Boundary layers

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Flows past a flat plate

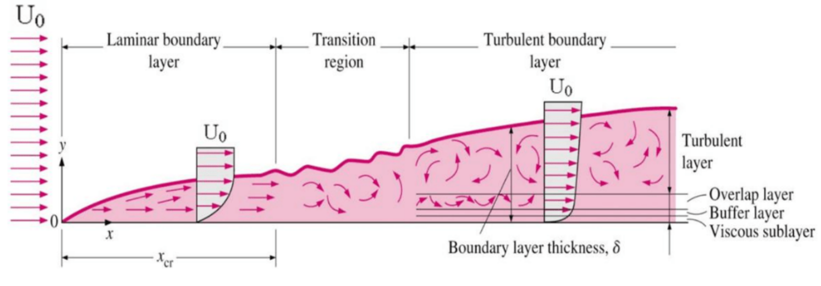
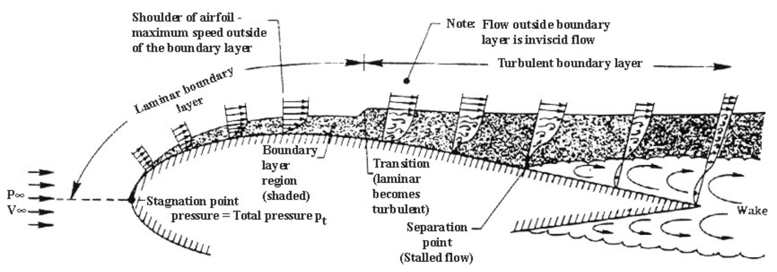


Figure: source internet.

Flows around an airfoil



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Hydrodynamics stability

- Helmholtz (1868), Kelvin (1871), Rayleigh (1879) on inviscid flows
- Reynolds (1883) on role of viscosity (e.g., flows in a pipe):

$$Re = \frac{UL}{\nu}.$$

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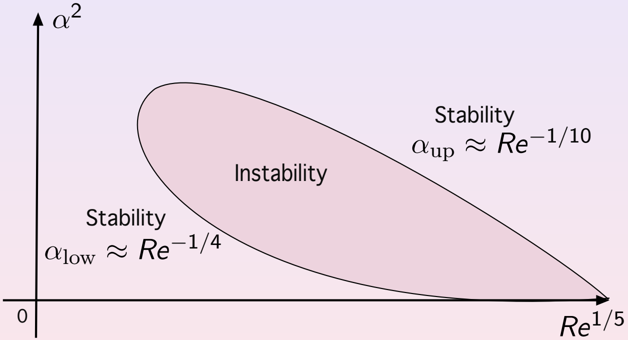
- Orr (1907), Sommerfeld (1908), Heisenberg (1924), Tollmien (1929), Schlichting (1933), C. C. Lin (1940s):

Viscosity may destabilize the flows!

All shear flows are unstable at large Reynolds numbers !



[Grenier-Guo-Toan, Duke Math J. 2016]

Inviscid limit problem

- For $\nu > 0$, Navier-Stokes equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \nu \Delta u \\ \nabla \cdot u &= 0 \\ u_{|\partial\Omega} &= 0. \end{aligned}$$

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- At $\nu = 0$, Euler equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \\ u \cdot n_{|_{\partial\Omega}} &= 0. \end{aligned}$$

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- Boundary layers appear.....Prandtl's 1904 Ansatz:

$$u^\nu = u^{\mathrm{Euler}}(t,x,y) + u^{\mathrm{BL}}(t,x,\frac{y}{\sqrt{\nu}}) + o(1)_{L^\infty}$$

(due to the boundary discrepancy and balancing $u \cdot \nabla u \approx \nu \Delta u$)

- More precisely, near the boundary,

$$u^\nu \approx \begin{pmatrix} u_1 \\ \sqrt{\nu} u_2 \end{pmatrix} (t, x, \frac{y}{\sqrt{\nu}})$$

(pertaining the divergence free condition: $\partial_x u_1 + \partial_Y u_2 = 0$)

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- Prandtl's famous equation (in blue!):

$$\begin{aligned} (\partial_t + u \cdot \nabla) u_1 + \partial_x p &= \partial_Y^2 u_1 + \nu \partial_x^2 u_1 \\ \nu(\partial_t + u \cdot \nabla) u_2 + \partial_Y p &= 0 + \nu(\partial_Y^2 + \nu \partial_x^2) u_2 \\ u_{1|_{Y=0}} = u_{2|_{Y=0}} &= 0 \\ \lim_{Y \to \infty} u_1 &= u^{\mathrm{Euler}}(t, x, 0) \end{aligned}$$

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• Great advantage: pressure is known! Blasius self-similar solutions! Reliable calculation of the drag (airplane works!). Boundary layer separation. Plus, many mathematical works.

Validity of the Prandtl's Analysis: $\nu \to 0$?

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- Sammartino-Caflisch '98 for analytic data
- Maekawa '14 for compactly supported data
- Gérard-Varet-Masmoudi-Maekawa '16, '20, for Gevrey data.

For Sobolev data ......

For Sobolev data ......Prandtl's Ansatz is FALSE !

Theorem (Grenier-Toan: The failure of Prandtl's Ansatz)

- *There are Prandtl's layers* u^{Prandtl} *so that*

$$\|(u^\nu - u^{\mathrm{Prandtl}})(t_\nu)\|_{L^\infty} \gtrsim 1$$

at time $t_\nu \sim \sqrt{\nu}\log\frac{1}{\nu} \to 0.$

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- *Even more: for "generic" stable Prandtl's layers:*

$$\|(u^\nu - u^{\mathrm{Prandtl}})(t_\nu)\|_{L^\infty} \gtrsim \nu^{\frac{1}{4}+\epsilon}$$

$t_\nu \sim \nu^{1/4}\log\frac{1}{\nu} \to 0$.

- Precisely, Prandtl's profiles

$$u^{\text{Euler}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u^{\text{Prandtl}} = \begin{pmatrix} U(t, y/\sqrt{\nu}) \\ 0 \end{pmatrix}$$

where $U(t,z)$ solves the heat equation

$$\partial_t U = \partial_z^2 U$$

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- Navier-Stokes near boundary layers

$$\omega_t + U\omega_x + u_2 U'' + u \cdot \nabla \omega = \nu \Delta \omega$$

together with the no-slip condition on $u = \nabla^\perp \Delta^{-1} \omega$.

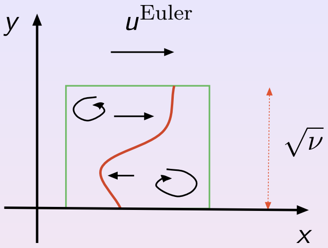
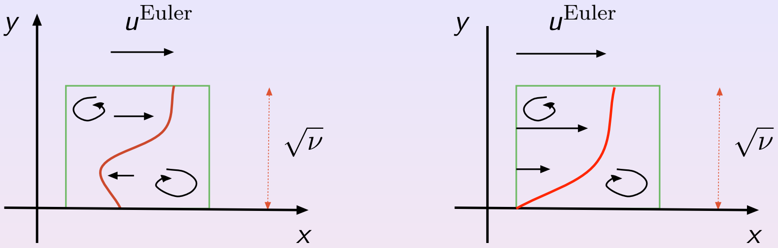


Figure: Sketched is a shape of $U(\cdot)$



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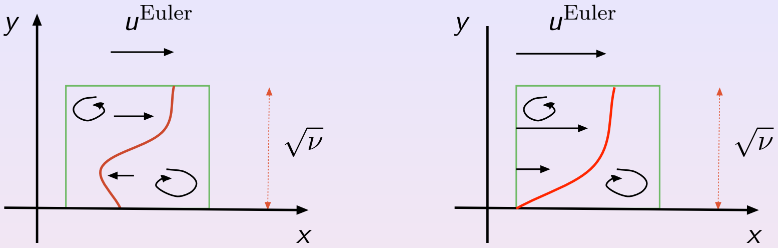

Figure: Sketched is a shape of $U(\cdot)$

- Within the hyperbolic scaling of size $\sqrt{\nu}$:
  - Euler around $U(y_1)$ is either spectrally stable or unstable:

$$u^{e,1} = e^{\lambda_0 t_1} \hat{u}^{e,1}, \qquad \hat{u}^{e,1} = \nabla^\perp (e^{i\alpha x_1} \phi(y_1)),$$

with ϕ solving the Rayleigh equation.

- Within the hyperbolic scaling of size $\sqrt{\nu}$:
  - Start with an unstable mode of Euler: $u^{\mathrm{e},1} = e^{\lambda_0 t_1} \hat{u}^{\mathrm{e},1}$, $\Re\lambda_0 > 0$.

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  - (local) Reynolds number: $\mathrm{Re}_1 = \frac{UL}{\nu} = \frac{1}{\sqrt{\nu}} \to \infty$.
  - Navier-Stokes around $U(y_1)$ is also unstable (linearly):

$$u^{\mathrm{NS},1} = \nu^N e^{\lambda_\nu t_1} \Big[\hat{u}^{\mathrm{e},1}(x_1, y_1) + \hat{u}^{\mathrm{BL},1}(x_1, \frac{y_1}{\nu^{1/4}}) \Big], \quad \Re\lambda_\nu \to \Re\lambda_0 > 0.$$

($\hat{u}^{\mathrm{BL},1}$ - the viscous boundary sublayer to correct the no-slip condition, computing $\nu^{1/4} = \sqrt{\sqrt{\nu}}$ and $y_1/\nu^{1/4} = y/\nu^{3/4}$).

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  - $u^{\mathrm{NS},1} \sim \nu^N e^{\lambda_\nu t_1}$, extremely large at $t = \sqrt{\nu} t_1 \sim \sqrt{\nu} \log \frac{1}{\nu} \to 0$.

- Linear to nonlinear: stability of the instability

$$u^\nu \quad \approx \quad u^{\mathrm{Prandtl}}(y_1) + \nu^N e^{\lambda_\nu t_1}\Big[\hat{u}^{\mathrm{e},1}(x_1, y_1) + \hat{u}^{\mathrm{BL},1}(x_1, \frac{y_1}{\nu^{1/4}})\Big] \quad ?$$

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  - YES, as long as

$$\|\nabla u^{\mathrm{NS,app}}\|_{L^\infty} \quad \lesssim \quad 1 + \nu^N e^{\Re \lambda_0 t_1}\Big[1 + \nu^{-1/4}\Big] \quad \lesssim \quad 1.$$

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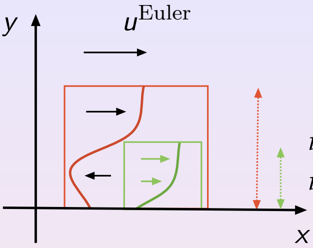
- Grenier, 2000: The instability of u^{Prandtl} is of order $\nu^{1/4}$:

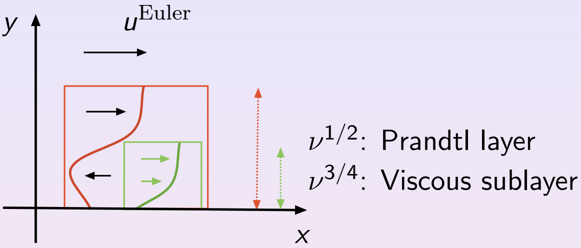
$$\|u^{\mathrm{NS}} - u^{\mathrm{Prandtl}}\|_{L^\infty} \gtrsim \nu^N e^{\Re\lambda_0 t_\nu} \sim \nu^{1/4}, \qquad t_\nu \sim \sqrt{\nu}\log\frac{1}{\nu} \to 0$$

(the sublayer velocity grows up to order $\nu^{1/4}$ in amplitude).



$\nu^{1/2}$: Prandtl layer
$\nu^{3/4}$: Viscous sublayer

- Going beyond the $\nu^{1/4}$-instability?



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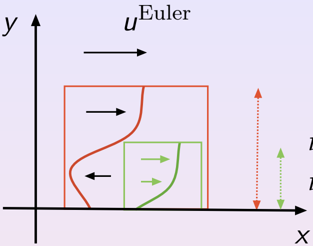

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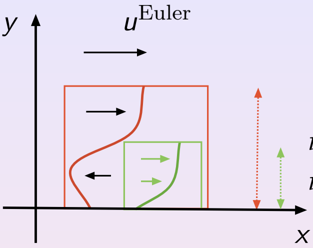

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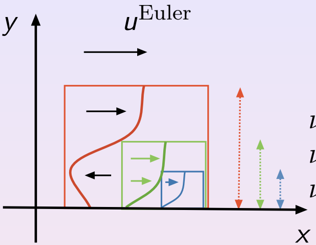
  - Sublayers themselves are unstable! Many instabilities.....!



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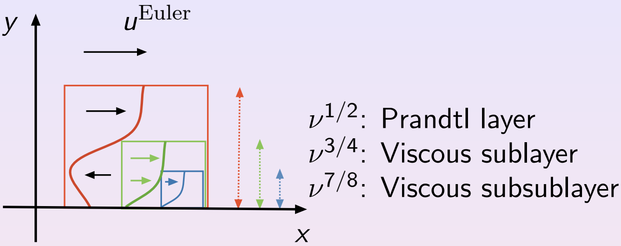
So, it seems hopeless for stability of the instability.....except:



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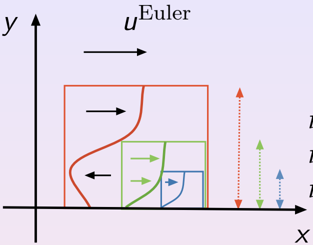
- Sammartino-Caflisch '98: analyticity prevents sublayers. Navier-Stokes solutions involve precisely Euler + Prandtl.



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- Program: use analyticity to prevent subsublayers. Navier- Stokes solutions involve precisely Euler + Prandtl + Sublayer.

$$u^\nu \quad \approx \quad u^{\mathrm{Prandtl}}(y_1) + \nu^N e^{\lambda_\nu t_1}\Big[\hat{u}^{\mathrm{e},1}(x_1, y_1) + \hat{u}^{\mathrm{BL},1}(x_1, \frac{y_1}{\nu^{1/4}})\Big]$$

- Issue:

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  - Large time analyticity framework: $t_1 \sim \log \frac{1}{\nu} \to \infty$.

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- Issue:
  - Large time analyticity framework: $t_1 \sim \log \frac{1}{\nu} \to \infty$.
  - Due to shearing, analyticity radius shrinks like $\frac{1}{t_1} \to 0$:

$$\omega_t + U(y)\omega_x + v_2 U'' = \sqrt{\nu}\Delta\omega, \qquad v_2 = -\partial_x \Delta^{-1}\omega$$

(analyticity is lost when needed!)

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(noting the exponential growth in time!),

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(noting the exponential growth in time!), which solves Navier-Stokes equations with forcing f^ν*, until the instability time:*

$$\nu^N e^{\Re\lambda_0 t_1} \sim 1,$$

where

$$\|(u^\nu - u^{\mathrm{Prandtl}})(0)\|_{H^s} + \|f^\nu\|_{H^s} \leq \nu^N.$$

- Some pains.....

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with vorticity ω^j iteratively solves:

$$(\partial_t + U\partial_x)\omega^j + u_2^j U'' - \sqrt{\nu}\Delta\omega^j = -\sum_{k+\ell=j} u^k \cdot \nabla\omega^\ell$$

together with the no-slip condition on $u^j = \nabla^\perp \Delta^{-1}\omega^j$.

Generator functions

$$Gen_\delta(\omega)(z) = \sum_{\alpha \in \mathbb{Z}} \sum_{\ell \geq 0} e^{z|\alpha|} \| D_y^\ell \hat{\omega}_\alpha \|_\delta \frac{z^\ell}{\ell!}, \qquad D_y = \frac{y}{1+y} \partial_y$$

where

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$$\|\hat\omega_\alpha\|_\delta = \sup_y |\hat\omega_\alpha(y)| \Big(\delta^{-1} e^{-y/\delta} + 1\Big)^{-1}, \qquad \delta = \nu^{1/4}.$$

- Good properties with products and derivatives:

$$Gen_\delta(fg) \le Gen_0(f) Gen_\delta(g),$$

$$Gen_\delta(\partial_x\omega) = \partial_z Gen_\delta(\omega), \qquad Gen_\delta(D_y\omega) = \partial_z Gen_\delta(\omega)$$

$$Gen_\delta(u\cdot\nabla\omega) \lesssim \partial_z Gen_0(u) \partial_z Gen_\delta(\omega)$$

Large time Cauchy-Kovalevskaya theory

- Derive a simple transport inequality for vorticity:

$$\partial_\tau Gen_\delta(\omega)(z) \quad \lesssim \quad Gen_\delta(\omega)(z) + Gen_\delta(\omega)\partial_z Gen_\delta(\omega)(z)$$

with $\tau = \nu^N e^{\Re\lambda_0 t_1}$, giving solution up to a time of order one:

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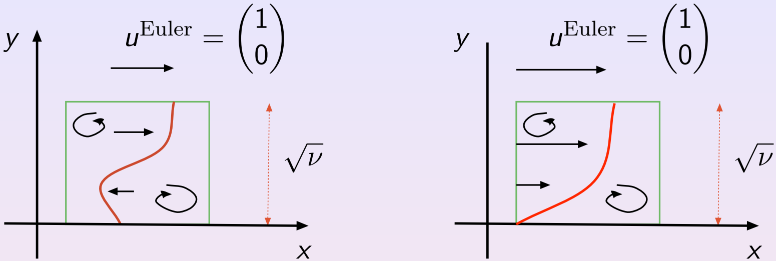
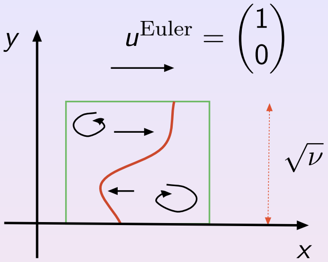
- An application:

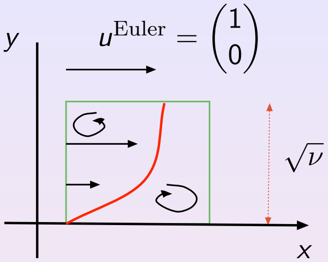
A simplification of Mouhot-Villani's proof of Landau damping

200 pages $\Longrightarrow$ an elementary proof (Grenier-Toan-Rodnianski '20).



Theorem (Grenier-Toan: The failure of Prandtl's Ansatz)





Theorem (Grenier-Toan: The failure of Prandtl's Ansatz)

Let $U(\cdot)$ be spectrally unstable or monotone and stable to Euler. Then, for any N, s, there are solutions u^ν of Navier Stokes with forcing terms f^ν so that

$$\|(u^\nu - u^{\text{Prandtl}})(0)\|_{H^s} + \|f^\nu\|_{L^\infty([0,t_\nu],H^s)} \le \nu^N,$$

$\|(u^\nu - u^{\text{Prandtl}})(t_\nu)\|_{L^\infty} \gtrsim 1$	$\|(u^\nu - u^{\text{Prandtl}})(t_\nu)\|_{L^\infty} \gtrsim \nu^{\frac{1}{4}+\epsilon}$
$t_\nu \sim \sqrt{\nu} \log \nu^{-1}.$	$t_\nu \sim \nu^{1/4} \log \nu^{-1}.$