Boundary layers

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Flows past a flat plate

Figure: source internet.
Flows around an airfoil

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Hydrodynamics stability

- Helmholtz (1868), Kelvin (1871), Rayleigh (1879) on inviscid flows
- Reynolds (1883) on role of viscosity (e.g., flows in a pipe):

\[ Re = \frac{UL}{\nu}. \]
Hydrodynamics stability

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  \[ Re = \frac{UL}{\nu}. \]

- Orr (1907), Sommerfeld (1908), Heisenberg (1924), Tollmien (1929), Schlichting (1933), C. C. Lin (1940s):

  Viscosity may destabilize the flows!
All shear flows are unstable at large Reynolds numbers!

\[ \alpha^2 \]

\[ \alpha_{\text{low}} \approx Re^{-1/4} \]

\[ \alpha_{\text{up}} \approx Re^{-1/10} \]

Inviscid limit problem

- For \( \nu > 0 \), Navier-Stokes equations

\[
\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \\
\nabla \cdot u = 0 \\
\left. u \right|_{\partial \Omega} = 0.
\]
Inviscid limit problem

• For $\nu > 0$, Navier-Stokes equations

\[ \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \]
\[ \nabla \cdot u = 0 \]
\[ u \mid_{\partial \Omega} = 0. \]

• At $\nu = 0$, Euler equations

\[ \partial_t u + u \cdot \nabla u + \nabla p = 0 \]
\[ \nabla \cdot u = 0 \]
\[ u \cdot n \mid_{\partial \Omega} = 0. \]
As $\nu \to 0$ and $\nu = 0$:

- The nature of equations changes: “Navier-Stokes to Euler”.

Boundary conditions change: $u_\nu(x, y) = 0$ to $u_{\text{Euler}}(x, y) = 0$ on $\partial \Omega$ (ideal fluids may “slip” on the boundary: $u_{\text{Euler}}(x, y) \neq 0$ on $\partial \Omega$). Boundary layers appear...
As $\nu \to 0$ and $\nu = 0$:

- The nature of equations changes: “Navier-Stokes to Euler”.
- Boundary conditions change: “$u^\nu = 0$ to $u_n^{\text{Euler}} = 0$ on $\partial \Omega$” (ideal fluids may “slip” on the boundary: $u_T^{\text{Euler}} \neq 0$ on $\partial \Omega$).
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As $\nu \to 0$ and $\nu = 0$:

- The nature of equations changes: “Navier-Stokes to Euler”.

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- Boundary layers appear.....Prandtl’s 1904 Ansatz:

$$u^n = u^{\text{Euler}}(t, x, y) + u^{\text{BL}}(t, x, \frac{y}{\sqrt{\nu}}) + o(1)_{L^\infty}$$

(due to the boundary discrepancy and balancing $u \cdot \nabla u \approx \nu \Delta u$)
• More precisely, near the boundary,

\[ u^\nu \approx \left( \frac{u_1}{\sqrt{\nu} u_2} \right) (t, x, \frac{y}{\sqrt{\nu}}) \]

(pertaining the divergence free condition: \( \partial_x u_1 + \partial_y u_2 = 0 \))
• More precisely, near the boundary,

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• Prandtl's famous equation (in blue!):

\[
\begin{align*}
(\partial_t + u \cdot \nabla)u_1 + \partial_x p &= \partial_Y^2 u_1 + \nu \partial_x^2 u_1 \\
\nu(\partial_t + u \cdot \nabla)u_2 + \partial_Y p &= 0 + \nu(\partial_Y^2 + \nu \partial_x^2)u_2
\end{align*}
\]

\[
u_1|_{Y=0} = u_2|_{Y=0} = 0
\]

\[
\lim_{Y \to \infty} u_1 = u_{\text{Euler}}(t, x, 0)
\]
• More precisely, near the boundary,

\[ u^\nu \approx \left( \frac{u_1}{\sqrt{\nu} u_2} \right) (t, x, \frac{y}{\sqrt{\nu}}) \]

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• Prandtl’s famous equation (in blue!):

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\nu (\partial_t + u \cdot \nabla) u_2 + \partial_y p &= 0 + \nu (\partial_y^2 + \nu \partial_x^2) u_2 \\
\end{align*}
\]

\[ u_1|_{Y=0} = u_2|_{Y=0} = 0 \]

\[ \lim_{Y \to \infty} u_1 = u_{Euler}^{\text{Euler}}(t, x, 0) \]

• Great advantage: pressure is known! Blasius self-similar solutions! Reliable calculation of the drag (airplane works!). Boundary layer separation. Plus, many mathematical works.
Validity of the Prandtl's Analysis: $\nu \to 0$?

\[ u^\nu = u^{\text{Euler}}(t, x, y) + u^{\text{BL}}(t, x, \frac{y}{\sqrt{\nu}}) + o(1)_{L^\infty} \]
Validity of the Prandtl's Analysis: $\nu \to 0$?

\[ u^\nu = u^{\text{Euler}}(t, x, y) + u^{\text{BL}}(t, x, \frac{y}{\sqrt{\nu}}) + o(1)_{L^\infty} \]

- Sammartino-Caflisch '98 for analytic data
- Maekawa '14 for compactly supported data
- Gérard-Varet-Masmoudi-Maekawa '16, '20, for Gevrey data.
For Sobolev data ......
For Sobolev data ...... Prandtl’s Ansatz is FALSE!
Theorem (Grenier-Toan: The failure of Prandtl’s Ansatz)

- There are Prandtl’s layers $u^{\text{Prandtl}}$ so that
  \[ \| (u^\nu - u^{\text{Prandtl}})(t^\nu) \|_{L^\infty} \gtrsim 1 \]

at time $t^\nu \sim \sqrt{\nu} \log \frac{1}{\nu} \to 0$. 
Theorem (Grenier-Toan: The failure of Prandtl’s Ansatz)

- There are Prandtl’s layers $u^{\text{Prandtl}}$ so that
  \[ \| (u^{\nu} - u^{\text{Prandtl}})(t^{\nu}) \|_{L^\infty} \gtrsim 1 \]
  at time $t^{\nu} \sim \sqrt{\nu} \log \frac{1}{\nu} \to 0$.
- Even more: for “generic” stable Prandtl’s layers:
  \[ \| (u^{\nu} - u^{\text{Prandtl}})(t^{\nu}) \|_{L^\infty} \gtrsim \nu^{1/4} + \epsilon \]
  $t^{\nu} \sim \nu^{1/4} \log \frac{1}{\nu} \to 0$. 

• Precisely, Prandtl’s profiles

\[
\begin{align*}
    u^{\text{Euler}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
    u^{\text{Prandtl}} &= \begin{pmatrix} U(t, y / \sqrt{\nu}) \\ 0 \end{pmatrix}
\end{align*}
\]

where \( U(t, z) \) solves the heat equation

\[
\partial_t U = \partial_z^2 U
\]

\( U \big|_{z=0} = 0, \quad \lim_{z \to \infty} U(t, z) = 1. \)
Precisely, Prandtl’s profiles

\[ u^{\text{Euler}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u^{\text{Prandtl}} = \begin{pmatrix} U(t, y/\sqrt{\nu}) \\ 0 \end{pmatrix} \]

where \( U(t, z) \) solves the heat equation

\[ \partial_t U = \partial_z^2 U \]

\[ U|_{z=0} = 0, \quad \lim_{z \to \infty} U(t, z) = 1. \]

Navier-Stokes near boundary layers

\[ \omega_t + U\omega_x + u_2 U'' + u \cdot \nabla \omega = \nu \Delta \omega \]

together with the no-slip condition on \( u = \nabla^\perp \Delta^{-1} \omega \).
The failure of Prandtl’s Ansatz

\[ u^{\text{Euler}} \]

Figure: Sketched is a shape of \( U(\cdot) \)
The failure of Prandtl’s Ansatz

Figure: Sketched is a shape of $U(\cdot)$

- Within the hyperbolic scaling of size $\sqrt{\nu}$:
The failure of Prandtl’s Ansatz

Within the hyperbolic scaling of size $\sqrt{\nu}$:

- Euler around $U(y_1)$ is either spectrally stable or unstable:

$$u^{e,1} = e^{\lambda_0 t_1} \hat{u}^{e,1}, \quad \hat{u}^{e,1} = \nabla_\perp (e^{i\alpha x_1} \phi(y_1)),$$

with $\phi$ solving the Rayleigh equation.

Figure: Sketched is a shape of $U(\cdot)$. 

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Boundary layers
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  - Start with an unstable mode of Euler: $u^{e,1} = e^{\lambda_0 t_1} \hat{u}^{e,1}$, $\Re \lambda_0 > 0$. 

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  • (local) Reynolds number: $\text{Re}_1 = \frac{UL}{\nu} = \frac{1}{\sqrt{\nu}} \to \infty$. 

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• Within the hyperbolic scaling of size $\sqrt{\nu}$:
  • Start with an unstable mode of Euler: $u^{e,1} = e^{\lambda_0 t_1} \hat{u}^{e,1}$, $\Re \lambda_0 > 0$.
  • (local) Reynolds number: $\text{Re}_1 = \frac{UL}{\nu} = \frac{1}{\sqrt{\nu}} \rightarrow \infty$.
  • Navier-Stokes around $U(y_1)$ is also unstable (linearly):
    
    $$u^{\text{NS},1} = \nu^N e^{\lambda_\nu t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{\text{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \right], \quad \Re \lambda_\nu \rightarrow \Re \lambda_0 > 0.$$ 

    ($\hat{u}^{\text{BL},1}$ - the viscous boundary sublayer to correct the no-slip condition, computing $\nu^{1/4} = \sqrt[4]{\nu}$ and $y_1/\nu^{1/4} = y/\nu^{3/4}$).
Within the hyperbolic scaling of size \( \sqrt{\nu} \):

- Start with an unstable mode of Euler: \( u^{e,1} = e^{\lambda_0 t_1} \hat{u}^{e,1} \), \( \Re \lambda_0 > 0 \).

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\]

(\( \hat{u}^{BL,1} \) - the viscous boundary sublayer to correct the no-slip condition, computing \( \nu^{1/4} = \sqrt{\sqrt{\nu}} \) and \( y_1/\nu^{1/4} = y/\nu^{3/4} \)).

- \( u^{NS,1} \sim \nu^N e^{\lambda_\nu t_1} \), extremely large at \( t = \sqrt{\nu} t_1 \sim \sqrt{\nu} \log \frac{1}{\nu} \to 0 \).
• Linear to nonlinear: stability of the instability

\[ u^\nu \approx u^{\text{Prandtl}}(y_1) + \nu^N e^{\lambda_\nu t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{\text{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \right] \]
• **Linear to nonlinear: stability of the instability**

\[
\begin{align*}
u' \approx & \quad u^{\text{Prandtl}}(y_1) + \nu N e^{\lambda \nu t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{\text{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \right] ? \\
\end{align*}
\]

• **YES, as long as**

\[
\| \nabla u^{\text{NS, app}} \|_{L^\infty} \lesssim 1 + \nu N e^{\Re \lambda_0 t_1} \left[ 1 + \nu^{-1/4} \right] \lesssim 1.
\]
• Linear to nonlinear: stability of the instability

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• Grenier, 2000: The instability of \( u^{\text{Prandtl}} \) is of order \( \nu^{1/4} \):

\[ \| u^{\text{NS}} - u^{\text{Prandtl}} \|_{L^\infty} \gtrsim \nu^N e^{\Re \lambda_0 t_\nu} \sim \nu^{1/4}, \quad t_\nu \sim \sqrt{\nu \log \frac{1}{\nu}} \to 0 \]

(the sublayer velocity grows up to order \( \nu^{1/4} \) in amplitude).
The failure of Prandtl’s Ansatz

- Going beyond the $\nu^{1/4}$-instability?

\[ u^\text{Euler} \]

\[ y \quad x \]

$\nu^{1/2}$: Prandtl layer
$\nu^{3/4}$: Viscous sublayer

\[ \nu^\frac{1}{4}: \text{instability?} \]
The failure of Prandtl’s Ansatz

Going beyond the $\nu^{1/4}$-instability?

(local) Reynolds number within sublayer: $Re_2 = \frac{UL}{\nu} = \frac{U_{\text{sub}}}{\nu^{1/4}} \to \infty$. 

- $\nu^{1/2}$: Prandtl layer
- $\nu^{3/4}$: Viscous sublayer
The failure of Prandtl’s Ansatz

Going beyond the $\nu^{1/4}$-instability?

- (local) Reynolds number within sublayer: $Re_2 = \frac{UL}{\nu} = \frac{U_{\text{sub}}}{\nu^{1/4}} \to \infty$.
- ALL shear flows, including stable ones to Euler, are linearly unstable to Navier-Stokes: viscous destabilization, Heisenberg ’24, C. C. Lin, Tollmien, Schlichting 40s (rigorously proved in Grenier-Guo-Toan ’16).
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- (local) Reynolds number within sublayer: $Re_2 = \frac{UL}{\nu} = \frac{U_{\text{sub}}}{\nu^{1/4}} \to \infty$.
- ALL shear flows, including stable ones to Euler, are linearly unstable to Navier-Stokes: viscous destabilization, Heisenberg ’24, C. C. Lin, Tollmien, Schlichting 40s (rigorously proved in Grenier-Guo-Toan ’16).
- Sublayers themselves are unstable! Many instabilities.....!
So, it seems hopeless for stability of the instability.....except:

\[ \nu^{1/2}: \text{Prandtl layer} \]
\[ \nu^{3/4}: \text{Viscous sublayer} \]
\[ \nu^{7/8}: \text{Viscous subsublayer} \]
So, it seems hopeless for stability of the instability.....except:

- Sammartino-Caflisch ’98: **analyticity** prevents sublayers. Navier-Stokes solutions involve precisely \( \text{Euler} + \text{Prandtl} \).
So, it seems hopeless for stability of the instability.....except:

- **Program**: use analyticity to prevent subsublayers. Navier-Stokes solutions involve precisely Euler + Prandtl + Sublayer.
The failure of Prandtl’s Ansatz

\[ u^{\nu} \approx u^{\text{Prandtl}}(y_1) + \nu^N e^{\lambda \nu t_1} \left[ \hat{u}^{\text{e},1}(x_1, y_1) + \hat{u}^{\text{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \right] \]

- Issue:
The failure of Prandtl’s Ansatz

\[ u^\nu \approx u^{\text{Prandtl}}(y_1) + \nu^N e^{\lambda t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{BL,1}(x_1, \frac{y_1}{\nu^{1/4}}) \right] \]

- Issue:
  - Large time analyticity framework: \( t_1 \sim \log \frac{1}{\nu} \to \infty \).
The failure of Prandtl's Ansatz

\[ u^{\nu} \approx u^{\text{Prandtl}}(y_1) + \nu^N e^{\lambda_\nu t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{\text{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \right] \]

- **Issue:**
  - Large time analyticity framework: \( t_1 \sim \log \frac{1}{\nu} \to \infty \).
  - Due to shearing, analyticity radius shrinks like \( \frac{1}{t_1} \to 0 \):

\[
\omega_t + U(y)\omega_x + v_2 U'' = \sqrt{\nu} \Delta \omega, \quad v_2 = -\partial_x \Delta^{-1} \omega
\]

(Analyticity is lost when needed!)
However,

**Theorem (Grenier-Toan: Annals of PDEs 2019)**

There is a complete instability solution

\[ u_\nu = u_{\text{Prandtl}}(y_1) + \sum_{j=1}^{\infty} \nu_j N [u_{e,j}(t_1, x_1, y_1) + u_{BL,j}(t_1, x_1, y_1)] \]

(noticing the exponential growth in time!),

which solves Navier-Stokes equations with forcing

\[ f_\nu \]

until the instability time:

\[ \nu_N e^{\Re \lambda_0 t_1} \sim 1, \]

where

\[ \| u_\nu - u_{\text{Prandtl}} \|_{H^s} + \| f_\nu \|_{H^s} \leq \nu_N. \]
However,

Theorem (Grenier-Toan: Annals of PDEs 2019)

There is a complete instability solution $u^\nu = \text{Euler} + \text{Prandtl} + \text{Sublayer}$:

$$u^\nu = u^{\text{Prandtl}}(y_1) + \sum_{j=1}^{\infty} \nu^j \left[u^{\text{e}}(t_1, x_1, y_1) + u^{\text{BL}}(t_1, x_1, \frac{y_1}{\nu^{1/4}})\right]$$

(notating the exponential growth in time!),
However,

**Theorem (Grenier-Toan: Annals of PDEs 2019)**

*There is a complete instability solution \( u^\nu = \text{Euler} + \text{Prandtl} + \text{Sublayer}:*

\[
    u^\nu = u^{\text{Prandtl}}(y_1) + \sum_{j=1}^{\infty} \nu^j \left[ u^{e,j}(t_1, x_1, y_1) + u^{BL,j}(t_1, x_1, \frac{y_1}{\nu^{1/4}}) \right]
\]

(notating the exponential growth in time!), which solves Navier-Stokes equations with forcing \( f^\nu \), until the instability time:

\[
    \nu^N e^{Re\lambda_0 t_1} \sim 1,
\]

where

\[
    \|(u^\nu - u^{\text{Prandtl}})(0)\|_{H^s} + \|f^\nu\|_{H^s} \leq \nu^N.
\]
Some pains.....

\[ u^\nu = u^{\text{Prandtl}}(y_1) + \sum_{j=1}^{\infty} \nu^j u^j(t_1, x_1, y_1) \]
Some pains.....

\[ u^{\nu} = u^{\text{Prandtl}}(y_1) + \sum_{j=1}^{\infty} \nu^j N^j u^j(t_1, x_1, y_1) \]

with vorticity \( \omega^j \) iteratively solves:

\[ (\partial_t + U \partial_x) \omega^j + u_2^j U'' - \sqrt{\nu} \Delta \omega^j = - \sum_{k+\ell=j} u^k \cdot \nabla \omega^\ell \]

together with the no-slip condition on \( u^j = \nabla^\perp \Delta^{-1} \omega^j \).
Generator functions

\[ \text{Gen}_\delta (\omega)(z) = \sum_{\alpha \in \mathbb{Z}} \sum_{\ell \geq 0} e^{z |\alpha|} \| D_y^{\ell} \hat{\omega}_\alpha \|_\delta \frac{z^\ell}{\ell!}, \quad D_y = \frac{y}{1 + y \, \partial_y} \]

where

\[ \| \hat{\omega}_\alpha \|_\delta = \sup_y |\hat{\omega}_\alpha(y)| \left( \delta^{-1} e^{-y/\delta} + 1 \right)^{-1}, \quad \delta = \nu^{1/4}. \]
Generator functions

\[ \text{Gen}_\delta(\omega)(z) = \sum_{\alpha \in \mathbb{Z}} \sum_{\ell \geq 0} e^{z|\alpha|} \| D_y^\ell \hat{\omega}_\alpha \|_\delta \frac{z^\ell}{\ell!}, \quad D_y = \frac{y}{1 + y} \partial_y \]

where

\[ \| \hat{\omega}_\alpha \|_\delta = \sup_y |\hat{\omega}_\alpha(y)| \left( \delta^{-1} e^{-y/\delta} + 1 \right)^{-1}, \quad \delta = \nu^{1/4}. \]

- Good properties with products and derivatives:

\[ \text{Gen}_\delta(fg) \leq \text{Gen}_0(f) \text{Gen}_\delta(g), \]

\[ \text{Gen}_\delta(\partial_x \omega) = \partial_z \text{Gen}_\delta(\omega), \quad \text{Gen}_\delta(D_y \omega) = \partial_z \text{Gen}_\delta(\omega) \]

\[ \text{Gen}_\delta(u \cdot \nabla \omega) \lesssim \partial_z \text{Gen}_0(u) \partial_z \text{Gen}_\delta(\omega) \]
Large time Cauchy-Kovalevskaya theory

- Derive a simple transport inequality for vorticity:

\[ \partial_{\tau} \text{Gen} \delta(\omega)(z) \lesssim \text{Gen} \delta(\omega)(z) + \text{Gen} \delta(\omega) \partial_z \text{Gen} \delta(\omega)(z) \]

with \( \tau = \nu^N e^{\Re \lambda_0 t_1} \), giving solution up to a time of order one:

\[ u^\nu \approx u^{\text{Prandtl}}(y_1) + \nu^N e^{\lambda \nu t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{\text{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \right]. \]
The failure of Prandtl’s Ansatz

Large time Cauchy-Kovalevskaya theory

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\[ \partial_\tau \text{Gen}_\delta(\omega)(z) \lesssim \text{Gen}_\delta(\omega)(z) + \text{Gen}_\delta(\omega)\partial_z \text{Gen}_\delta(\omega)(z) \]

with \( \tau = \nu^N e^{R\lambda_0 t_1} \), giving solution up to a time of order one:

\[ u^\nu \approx u^\text{Prandtl}(y_1) + \nu^N e^{\lambda_\nu t_1} \left[ \hat{u}^{e,1}(x_1, y_1) + \hat{u}^{BL,1}(x_1, \frac{y_1}{\nu^{1/4}}) \right]. \]

- An application:

A simplification of Mouhot-Villani’s proof of Landau damping

200 pages \(\implies\) an elementary proof (Grenier-Toan-Rodnianski ’20).
Theorem (Grenier-Toan: The failure of Prandtl’s Ansatz)

\[ u^{\text{Euler}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \sqrt{\nu} \]

\[ \| (u^{\nu} - u^{\text{Prandtl}})(0) \|_{H^s} + \| f^{\nu} \|_{L^\infty([0, t^{\nu}], H^s)} \leq \nu N \]

\[ \| (u^{\nu} - u^{\text{Prandtl}})(t^{\nu}) \|_{L^\infty} \gtrsim 1 \]

\[ \| (u^{\nu} - u^{\text{Prandtl}})(t^{\nu}) \|_{L^\infty} \gtrsim \nu^{1/4} + \epsilon \]

\[ t^{\nu} \sim \nu^{1/4} \log \nu - 1. \]
Theorem (Grenier-Toan: The failure of Prandtl’s Ansatz)

Let \( U(\cdot) \) be spectrally \textbf{unstable} or \textbf{monotone and stable} to Euler. Then, for any \( N, s \), there are solutions \( u^\nu \) of Navier Stokes with forcing terms \( f^\nu \) so that

\[
\| (u^\nu - u^{\text{Prandtl}})(0) \|_{H^s} + \| f^\nu \|_{L^\infty([0,t^\nu],H^s)} \leq \nu^N,
\]

\[
\| (u^\nu - u^{\text{Prandtl}})(t^\nu) \|_{L^\infty} \gtrsim 1 \quad \text{with} \quad t^\nu \sim \sqrt{\nu} \log \nu^{-1},
\]

\[
\| (u^\nu - u^{\text{Prandtl}})(t^\nu) \|_{L^\infty} \gtrsim \nu^{1/4+\epsilon} \quad \text{with} \quad t^\nu \sim \nu^{1/4} \log \nu^{-1}.
\]