, v_*)}_{Loss} \Big] dv_* dv' dv'_*$$

with $f_2(x, v, v_*)$ the probability distribution of finding two particles with respective momenta.

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- Molecular chaos assumption:

$$f_2(x, v, v_*) = f(x, v) f(x, v_*)$$

asserting velocities are uncorrelated before the collision (unlike Hamiltonian, here it introduces a notion of time arrow!).

- Conservation laws: $v', v'_* \in S_{vv_*}$ and hence

$$dv' dv'_* = d\sigma(S_{vv_*}) = |v - v_*| d\sigma(n), \qquad n \in S_2.$$

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$$\omega(v, v_*|v', v'_*) = \frac{|n \cdot (v - v_*)|}{|v - v_*|} \qquad \text{(angle of collision)}$$

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- The Boltzmann equation (1872) for hard-sphere gases:

$$\partial_t f - \{H, f\} = \int_{\mathbb{R}^3} \int_{S_2} |n \cdot (v - v_*)| \Big[f' f'_* - f f_* \Big] dndv_*.$$

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Is it solvable ? Di Perna-Lions '89, Guo 2010. There remain many outstanding open problems.

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- Entropy is decreasing ! (unlike Hamiltonian, there is a time arrow!).

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and so iff (presentation?): Maxwellian distribution

$$f(x,v) = Ce^{-\beta|v-u|^2} = \frac{\rho(x)}{(2\pi\theta(x))^{3/2}} e^{-\frac{|v-u(x)|^2}{2\theta(x)}}$$

in which (ρ, u, θ) denotes macroscopic density, velocity, and temperature of gases.

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- Some of my works (available on my homepage):
  - On justification of Maxwell-Boltzmann distribution (w/ Bardos, Golse, Sentis)
  - on Quantum Boltzmann, interaction between fermions, bosons, and Bose-Einstein condenstate (w/ Tran)