The Maxwell-Boltzmann approximation for ion kinetic modeling

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Abstract

The aim of this paper is to provide a justification of the Maxwell-Boltzmann approximation of electron density from kinetic models. First, under reasonable regularity assumption, we rigorously derive a reduced kinetic model for the dynamics of ions, while electrons satisfy the Maxwell-Boltzmann relation. Second, we prove that equilibria of the electrons distribution are local Maxwellians, and they can be uniquely determined from conserved mass and energy constants. Finally, we prove that the reduced kinetic model for ions is globally well-posed. The constructed weak solutions conserve energy.

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1 Introduction

1.1 Physical framework for the modeling

Consider a plasma consisting of electrons and one kind of ions, which are charged particles moving in an electromagnetic field. Let $\widetilde{f}_+(x,v,t)$ and $\widetilde{f}_-(x,w,t)$ be the corresponding nonnegative density distribution functions for ions and electrons, respectively; here, (v,w) represent particle velocity variables for ions and electrons belonging to $\mathbb{R}^d$ (here $d=2$ or 3). Moreover x is the spatial variable belonging to an open set of $\mathbb{R}^d$ and t is the time. The dynamics of the plasma in the case where there is no magnetic field is modeled by the following well-known Vlasov-Poisson-Boltzmann system

$$\begin{aligned} \partial_t\widetilde{f}_- + w\cdot\nabla_x\widetilde{f}_- - \frac{q_e}{m_e}\widetilde{E}\cdot\nabla_w\widetilde{f}_- &= \widetilde{Q}_-(\widetilde{f}_-) \\ \partial_t\widetilde{f}_+ + v\cdot\nabla_x\widetilde{f}_+ + \frac{q_e}{m_i}\widetilde{E}\cdot\nabla_v\widetilde{f}_+ &= \widetilde{Q}_+(\widetilde{f}_+) \end{aligned} \tag{1.1}$$

in which m_e, m_i denotes the electrons and ions mass, q_e the elementary charge (for the sake of simplicity we assume that the ions charge is equal to 1). The electrostatic field is given by $\widetilde{E} = -\nabla_x\widetilde{\phi}$ and solves the Poisson equation, which reads as

$$-\epsilon^0\Delta_x\widetilde{\phi} = \langle\widetilde{f}_+\rangle - \langle\widetilde{f}_-\rangle$$

with ϵ^0 being the vacuum permittivity. Here and in the sequel $\langle\cdot\rangle$ denotes the integral over the velocity space, that is

$$\langle F\rangle := \int_{\mathbb{R}^d} F(v)dv.$$

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In equation (1.1), $\widetilde{Q}_-(\widetilde{f}_-)$ accounts for the non-linear collision operator of electrons with themselves (for example, a binary Boltzmann or Fokker-Planck operator). Due to complexity of the problem, we have ignored the magnetic effects and the collision between electrons and ions.

Such a model has been widely used in plasma physics from a theoretical point of view; see, for instance, [14, 26, 31, 30]. But, since the electron/ion mass ratio is relatively small, the characteristic time scale of the dynamics of ions is significantly larger than that of electrons. As a consequence, if one addresses a model for ions dynamics, it is very classical to use a fluid modeling for the electrons and to assume that they have reached the thermal equilibrium; that is to say the distribution function is a Maxwellian function with an electron temperature $\widetilde{\theta}$ and a density given by the well-known *Maxwell-Boltzmann relation*

$$\langle \widetilde{f}_- \rangle = e^{q_e \widetilde{\phi}/\widetilde{\theta}} \tag{1.2}$$

(here the temperature $\widetilde{\theta}$ can be expressed in energy units).

The ion dynamics described by a kinetic model, together with the Maxwell-Boltzmann relation, has been addressed in a number of works with a constant electrons temperature; see, for instance, [3, 18, 17]. For relevant studies on the massless electrons limit or particles with disparate masses, see [5, 6, 7, 16, 22, 8, 9] and the references therein. In this paper, we are interested in justifying the ions model at the kinetic level, together with the Maxwell-Boltzmann approximation (1.2), directly from the kinetic system (1.1) in the limit as $m_e/m_i \to 0$.

1.2 The non-dimensional form

Let us first derive the non-dimensional form of (1.1). Denote by θ_{ref} and by N_{ref} the characteristic values of the electrons temperature and of the electrons density. We introduce the non-dimensional parameter

$$\epsilon = \sqrt{\frac{m_e}{m_i}}$$

which is assumed to be sufficiently small. In addition, we rescale the velocity of electrons and their distribution functions as follows:

$$w = v/\epsilon, \qquad f_-(v) = \frac{1}{\epsilon^3 N_{ref}} \widetilde{f}_-(v/\epsilon), \qquad f_+(v) = \frac{1}{N_{ref}} \widetilde{f}_+(v),$$

while the electric potential and temperature is scaled as $\phi = q_e\widetilde{\phi}/\theta_{ref}$ and $\theta = \widetilde{\theta}/\theta_{ref}$.

The system (1.1) then becomes

$$\begin{aligned} \epsilon\partial_t f_- + v \cdot \nabla_x f_- + \nabla_x \phi \cdot \nabla_v f_- &= \eta_\epsilon Q_-(f_-) \\ \partial_t f_+ + v \cdot \nabla_x f_+ - \nabla_x \phi \cdot \nabla_v f_+ &= \sigma_\epsilon Q_+(f_+) \end{aligned} \tag{1.3}$$

and the Poisson equation now reads

$$-\lambda_D^2 \Delta_x \phi = \langle f_+ \rangle - \langle f_- \rangle \,. \tag{1.4}$$

In the above system, $\eta_\epsilon, \sigma_\epsilon$ denote the collision parameters, and λ_D is the Debye length (e.g., see [31]), defined by

$$\lambda_D = \sqrt{\epsilon^0 \theta_{ref}/(q_e^2 N_{ref})}.$$

For sake of simplicity, we take $\lambda_D = 1$; see, however, Remark 1.3.

The system (1.3) is posed on the phase space $\Omega \times \mathbb{R}^d$, with Ω being the periodic box $\mathbb{T}^d$ or a bounded open subset of $\mathbb{R}^d$. In case of a boundary, we assume that both ions and electrons reflect specularly on the boundary, that is to say

$$f_\pm(x, v, t) = f_\pm(x, v - 2(v \cdot n(x))n(x), t), \qquad n(x) \cdot v < 0 \tag{1.5}$$

at each point $x \in \partial\Omega$, in which $n(x)$ denotes the outward normal vector of $\partial\Omega$. We also assume the Neumann boundary condition for the Poisson problem (1.4)

$$\frac{\partial \phi}{\partial n}_{|\partial\Omega} = 0. \tag{1.6}$$

In view of (1.4), this latter boundary condition limits initial data $f_\pm(0)$ to satisfy

$$\iint_{\Omega \times \mathbb{R}^d} f_+(0)\, dxdv = \iint_{\Omega \times \mathbb{R}^d} f_-(0)\, dxdv = m_0.$$

As for the collision operator, we assume that for each continuous, nonnegative, and rapidly decaying function $f(v)$, $Q_\pm(\cdot)$ satisfies the following classical properties:

$$\langle Q_\pm(f) \rangle = 0, \qquad \langle v Q_\pm(f) \rangle = 0, \qquad \langle |v|^2 Q_\pm(f) \rangle = 0, \tag{1.7}$$

$$\langle Q_\pm(f) \log f \rangle \le 0, \tag{1.8}$$

with equality in (1.8) implying that such functions are Maxwellians.

1.3 Main results

1.3.1 Maxwell-Boltzmann approximation

In the sequel, instead of electrons temperature θ, we shall work with its inverse $\beta = \frac{1}{\theta}$. As mentioned, our goal is to justify the Maxwell-Boltzmann approximation (1.2) from the Vlasov-Poisson-Boltzmann system (1.3)-(1.4). This will be done under an assumption on existence and regularity of solutions to the system. Precisely, our main results are as follows.

Theorem 1.1. *Assume Ω is non-axisymmetric, and the collisional coefficients $\eta_\epsilon, \sigma_\epsilon$ satisfy*

$$\lim_{\epsilon \to 0} \eta_\epsilon \epsilon^{-1} = \infty, \qquad \lim_{\epsilon \to 0} \eta_\epsilon < +\infty, \qquad \sigma_0 := \lim_{\epsilon \to 0} \sigma_\epsilon < +\infty \tag{1.9}$$

Let $(f_+^\epsilon, f_-^\epsilon, \phi^\epsilon)$ be a sequence of smooth solutions to the system (1.3)-(1.4) *so that the followings hold*

- *$f^\epsilon_\pm$ are nonnegative and satisfy*

$$f^\epsilon_\pm(x,v,t) \le Ce^{-|v|^\gamma} \tag{1.10}$$

for some positive constants C, γ, uniformly in x, v, t and in ϵ.

- *as $\epsilon \to 0$, $f^\epsilon_\pm$ converges almost everywhere to nonnegative functions $f_\pm$.*
- *the entropy dissipation rate satisfies*

$$\limsup_{\epsilon\to 0} \iint_{\Omega\times\mathbb{R}^d} Q_-(f^\epsilon_-)\log f^\epsilon_-\, dxdv \le \iint_{\Omega\times\mathbb{R}^d} Q_-(f_-)\log f_-\, dxdv. \tag{1.11}$$

Then, as $\epsilon \to 0$, the limit f_- is a local Maxwellian of the form

$$f_-(x,v,t) = n_e(x,t)\Big(\frac{\beta(t)}{2\pi}\Big)^{\frac{d}{2}} e^{-\beta(t)\frac{|v|^2}{2}}, \qquad n_e(x,t) = e^{\beta(t)\phi(x,t)},$$

in which the electric potential ϕ and the inverse of electron temperature β are determined through the following **reduced ion kinetic** *system*

$$\begin{aligned} &\text{i)} \quad \partial_t f_+ + v\cdot\nabla_x f_+ - \nabla_x\phi\cdot\nabla_v f_+ = \sigma_0 Q_+(f_+),\\ &\text{ii)} \quad -\Delta\phi + e^{\beta\phi} = \langle f_+\rangle,\\ &\text{iii)} \quad \frac{m_0 d}{2\beta} + \frac{1}{2}\iint_{\Omega\times\mathbb{R}^d} |v|^2 f_+(x,v,t)\, dxdv + \frac{1}{2}\int_\Omega |\nabla_x\phi(x,t)|^2 dx = \mathcal{E}_0, \end{aligned} \tag{1.12}$$

with m_0 and $\mathcal{E}_0$ being the total mass and energy, respectively.

In the literature, the equation (ii) in (1.12) is often referred to as the Poisson-Poincaré equation. We stress that there is no time-dynamics for the electrons in the limit as $\epsilon \to 0$. The time-dependence is precisely determined through the dynamics of ions.

Theorem 1.2. *Assume Ω is non-axisymmetric, and the collisional coefficients $\eta_\epsilon, \sigma_\epsilon$ satisfy*

$$\lim_{\epsilon\to 0} \eta_\epsilon \epsilon^{-1} = \infty, \qquad \lim_{\epsilon\to 0} \eta_\epsilon < +\infty, \qquad \lim_{\epsilon\to 0} \sigma_\epsilon = +\infty \tag{1.13}$$

Let $(f^\epsilon_+, f^\epsilon_-, \phi^\epsilon)$ be a sequence of smooth solutions to the system (1.3)-(1.4) so that the same assumptions as made in Theorem 1.1 hold. Assume additionally that the entropy dissipation rate satisfies

$$\limsup_{\epsilon\to 0} \iint_{\Omega\times\mathbb{R}^d} Q_+(f^\epsilon_+)\log f^\epsilon_+\, dxdv \le \iint_{\Omega\times\mathbb{R}^d} Q_+(f_+)\log f_+\, dxdv. \tag{1.14}$$

Then, as $\epsilon \to 0$, the limit $f_\pm$ are local Maxwellians of the following form:

$$\begin{aligned} f_+(x,v,t) &= n_I(x,t)\Big(\frac{1}{2\pi\theta_I(x,t)}\Big)^{\frac{d}{2}} e^{-\frac{|v-u_I(x,t)|^2}{2\theta_I(x,t)}}, \qquad n_I(x,t) = \langle f_+\rangle(x,t),\\ f_-(x,v,t) &= n_e(x,t)\Big(\frac{\beta(t)}{2\pi}\Big)^{\frac{d}{2}} e^{-\beta(t)\frac{|v|^2}{2}}, \qquad n_e(x,t) = e^{\beta(t)\phi(x,t)}, \end{aligned} \tag{1.15}$$

in which (n_I, u_I, θ_I) *solve the following compressible Euler system*

$$\begin{aligned} \partial_t n_I + \nabla\cdot(n_I u_I) = 0, \\ \partial_t(n_I u_I) + \nabla\cdot(n_I u_I \otimes u_I) + \nabla(n_I\theta_I) + n_I\nabla\phi = 0, \\ \partial_t\Big(n_I(\frac{|u_I|^2}{2} + \frac{d}{2}\theta_I)\Big) + \nabla\cdot\Big(n_I u_I\Big(\frac{|u_I|^2}{2} + \frac{d+2}{2}\theta_I\Big)\Big) + n_I u_I\cdot\nabla\phi = 0, \end{aligned} \tag{1.16}$$

coupled with the following Poisson equations

$$\begin{aligned} -\Delta\phi + e^{\beta\phi} = n_I, \\ \frac{m_0 d}{2\beta} + \int_\Omega n_I(x,t)\Big(\frac{|u_I|^2}{2} + \frac{d}{2}\theta_I\Big)\,dx + \frac{1}{2}\int_\Omega |\nabla\phi(x,t)|^2\,dx = \mathcal{E}_0, \end{aligned} \tag{1.17}$$

with m_0 *and* $\mathcal{E}_0$ *denoting the total mass and energy, respectively.*

The proof of Theorems 1.1 and 1.2 is classical and will be given in Section 2. We point out that the local Maxwellians obtained in the limit for electrons are precisely due to the presence of collision operator $Q_-(f_-)$, without which electron equilibria are not unique and of the general form $\Psi(\frac{|v|^2}{2} - \phi)$, for arbitrary functions $\Psi(\cdot)$. Finally, we note that the appearance of the compressible Euler-Poisson equations as a possible limit implies that rigorous results concerning the convergence may not be available without supplementary constraints.

Remark 1.3. Letting $\lambda_D \to 0$ in (1.4) corresponds to the so-called quasi-neutral approximation, which formally leads to the relation

$$\beta\nabla\phi \simeq \nabla(\log n_I)\,. \tag{1.18}$$

Moreover from (1.18) one deduces the formula

$$n_I\nabla\phi \simeq \nabla(n_I\beta^{-1}) \tag{1.19}$$

which means that the gradient of potential is the gradient of the electrons pressure. The approximation (1.18) is well established at the level of physics (cf. [30]). On the other hand the mathematical (with full rigor) justification of (1.18) is the object of many recent researches (cf. for instance [21, 19, 20] and the references therein).

1.3.2 Unique determination of Maxwellians

In view of Theorems 1.1 and 1.2, the next natural question is whether the local Maxwellians are uniquely determined in term of conserved quantities, namely the total mass and energy. In this section, we shall present our next main result, concerning the unique determination of Maxwellians for the electron density, when the ions density is given. Precisely, we consider the following Vlasov-Poisson-Boltzmann system

$$\begin{aligned} \epsilon\partial_t f_- + v\cdot\nabla_x f_- + \nabla_x\phi\cdot\nabla_v f_- = \eta_\epsilon Q_-(f_-), \qquad \eta_\epsilon > 0, \\ -\Delta\phi + \langle f_-(x,t)\rangle = n_I(x), \qquad \frac{\partial\phi}{\partial n}_{|\partial\Omega} = 0 \end{aligned} \tag{1.20}$$

for a fixed ion density $n_I(x)$, together with the specular boundary condition for f_- on $\partial\Omega$, and the mass and energy constraints

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^d} f_-(x,v,t)\,dxdv &= \int_\Omega n_I(x)\,dx = m_0\\ \frac{1}{2}\iint_{\Omega\times\mathbb{R}^d} |v|^2 f_-(x,v,t)\,dxdv + \frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx &= \mathcal{E}_1\end{aligned}\tag{1.21}$$

for some positive constant $\mathcal{E}_1$.

Our next main result reads as follows.

Theorem 1.4. *Let Ω be smooth, bounded, and non-axisymmetric, and let $f_-(x,v)$ be a steady solution to the Vlasov-Poisson-Boltzmann system* (1.20). *Assume that f_- is continuous and rapidly decaying, and $-\log f_-$ grows at most polynomially in $|v|$, as $|v|\to\infty$. Then, there hold the followings:*

i) the solution $f_-(x,v)$ is given by the formula:

$$f_-(x,v) = \Big(\frac{\beta}{2\pi}\Big)^{d/2} e^{-\beta\frac{|v|^2}{2}} e^{\beta\phi(x)}, \qquad n_e(x) = e^{\beta\phi(x)}\tag{1.22}$$

for some positive constant β and for ϕ being a solution to the Poisson-Poincaré equation

$$-\Delta\phi + e^{\beta\phi} = n_I(x), \qquad \frac{\partial\phi}{\partial n}_{|\partial\Omega} = 0.\tag{1.23}$$

ii) for $n_I \in L^2(\Omega)$, $m_0 > 0$, and $\mathcal{E}_1 > 0$, there exists a unique solution (β,ϕ) to the elliptic problem (1.23) *coupled with the energy constraint*

$$\frac{m_0 d}{2\beta} + \frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx = \mathcal{E}_1.\tag{1.24}$$

The theorem will be proved in Section 3. In addition, we will prove that the steady Maxwellian constructed in Theorem 1.4 is nonlinearly stable to (1.20) in the sense of Arnold.

1.3.3 The reduced ion kinetic model

Our final main result concerns the well-posedness of the reduced ion kinetic model, derived in Theorem 1.1. Precisely, we study the following system for (f_+,ϕ,β):

$$\begin{aligned}&\text{i)}\quad \partial_t f_+ + v\cdot\nabla_x f_+ - \nabla_x\phi\cdot\nabla_v f_+ = 0\\ &\text{ii)}\quad -\Delta\phi + e^{\beta(t)\phi} = \langle f_+\rangle\\ &\text{iii)}\quad \frac{m_0 d}{2\beta(t)} + \frac{1}{2}\iint_{\Omega\times\mathbb{R}^d} |v|^2 f_+(x,v,t)\,dxdv + \frac{1}{2}\int_\Omega |\nabla_x\phi(x,t)|^2 dx = \mathcal{E}_0\end{aligned}\tag{1.25}$$

in which we have dropped the collision operator in the above kinetic model. Roughly speaking, we construct global-in-time weak solutions (f_+, ϕ, β) satisfying

$$f_+ \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty(\Omega \times \mathbb{R}^d)), \qquad n_I \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty(\Omega)),$$
$$E = -\nabla\phi \in L^\infty(\mathbb{R}_+; L^\infty(\Omega)),$$

and $\beta \in L^\infty(\mathbb{R}_+)$. We emphasize that the constructed weak solutions conserve energy. Under additional assumptions on the regularity of f_+, we prove that the solution (f_+, ϕ, β) is unique. For the precise statement of the main result, see Section 4.

1.4 Outline of the paper

The rest of the paper is devoted to proving the main results listed above, accordingly in each of following sections.

2 A justification of Maxwell-Boltzmann relation

In this section, we consider the full Vlasov-Poisson-Boltzmann system (1.3)-(1.4), which we recall

$$\begin{aligned} \epsilon\partial_t f_-^\epsilon + v\cdot\nabla_x f_-^\epsilon + \nabla_x\phi^\epsilon\cdot\nabla_v f_-^\epsilon &= \eta_\epsilon Q_-(f_-^\epsilon)\\ \partial_t f_+^\epsilon + v\cdot\nabla_x f_+^\epsilon - \nabla_x\phi^\epsilon\cdot\nabla_v f_+^\epsilon &= \sigma_\epsilon Q_+(f_+^\epsilon)\\ -\Delta_x\phi^\epsilon &= \langle f_+^\epsilon\rangle - \langle f_-^\epsilon\rangle \end{aligned} \tag{2.1}$$

together with specular boundary condition for $f_\pm^\epsilon$ and the Neumann boundary condition for ϕ^ϵ (in the case when $\partial\Omega \neq \emptyset$).

2.1 Conservation properties

Let us first recall a few basic properties of solutions of (2.1). Assume that f_-^ϵ and f_+^ϵ have sufficient regularity and rapidly decay to zero as $v \to \infty$. The first property of $Q_\pm$ in (1.7) immediately yields the conservation of mass:

$$\partial_t\langle f_+^\epsilon\rangle + \nabla_x\cdot\langle v f_+^\epsilon\rangle = 0, \qquad \partial_t\langle f_-^\epsilon\rangle + \epsilon^{-1}\nabla_x\cdot\langle v f_-^\epsilon\rangle = 0. \tag{2.2}$$

Together with the specular reflection boundary condition for $f_\pm^\epsilon$, this yields the global conservation of mass:

$$\int\langle f_+^\epsilon(t)\rangle dx = \int\langle f_-^\epsilon(t)\rangle dx = m_0, \qquad \forall t \geq 0. \tag{2.3}$$

For the momentum conservation, we get

$$\begin{aligned} \partial_t\langle v f_+^\epsilon\rangle + \nabla_x\cdot\langle v\otimes v f_+^\epsilon\rangle &= -\langle f_+^\epsilon\rangle\nabla_x\phi^\epsilon,\\ \partial_t\langle v f_-^\epsilon\rangle + \frac{1}{\epsilon}\nabla_x\cdot\langle v\otimes v f_-^\epsilon\rangle &= \frac{1}{\epsilon}\langle f_-^\epsilon\rangle\nabla_x\phi^\epsilon. \end{aligned} \tag{2.4}$$

Moreover, for the ions and electrons energy conservation, we get

$$
\begin{aligned}
\partial_t\langle \frac{|v|^2}{2} f_+^\epsilon\rangle + \nabla_x\cdot\langle v\frac{|v|^2}{2} f_+^\epsilon\rangle &= -\nabla_x\phi^\epsilon\cdot\langle v f_+^\epsilon\rangle \\
\partial_t\langle \frac{|v|^2}{2} f_-^\epsilon\rangle + \frac{1}{\epsilon}\nabla_x\cdot\langle v\frac{|v|^2}{2} f_-^\epsilon\rangle &= \frac{1}{\epsilon}\nabla_x\phi^\epsilon\cdot\langle v f_-^\epsilon\rangle.
\end{aligned}
\tag{2.5}
$$

Hence, a direct computation yields

$$
\begin{aligned}
\frac{d}{dt}\int \langle \tfrac12 |v|^2 f_+^\epsilon\rangle + \langle \tfrac12 |v|^2 f_-^\epsilon\rangle dx &= \int_\Omega \nabla_x\phi^\epsilon\cdot\Big(-\langle v f_+^\epsilon\rangle + \frac{1}{\epsilon}\langle v f_-^\epsilon\rangle\Big)dx \\
&= \int_\Omega \phi^\epsilon \nabla_x\cdot\Big(\langle v f_+^\epsilon\rangle - \frac{1}{\epsilon}\langle v f_-^\epsilon\rangle\Big)dx \\
&= \int_\Omega \phi^\epsilon \partial_t\Big(\langle f_-^\epsilon\rangle - \langle f_+^\epsilon\rangle\Big)dx,
\end{aligned}
$$

in which the conservation (2.2) of mass was used. Using the Poisson equation (1.4) and taking the integration by parts $\int \phi^\epsilon\Delta(\partial_t\phi^\epsilon)dx = -\int \nabla\phi^\epsilon\cdot(\partial_t\nabla\phi^\epsilon)dx$, we thus obtain the conservation of energy

$$
\int \langle \frac{|v|^2}{2} f_-^\epsilon\rangle + \langle \frac{|v|^2}{2} f_+^\epsilon\rangle dx + \frac12\int_\Omega |\nabla_x\phi^\epsilon|^2\,dx = \mathcal{E}_0, \qquad \forall t\ge 0 \tag{2.6}
$$

with $\mathcal{E}_0$ being a positive constant. Next, multiplying equation (1.3) by $\log f_-^\epsilon$, we obtain

$$
\frac{d}{dt}\int \langle f_-^\epsilon \log f_-^\epsilon\rangle dx = \frac{1}{\epsilon}\eta_\epsilon \int \langle Q_-(f_-^\epsilon)\log f_-^\epsilon\rangle dx. \tag{2.7}
$$

Thus, by (1.8), minus the entropy of f_-^ϵ is decreasing in time:

$$
\frac{d}{dt}\int \langle f_-^\epsilon \log f_-^\epsilon\rangle dx \le 0.
$$

2.2 Proof of Theorems 1.1 and 1.2

We shall use the following lemma (cf. [11] or [10, Proposition 13] for discussions on more general setting).

Lemma 2.1 (Korn's inequality)**.** *Let Ω be a smooth bounded subset of $\mathbb{R}^d$, $d\ge 2$. Then, there exists a constant $\overline{K}(\Omega)>0$ such that for any vector fields $u:\Omega\mapsto\mathbb{R}^d$, one has*

$$
\Big\|\frac{\nabla u + (\nabla u)^t}{2}\Big\|_{L^2(\Omega)} \ge \overline{K}(\Omega) \inf_{R\in\mathcal{R}(\Omega)} \|\nabla(u-R)\|^2_{L^2(\Omega)}, \tag{2.8}
$$

in which $\mathcal{R}(\Omega)$ denotes the space that consists of all affine maps $R:\Omega\mapsto\mathbb{R}^d$ whose linear part is anti-symmetric. In particular, if Ω is non-axisymmetric and if $u\cdot n = 0$ on $\partial\Omega$, then the Korn's inequality (2.8) *holds for $R\equiv 0$.*

Proof of Theorem 1.1. We first find the limiting system of (2.1) as $\epsilon \to 0$. Thanks to the uniform decay assumption (1.10) on $f^\epsilon_\pm$, the ion and electron densities $\langle f^\epsilon_\pm \rangle$ are uniformly bounded. It thus follows from the Poisson equation that ϕ^ϵ is uniformly bounded in $W^{2,p}(\Omega)$ for all $p \in [1,\infty$. In particular, $\nabla\phi^\epsilon$ converges to $\nabla\phi$ strongly in $L^2(\Omega)$. Together with the pointwise convergence and the uniform decay assumption of $f^\epsilon_\pm$, this proves

$$\nabla_x \phi^\epsilon \cdot \nabla_v f^\epsilon_\pm \to \nabla_x \phi \cdot \nabla_v f_\pm,$$

weakly in $L^1(\Omega \times \mathbb{R}^d)$. Similarly, thanks to the decay assumption (1.10), the Lebesgue's dominated convergence theorem $Q_\pm(f^\epsilon_\pm)$ converges almost everywhere to $Q_\pm(f_\pm)$. Taking the limit as $\epsilon \to 0$ in (2.1), we thus obtain the following system for the limit $(f_\pm, \phi)$:

$$\begin{aligned} v \cdot \nabla_x f_- + \nabla_x \phi \cdot \nabla_v f_- &= \eta_0 Q_-(f_-) \\ \partial_t f_+ + v \cdot \nabla_x f_+ - \nabla_x \phi \cdot \nabla_v f_+ &= \sigma_0 Q_+(f_+) \\ -\Delta_x \phi &= \langle f_+ \rangle - \langle f_- \rangle \end{aligned} \tag{2.9}$$

together with the energy conservation

$$\int \langle \frac{|v|^2}{2} f_- \rangle + \langle \frac{|v|^2}{2} f_+ \rangle dx + \frac{1}{2} \int_\Omega |\nabla_x \phi|^2 \, dx = \mathcal{E}_0, \qquad \forall t \geq 0. \tag{2.10}$$

It remains to check that the limit f_- is of the form of a local Maxwellian as claimed in the theorem. Indeed, in view of (2.7), for any positive T, we obtain

$$\int_0^T \int \langle Q_-(f^\epsilon_-) \log f^\epsilon_- \rangle dx dt = \epsilon \eta_\epsilon^{-1} \int \langle f^\epsilon_-(T) \log f^\epsilon_-(T) \rangle dx - \epsilon \eta_\epsilon^{-1} \int \langle f^\epsilon_-(0) \log f^\epsilon_-(0) \rangle dx.$$

Thanks to (1.10), the right hand side of the above is uniformly bounded. Thus, we obtain

$$0 \leq -\int_0^T \iint Q_-(f^\epsilon_-) \log f^\epsilon_- \, dx dv dt \leq C_0 \epsilon \eta_\epsilon^{-1},$$

uniformly in T and ϵ, for some universal constant $C_0 > 0$. Taking $\epsilon \to 0$ and using (1.11), we obtain

$$0 \leq -\int_0^T \iint_{\Omega \times \mathbb{R}^d} Q_-(f_-) \log f_- \, dv dx dt \leq C_0 \lim_{\epsilon \to 0} \epsilon \eta_\epsilon^{-1} = 0.$$

This proves that the entropy dissipation rate for f_- vanishes, and by the H-theorem, f_- is a local Maxwellian of the form

$$f_-(x,v,t) = n_e \Big(\frac{\beta}{2\pi} \Big)^{\frac{d}{2}} e^{-\beta \frac{|v-u_-|^2}{2}}$$

in which (n_e, u_-, β) depend on (x,t). In particular, $Q_-(f_-) = 0$. The limiting Vlasov equation in (2.9) for the electron density distribution becomes

$$v \cdot \nabla_x f_- + \nabla_x \phi \cdot \nabla_v f_- = 0, \qquad \forall \, (x,v) \in \Omega \times \mathbb{R}^d. \tag{2.11}$$

Direct computations yield

$$v \cdot \nabla_x f_- = v \cdot \Big[\nabla \log n_e - \frac{d}{2}\nabla\beta + \frac{\beta|v-u_-|^2}{2}\nabla\beta + \beta\sum_k (v_k - u_{k,-})\nabla u_{k,-}\Big] f_-$$

and

$$\nabla_x\phi \cdot \nabla_v f_- = -\beta\nabla_x\phi \cdot (v-u_-)f_-.$$

We write (2.11) as a polynomial with variable $v-u$, and set its coefficients to be zero. From the cubic term, we get $\nabla\beta = 0$ and so $\beta = \beta(t)$. The quadratic term is

$$\begin{aligned} f_-\beta[(v-u_-)\otimes(v-u_-)] &: \frac{\nabla u_- + (\nabla u_-)^t}{2} \\ &= f_-\beta\sum_{jk}(v_j - u_{j,-})(v_k - u_{k,-})\frac{\partial_{x_j}u_{k,-} + \partial_{x_k}u_{j,-}}{2} \end{aligned}$$

which implies that $\nabla u_- + (\nabla u_-)^t = 0$. In addition, since f_- is an even function with respect to variable $v - u_-$, we get

$$u_-(x,t) = \frac{1}{n_e(x,t)}\langle v f_-(x,v,t)\rangle \tag{2.12}$$

This gives $u_- \cdot n = 0$ on $\partial\Omega$, thanks to the specular boundary condition on f_-. By Korn's inequality, $\nabla u_- = 0$ and so $u_- = 0$. The equation (2.11) simply reduces to

$$0 = \nabla \log n_e - \beta\nabla_x\phi.$$

This proves that $n_e(x,t) = e^{\beta(t)\phi(x,t)}$ and $f_-(x,v,t)$ is of the form as claimed. The relation (1.12 -iii)) comes from the energy conservation property and the fact that

$$\langle \frac{|v|^2}{2} f_-\rangle = n_e\frac{d}{2\beta}$$

This completes the proof. □

Proof of Theorem 1.2. The proof is similar, yielding the same Maxwellian for f_-. In addition, (1.14) and the assumption $\lim_{\epsilon\to 0}\sigma_\epsilon = \infty$ imply that f_+ is also a local Maxwellian, as claimed. The macroscopic equations (1.16) are obtained by taking the moments of f_+, upon recalling that

$$n_I = \langle f_+\rangle, \qquad n_I u_I = \langle v f_+\rangle, \qquad n_I(\frac{|u_I|^2}{2} + \frac{d}{2}\theta_I) = \langle\frac{|v|^2}{2}f_+\rangle.$$

Indeed, same relations hold for f_+^ϵ. By multiplying the Vlasov-Boltzmann equation for f_+^ϵ by $1, v$ and $\frac{|v|^2}{2}$ and integrating over $\mathbb{R}^d$ with respect to v, we obtain the following local conservation laws, respectively

$$\partial_t n_I^\epsilon + \nabla_x \cdot (n_I^\epsilon u_I^\epsilon) = 0$$

$$\partial_t(n_I^\epsilon u_I^\epsilon) + \nabla_x \cdot \langle v \otimes v f_+^\epsilon \rangle + n_I^\epsilon \nabla_x \phi^\epsilon = 0$$

$$\partial_t \Big[n_I^\epsilon (\frac{|u_+^\epsilon|^2}{2} + \frac{d}{2}\theta_+^\epsilon) \Big] + \nabla_x \cdot \langle v \frac{|v|^2}{2} f_+^\epsilon \rangle + n_I^\epsilon u_I^\epsilon \cdot \nabla_x \phi^\epsilon = 0.$$

Passing to the limit as $\epsilon \to 0$ and using the fact that the limiting distribution f_+ is the Maxwellian (which is an even function in $v - u_I$), we compute

$$\begin{aligned} \nabla_x \cdot \langle v \otimes v f_+ \rangle &= \nabla_x \cdot \langle u_I \otimes u_I f_+ \rangle + \nabla_x \cdot \langle (v - u_I) \otimes (v - u_I) f_+ \rangle \\ &= \nabla_x \cdot (n_I u_I \otimes u_I) + \nabla_x (n_I \theta_I). \end{aligned}$$

Similarly, repeatedly using the evenness of f_+ in $v - u_I$, we compute

$$\begin{aligned} \nabla_x \cdot \langle v \frac{|v|^2}{2} f_+ \rangle &= \nabla_x \cdot \langle u_I \frac{|v|^2}{2} f_+ \rangle + \nabla_x \cdot \langle (v - u_I) \Big[\frac{|v - u_I|^2}{2} + u_I \cdot v - \frac{|u_I|^2}{2} \Big] f_+ \rangle \\ &= \nabla_x \cdot \Big(n_I u_I (\frac{|u_I|^2}{2} + \frac{d}{2}\theta_I) \Big) + \nabla_x \cdot \langle (v - u_I) u_I \cdot (v - u_I) f_+ \rangle \\ &= \nabla_x \cdot \Big(n_I u_I (\frac{|u_I|^2}{2} + \frac{d}{2}\theta_I) \Big) + \frac{2}{d} \nabla_x \cdot \langle u_I \frac{|v - u_I|^2}{2} f_+ \rangle \\ &= \nabla_x \cdot \Big(n_I u_I (\frac{|u_I|^2}{2} + \frac{d}{2}\theta_I) \Big) + \nabla_x \cdot (n_I u_I \theta_I). \end{aligned}$$

This yields (1.16), and thus completes the proof of the theorem. □

3 Analysis of electron dynamics

In this section, we focus on the electrons dynamics, modeled by the following Vlasov-Poisson-Boltzmann system

$$\epsilon \partial_t f_- + v \cdot \nabla_x f_- + \nabla_x \phi \cdot \nabla_v f_- = \eta_\epsilon Q_-(f_-) \tag{3.1}$$

$$-\Delta \phi + \langle f_-(x,t) \rangle = n_I(x), \qquad \frac{\partial \phi}{\partial n}_{|\partial\Omega} = 0 \tag{3.2}$$

for a fixed ions density $n_I(x)$, together with the specular boundary condition for f_- on $\partial\Omega$, and the mass and energy constraints (1.21).

3.1 Uniqueness of local Maxwellians

In this section, we shall prove Theorem 1.4. Indeed, let $f_-(x,v)$ be a steady state of (3.1)-(3.2). Following exactly the proof of Theorem 1.1 in deriving the form of Maxwellian for electrons, the solution $f_-(x,v)$ is given by the formula:

$$f_-(x,v) = \Big(\frac{\beta}{2\pi} \Big)^{d/2} e^{-\beta(\frac{|v|^2}{2} - \phi(x))} \tag{3.3}$$

for some positive β. Observe that

$$\langle f_-(x)\rangle = e^{\beta\phi(x)}, \qquad \frac{1}{2}\iint_{\Omega\times\mathbb{R}^d} |v|^2 f_-(x,v)\,dxdv = \frac{m_0 d}{2\beta}.$$

Hence, the Poisson problem (3.2) is reduced to

$$-\Delta\phi + e^{\beta\phi} = n_I(x), \qquad \frac{\partial\phi}{\partial n}\Big|_{\partial\Omega} = 0, \tag{3.4}$$

together with the energy constraint

$$\mathcal{E}(\beta) := \frac{m_0 d}{2\beta} + \frac{1}{2}\int_\Omega |\nabla\phi|^2\,dx = \mathcal{E}_1.$$

It remains to prove that the elliptic problem (with the energy constraint) has a unique solution (ϕ,β), for each $m_0,\mathcal{E}_1 > 0$. Indeed, for each fixed $\beta > 0$, the mapping $\phi \mapsto e^{\beta\phi}$ is strictly increasing and hence by the standard elliptic theory, the problem (3.4) has a unique solution $\phi^\beta \in H^2(\Omega)$. Next, to study the β-dependence, we consider the following linear problem for $\partial_\beta\phi^\beta$:

$$-\Delta\partial_\beta\phi^\beta + \beta e^{\beta\phi^\beta}\partial_\beta\phi^\beta = -e^{\beta\phi^\beta}\phi^\beta\,, \qquad \frac{\partial\partial_\beta\phi^\beta}{\partial n}\Big|_{\partial\Omega} = 0 \tag{3.5}$$

whose solution exists and is unique, with $\partial_\beta\phi^\beta \in H^2(\Omega)$. The uniqueness proves that $\partial_\beta\phi^\beta$ is indeed the derivative of ϕ^β with respect to β.

Next, to determine β, we use the energy constraint. Taking the β-derivative of the energy, we have

$$\partial_\beta\mathcal{E}(\beta) = -\frac{m_0 d}{2\beta^2} + \int_\Omega \nabla_x\phi^\beta\cdot\nabla_x\partial_\beta\phi^\beta dx = -\frac{m_0 d}{2\beta^2} - \int_\Omega \phi^\beta\Delta_x\partial_\beta\phi^\beta dx\,. \tag{3.6}$$

To compute the last term, from (3.5), we write

$$\phi^\beta = e^{-\beta\phi^\beta}(\Delta_x\partial_\beta\phi^\beta) - \beta\partial_\beta\phi_\beta$$

which yields at once

$$\partial_\beta\mathcal{E}(\beta) = -\frac{m_0 d}{2\beta^2} - \int_\Omega e^{-\beta\phi^\beta}|\Delta_x\partial_\beta\phi^\beta|^2 dx - \beta\int_\Omega |\nabla_x\partial_\beta\phi^\beta|^2 dx.$$

This proves that $\beta \mapsto \mathcal{E}(\beta)$ is a strictly decreasing function. Clearly, $\lim_{\beta\to 0}\mathcal{E}(\beta) = \infty$, which follows from the term $\frac{m_0 d}{2\beta}$. On the other hand, from the elliptic equation for ϕ^β, we obtain

$$\begin{aligned}\int_\Omega |\nabla\phi^\beta|^2 dx &= \int_\Omega \Big(n_I(x)\phi^\beta(x) - e^{\beta\phi^\beta}\phi^\beta\Big)dx \\ &\le \int_{\{\phi^\beta\ge 0\}} (n_I(x)\phi^\beta(x) - e^{\beta\phi^\beta}\phi^\beta)dx - \frac{1}{\beta}\int_{\{\phi^\beta\le 0\}} e^{\beta\phi^\beta}\beta\phi^\beta dx.\end{aligned}$$

Using the fact that $e^x \geq x$ for $x \geq 0$ and $-xe^x \leq e^{-1}$ for $x \leq 0$, we obtain

$$\int_{\{\phi^\beta \geq 0\}} (n_I(x)\phi^\beta(x) - e^{\beta\phi^\beta}\phi^\beta)dx \leq \|n_I\|_{L^2}\|\phi^\beta\|_{L^2} - \beta\|\phi^\beta\|_{L^2}^2 \leq \frac{1}{2\beta}\|n_I\|_{L^2}^2$$

and

$$\frac{1}{\beta}\int_{\{\phi^\beta \leq 0\}} e^{\beta\phi^\beta}\beta\phi^\beta dx \leq \frac{|\Omega|e^{-1}}{\beta}.$$

This proves that $\mathcal{E}(\beta) \to 0$ as $\beta \to \infty$. The existence and uniqueness of β so that $\mathcal{E}(\beta) = \mathcal{E}_1$ follows from the strict monotonicity of $\mathcal{E}(\beta)$ in $\beta \in (0, \infty)$. Theorem 1.4 is thus proved.

Remark 3.1. In the case when Ω is axisymmetric, nonzero macroscopic velocity is allowed. For instance, when $\Omega = Q \times \mathbb{T}^k$ with $Q \subset \mathbb{R}^{d-k}$ being non axisymmetric, the failure of the Korn's inequality yields the following form of Maxwellian for $f(x, v)$

$$f_-(x,v) = \Big(\frac{\beta}{2\pi}\Big)^{d/2} e^{-\beta\frac{|v-u|^2}{2}} e^{\beta\phi(x)}$$

in which $u = (0, u_k)$ is a vector constant in $\mathbb{R}^{d-k} \times \mathbb{R}^k$. In the case that $u_k \neq 0$, it is necessary that $\phi = \phi(x_q)$, for $x_q \in Q$.

Remark 3.2. Consider Ω to be a solid torus ([25]), defined by

$$\Omega = \Big\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ \Big(a - \sqrt{x_1^2 + x_2^2}\Big)^2 + x_3^2 < 1\Big\}, \qquad a > 1, \tag{3.7}$$

which can be parametrized with the following toroidal coordinates (r, θ, φ):

$$x_1 = (a + r\cos\theta)\cos\varphi, \quad x_2 = (a + r\cos\theta)\sin\varphi, \quad x_3 = r\sin\theta.$$

Here, $0 \leq r \leq 1$ is the radial coordinate in the minor cross-section, $0 \leq \theta < 2\pi$ is the poloidal angle, and $0 \leq \varphi < 2\pi$ is the toroidal angle. Let e_φ be the toroidal direction with respect to the angle φ. Then, the Maxwellian of $f(x, v)$ is of the form

$$f_-(x,v) = \Big(\frac{\beta}{2\pi}\Big)^{3/2} e^{-\beta\frac{|v-u_\varphi e_\varphi|^2}{2}} e^{\beta\phi(x)},$$

for $u_\varphi = \gamma_\varphi(a + r\cos\theta)$, with γ_φ being a constant. In the case when $\gamma_\varphi \neq 0$, it is necessary that $\phi = \phi(r, \theta)$.

3.2 Arnold's nonlinear stability for electrons

In this section, we study the stability of the steady solution (F, Φ) given by

$$F(x,v) = \Big(\frac{\beta}{2\pi}\Big)^{3/2} e^{-\beta\Big(\frac{|v|^2}{2} - \Phi(x)\Big)} \tag{3.8}$$

with Φ solving the Poisson equation (3.2). We shall establish an entropic stability of the stationary solution in the sense of Arnold in his stability theory for two-dimensional Euler flows (see, for instance, [1, Section 4, Chapter 2]). We introduce the notion of relative entropy:

$$\mathcal{H}(f|F) := \iint_{\Omega\times\mathbb{R}^3} \Big[f \log\Big(\frac{f}{F}\Big) - f + F\Big](x,v)\, dxdv,$$

for measurable functions $f \geq 0$ and $F > 0$. One observes that $\mathcal{H}(f|F) = 0$ if and only if $f = F$ almost everywhere.

Theorem 3.3. *Let* (F, Φ) *be any stationary solution given by* (3.8)*. Let* (f_-, ϕ) *be any smooth solution of the Vlasov-Poisson-Boltzmann system* (3.1)-(3.2) *so that* f_- *is rapidly decaying and* $\log f_-$ *grows at most polynomially in* v *as* $|v| \to \infty$*. There holds*

$$\epsilon\frac{d}{dt}\mathcal{H}(f_-|F) + \frac{\beta}{2}\frac{d}{dt}\int_\Omega |\nabla\phi - \nabla\Phi|^2\, dx = D(f_-) \tag{3.9}$$

in which $D(f_-)$ *denotes the entropy dissipation, defined by*

$$D(f_-) := \eta_\epsilon \iint_{\Omega\times\mathbb{R}^3} Q_-(f_-)\log f_-\, dxdv \leq 0.$$

Proof. We set $f_-(t,x,v) = f(t,x,v)$. Multiplying the Vlasov equation (3.1) for electrons by $\log f$, integrating over $\Omega\times\mathbb{R}^3$, and using the specular boundary condition on f, we get

$$\epsilon\frac{d}{dt}\mathcal{H}(f) = D(f).$$

Hence, by definition,

$$\begin{aligned}\epsilon\frac{d}{dt}\mathcal{H}(f|F) - D(f) &= -\iint_{\Omega\times\mathbb{R}^3}(1+\log F)\partial_t f(x,v,t)\, dxdv\\ &= \iint_{\Omega\times\mathbb{R}^3}(1+\log F)\Big[v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f - \eta_\epsilon Q_-(f)\Big]\, dxdv\\ &= \iint_{\Omega\times\mathbb{R}^3}\Big(\mu - \beta(\frac{|v|^2}{2} - \Phi)\Big)\Big[v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f - \eta_\epsilon Q_-(f)\Big]\, dxdv,\end{aligned}$$

with $\mu = 1 + \frac{3}{2}\log(\frac{\beta}{2\pi})$, in which we have used the explicit form of F as in (3.8). Using the property of $Q_-(f)$, stated in (1.7), we have

$$\iint_{\Omega\times\mathbb{R}^3}\Big(\mu - \beta(\frac{|v|^2}{2} - \Phi)\Big)Q_-(f)\, dxdv = 0.$$

Next, integrating by parts with respect to x and v and using the specular boundary condition on f, we get

$$\begin{aligned}\epsilon\frac{d}{dt}\mathcal{H}(f|F)-D(f)&=\beta\iint_{\Omega\times\mathbb{R}^3}(\nabla_x\phi-\nabla_x\Phi)\cdot vf\;dxdv\\&=-\beta\int_\Omega(\phi-\Phi)\nabla_x\cdot\langle vf\rangle\;dx=\beta\int_\Omega(\phi-\Phi)\partial_t\langle f\rangle\;dx\\&=\beta\int_\Omega(\phi-\Phi)\partial_t\Delta\phi\;dx=-\frac{\beta}{2}\frac{d}{dt}\int_\Omega|\nabla\phi-\nabla\Phi|^2\;dx\end{aligned}\tag{3.10}$$

in which the local conservation of mass was used. This proves the theorem. □

3.3 Weak-strong property.

Theorem 3.3 in fact holds for any weak limit of smooth solutions. Precisely, we have the following corollary.

Corollary 3.4. *Let* (F,Φ) *be any stationary solution given by* (3.8)*. Let* (f_n,ϕ_n) *be a sequence of smooth solutions of the Vlasov-Poisson-Boltzmann system* (3.1)-(3.2)*, and let* (f_-,ϕ) *be any weak limit* (f_n,ϕ_n) *in the sense that* $\nabla\phi^n\rightharpoonup\nabla\phi$ *weakly in* $L^\infty(\mathbb{R}_+;L^2(\Omega))$ *and* $f_-^n\rightharpoonup f_-$ *weakly in* $L^1_{loc}(\mathbb{R}_+\times\Omega\times\mathbb{R}^3)$*. Assume that* f_- *is rapidly decaying and* $\log f_-$ *grows at most polynomially in* v *as* $|v|\to\infty$*. Then, there holds*

$$\epsilon\frac{d}{dt}\mathcal{H}(f_-|F)+\frac{\beta}{2}\frac{d}{dt}\int_\Omega|\nabla\phi-\nabla\Phi|^2\;dx\le 0.\tag{3.11}$$

Proof. The proof is identical to that of Theorem 3.3, upon noting that the smooth solutions (f_-^n,ϕ^n) satisfy

$$\iint_{\Omega\times\mathbb{R}^3}|v|^2f_-^n\;dxdv+\int_\Omega|\nabla\phi^n|^2\;dx\le C_0,$$

$$\iint_{\Omega\times\mathbb{R}^3}f_-^n\log f_-^n\;dxdv\le C_0,\qquad\sup_{(x,v)\in\Omega\times\mathbb{R}^3}|f_-^n(t,x,v)|\le C_0,$$

for some universal constant C_0, for almost everywhere $t\ge 0$. □

4 The reduced ion kinetic model

In this final section, we study the well-posedness of the reduced ions kinetic model derived in Theorem 1.1. Precisely, we consider the following system

$$\begin{aligned}&\partial_t f_+ + v\cdot\nabla_x f_+ - \nabla_x\phi\cdot\nabla_v f_+ = 0,\\&-\Delta\phi+e^{\beta(t)\phi}=n_I(x,t),\\&\int_\Omega e^{\beta(t)\phi}\;dx=\int_\Omega n_I(x,t)\;dx=m_0,\end{aligned}\tag{4.1}$$

with the boundary conditions (1.5) and (1.6), for a given positive m_0, in which $\beta(t)$ is determined through the conservation of energy

$$\frac{m_0 d}{2\beta(t)} + \frac{1}{2}\int_\Omega |\nabla\phi|^2\, dx + \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_+(x,v,t)\, dvdx = \mathcal{E}_0 \tag{4.2}$$

for some fixed $\mathcal{E}_0 > 0$. This is a weakly nonlinear modification of the Vlasov-Poisson system. The classical results there can be adapted to the above reduced ion problem. Here, Ω is either a bounded open domain or periodic box in $\mathbb{R}^d$, $d \geq 2$. In the former case, we use the specular boundary condition for f_+ and the zero Neumann boundary condition for ϕ.

4.1 Local weak solutions

We shall construct weak solutions to the reduced ion problem with finite moments. Precisely, for $k > 0$, let us introduce the k^{th}-moment

$$M_k[f_+](t) := \iint_{\Omega\times\mathbb{R}^d} |v|^k f_+(x,v,t)\, dxdv. \tag{4.3}$$

Our result in this section is as follows.

Theorem 4.1 (Existence of weak solutions)**.** *Assume that the initial data $f_{0,+} \in L^1 \cap L^\infty$ with finite moments $M_k[f_{0,+}] < \infty$, for some $k > d(d-1)$. In addition, we assume that*

$$\iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_{0,+}(x,v)dxdv \leq a\mathcal{E}_0, \quad \text{with } a < 1\,. \tag{4.4}$$

Then, there exists a time $T > 0$ so that the reduced ion problem (4.1)-(4.2) *has a local weak solution (f_+, ϕ, β), satisfying*

$$f_+ \in L^\infty([0,T]; L^1 \cap L^\infty(\Omega\times\mathbb{R}^d)), \qquad M_k[f_+] \in L^\infty([0,T]),$$

the electric field $E = -\nabla\phi \in L^\infty([0,T]; L^\infty(\Omega))$, and $\beta \in L^\infty([0,T])$.

Remark 4.2. We stress that weak solutions constructed in Theorem 4.1 conserve energy (4.2). In addition, we note that the initial density distribution needs not to be compactly supported in v.

As usual, the construction of weak solutions relies on a priori estimates, which will be derived in several steps. We first remark that if the solution f_+ satisfies

$$\iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_+(x,v,t)\, dvdx \leq \mathcal{E}_0, \qquad \iint_{\Omega\times\mathbb{R}^d} f_+(x,v,t)\, dvdx = m_0, \tag{4.5}$$

for all $t \geq 0$, it is then straightforward to check that

$$\begin{aligned}\sup_{t\geq 0} \|n_I(\cdot,t)\|_{L^{\frac{d+2}{d}}(\Omega)} &\leq 2^{\frac{d+2}{2}} |\mathbb{S}^{d-1}|^{\frac{d+2}{d}} \|f_+\|_{L^\infty(\Omega\times\mathbb{R}^d)}^{\frac{2}{d+2}} \left(\iint_{\Omega\times\mathbb{R}^d} |v|^2 |f_+(x,v,t)\, dvdx\right)^{\frac{d}{d+2}} \\ &\leq C_0.\end{aligned}$$

4.1.1 A priori bound on $\beta(t)$

In view of (4.2), we observe that $\beta(t)$ is bounded below from zero. The fact that $\beta(t)$ also bounded from above follows from the next lemma.

Lemma 4.3. *For (β, ϕ, f_+) solution of the ion problem (4.1), the conservation of energy (4.2) is equivalent to the following relation:*

$$\beta(t) = \exp \frac{1}{m_0 d}\Big(C_0 - 2\int_\Omega \beta(t)\phi(x,t)e^{\beta(t)\phi(x,t)}dx\Big) \tag{4.6}$$

in which $C_0 := m_0 d \log \beta(0) + 2\int_\Omega \beta(0)\phi(x,0)e^{\beta(0)\phi(x,0)}\,dx$.

Corollary 4.4. *For (β, ϕ, f_+) solution of the ion problem, $\beta(t)$ is uniformly bounded; precisely,*

$$0 < \frac{m_0 d}{2\mathcal{E}_0} \le \beta(t) \le e^{\frac{1}{m_0 d}(C_0 + 2|\Omega| e^{-1})} < \infty. \tag{4.7}$$

Proof. The lower bound in the estimate (4.7) is a direct consequence of (4.2), whereas the upper bound follows from (4.6) with the estimate:

$$-2\int_\Omega (\beta\phi)e^{\beta\phi}\,dx \le -2\int_{\Omega\cap\{\beta\phi<0\}} (\beta\phi)e^{\beta\phi}\,dx + C_0 \le 2e^{-1}|\Omega| + C_0. \tag{4.8}$$

Given Lemma 4.3, the corollary is proved. □

Proof of Lemma 4.3. The existence and uniqueness of $(\beta(t), \phi(t))$ given $f_+(t)$ (in particular for $t = 0$) is proven in Theorem 1.4. To prove (4.6), we compute

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\phi|^2\,dx = -\int_\Omega \phi\Delta\phi_t\,dx = \int_\Omega \phi\partial_t n_I\,dx - \int_\Omega \phi\partial_t e^{\beta\phi}\,dx.$$

On the other hand, using the local conservation of mass, we have

$$\int_\Omega \phi\partial_t n_I\,dx = -\int_\Omega \phi\nabla\cdot(n_I u_I)\,dx = -\int_\Omega E\cdot n_I u_I\,dx.$$

Similarly, we compute

$$\frac{d}{dt}\iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_+(x,v,t)\,dvdx = \int_\Omega E\cdot n_I u_I\,dx.$$

This yields

$$\begin{aligned}
&\frac{d}{dt}\iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_+(x,v,t)\,dvdx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\phi|^2\,dx \\
&\quad = -\int_\Omega \phi\partial_t e^{\beta\phi} = -\frac{1}{\beta}\partial_t \int_\Omega (\beta\phi - 1)e^{\beta\phi}\,dx \\
&\quad = -\frac{1}{\beta}\partial_t \int_\Omega \beta\phi e^{\beta\phi}\,dx
\end{aligned}$$

in which the last equality is due to the conservation of mass. The constraint (4.2) now reads

$$-\frac{m_0 d}{2\beta^2}\partial_t\beta - \frac{1}{\beta}\partial_t \int_\Omega \beta\phi e^{\beta\phi}\, dx = 0\,. \tag{4.9}$$

Or equivalently,

$$m_0 d \log\beta + \int_\Omega \beta\phi e^{\beta\phi}\, dx = C_0, \qquad \forall t \geq 0. \tag{4.10}$$

and then (4.6) follows by integration. □

4.1.2 Bounds on the electric field

Let f_+ satisfy (4.5). We start with a priori estimates to the following elliptic problem

$$\begin{aligned} &-\Delta_x\phi + e^{\beta(t)\phi} = n_I(x,t), \qquad \int_\Omega e^{\beta(t)\phi}\, dx = m_0, \\ &\text{with} \quad \partial_n\phi_{|\partial\Omega} = 0 \quad \text{whenever} \quad \partial\Omega \neq \emptyset \end{aligned} \tag{4.11}$$

with the constraint (4.2). For any $p \geq 1$, multiplying the elliptic equation by $e^{(p-1)\beta(t)\phi}$, and integrating by parts, we get

$$(p-1)\beta(t)\int_\Omega e^{(p-1)\beta(t)\phi}|\nabla_x\phi|^2\, dx + \int_\Omega e^{p\beta(t)\phi}\, dx \leq \|n_I(\cdot,t)\|_{L^p}\|e^{p\beta(t)\phi}\|_{L^1}^{\frac{p-1}{p}}$$

which implies

$$\|e^{\beta(t)\phi}\|_{L^p} \leq \|n_I(\cdot,t)\|_{L^p},, \qquad \forall p \in [1,\infty[\tag{4.12}$$

uniformly in $t \geq 0$ and $p \geq 1$. Eventually by taking $p \to \infty$ in the above inequality, we have also

$$\|e^{\beta(t)\phi(\cdot,t)}\|_{L^p} \leq \|n_I(\cdot,t)\|_{L^p}, \qquad \forall p \in [1,\infty], \tag{4.13}$$

uniformly in $t \geq 0$ and in $\beta(t)$, as long as the right hand side is finite. This yields

$$-\Delta_x\phi = n_I - e^{\beta\phi} \in L^{\frac{d+2}{d}}(\Omega).$$

The standard elliptic problem then yields $\phi \in W^{2,\frac{d+2}{d}}$, whose norm is uniformly bounded in time. In particular, by Sobolev embedding, ϕ is uniformly bounded, for $d = 2$ or 3.

We now write the solution to the elliptic problem as

$$\phi = \int_\Omega K(x,y)\Big[n_I(y,t) - e^{\beta(t)\phi(y,t)}\Big]\, dy \tag{4.14}$$

in which $K(x,y)$ denotes the Green kernel of the Laplacian on Ω with the Neumann boundary condition or periodic boundary condition. It is classical that

$$|\partial_x^k K(x,y)| \leq C_0|x-y|^{2-d-k}, \qquad k \geq 0 \tag{4.15}$$

for $d \geq 3$. For $d = 2$, $K(x,y)$ is of order of $\log|x-y|$.

Lemma 4.5. *Let $k > d^2 - d$. With $E = -\nabla\phi$, there hold*

$$\|E(\cdot,t)\|_{L^\infty} \le C_k \|n_I(\cdot,t)\|_{L^1}^{1-\frac{d+k}{k}\frac{d-1}{d}} \|n_I(\cdot,t)\|_{L^{\frac{d+k}{d}}}^{\frac{d+k}{k}\frac{d-1}{d}}$$

uniformly in $t \ge 0$.

Proof. The proof is straightforward, using (4.14) and (4.15). Indeed, setting $\widetilde{n}_I = n_I(y,t) - e^{\beta(t)\phi(y,t)}$, we estimate

$$\begin{aligned}|E(x,t)| &\le C_0 \int_\Omega |x-y|^{1-d} |\widetilde{n}_I(y)|\, dy \\ &\le C_0 \int_{\{|x-y|\ge A\}} |x-y|^{1-d}|\widetilde{n}_I(y)|\, dy + C_0 \int_{\{|x-y|\le A\}} |x-y|^{1-d}|\widetilde{n}_I(y)|\, dy \\ &\le C_0 A^{1-d} \|\widetilde{n}_I\|_{L^1} + C_0 A^{1-d+\frac{dk}{d+k}} \|\widetilde{n}_I\|_{L^{\frac{d+k}{d}}}.\end{aligned}$$

The lemma follows at once from optimizing the constant A and using (4.13). □

In addition, we have the following.

Lemma 4.6. *Define the k^{th}-moment as in* (4.3)*. There holds*

$$\|n_I(\cdot,t)\|_{L^{\frac{d+k}{d}}} \le C_k M_k[f_+](t)^{\frac{d}{d+k}}. \tag{4.16}$$

Proof. By definition, we write

$$\begin{aligned}n_I(x,t) &= \int_{\mathbb{R}^d} f_+(x,v,t)\, dv = \int_{\{|v|\le A\}} f_+(x,v,t)\, dv + \int_{\{|v|\ge A\}} f_+(x,v,t)\, dv \\ &\le A^d \|f_+(t)\|_{L^\infty} + A^{-k} \int_{\mathbb{R}^d} |v|^k f_+(x,v,t)\, dv.\end{aligned}$$

The lemma follows from optimizing the constant A. □

Combining Lemmas 4.5 and (4.6), together with the mass conservation, we obtain

Corollary 4.7. *For $k > d^2 - d$, there holds*

$$\|E(\cdot,t)\|_{L^\infty} \le C_k M_k[f_+](t)^{\frac{d-1}{k}}, \qquad \forall\ t \ge 0.$$

4.1.3 A priori bounds on moments

Given the field $E(x,t)$, starting from $(x,v) \in \Omega \times \mathbb{R}^d$, the particle trajectories $(X(t), V(t))$ are defined by the ODEs

$$\dot{X} = V, \qquad \dot{V} = E(X(t),t)$$

as long as $X(t)$ remains in the interior of Ω. In the case Ω has a boundary, we let t_0 be the positive time when $X(t_0)$ hits the boundary, that is $X(t_0) \in \partial\Omega$. The trajectory is then continued by the ODE dynamics, with the new "initial" condition:

$$X(t_0) = \lim_{t \to t_0^-} X(t), \qquad V(t_0) := \lim_{t \to t_0^-} \Big[V(t) - 2(V(t) \cdot n(X(t)))n(X(t)) \Big],$$

which of course correspond to the specular boundary condition of particles, and so on, in case of multiple reflections. The backward trajectory $(X(t), V(t))$ is defined in the similar way, for $0 < t < t_0$.

Then, the solution f_+ to the Vlasov equation is constructed through

$$f_+(x, v, t) = f_{0,+}(X(-t), V(-t)), \qquad \forall t \geq 0, \quad \forall (x, v) \in \Omega \times \mathbb{R}^d, \tag{4.17}$$

with $(X(0), V(0)) = (x, v)$. By definition, as long as $X(t) \in \Omega$, there holds

$$\frac{d}{dt}|V|^2 = 2E \cdot V.$$

When $X(t)$ meets $\partial\Omega$, $|V(t)|$ is conserved under the specular reflection. Hence, for all $(x, v) \in \Omega \times \mathbb{R}^d$, we have

$$|V(t)| \leq |v| + \int_0^{|t|} \|E(s)\|_{L^\infty} \, ds \tag{4.18}$$

for all $t \in \mathbb{R}$.

Now using the characteristic equation (4.17) and estimate (4.18), we can compute the k^{th}-moment of f_+, yielding

$$\begin{aligned} M_k[f_+](t) &= \iint_{\Omega \times \mathbb{R}^d} |v|^k f_+(x, v, t) \, dxdv \\ &= \iint_{\Omega \times \mathbb{R}^d} |V(-t)|^k f_{0,+}(X(-t), V(-t)) \, dxdv \\ &\leq C_0 \iint_{\Omega \times \mathbb{R}^d} \Big[|v|^k + \Big(\int_0^t \|E(s)\|_{L^\infty} \, ds \Big)^k \Big] f_{0,+}(X(-t), V(-t)) \, dxdv \\ &\leq C_0 M_k[f_{0,+}] + C_0 \Big(\int_0^t \|E(s)\|_{L^\infty} \, ds \Big)^k. \end{aligned} \tag{4.19}$$

Together with Corollary (4.7), we obtain

$$M_k[f_+](t) \leq C_0 M_k[f_{0,+}] + C_k \Big(\int_0^t M_k[f_+](s)^{\frac{d-1}{k}} \, ds \Big)^k. \tag{4.20}$$

Hence, the Gronwall's inequality gives

$$M_k[f_+](t) \leq C_T \tag{4.21}$$

for all $t \in [0, T]$, for some positive T. In particular, T can be arbitrarily large in the two dimensional case. This yields the boundedness of $E(x, t)$ on $[0, T]$, thanks to Corollary 4.7.

4.1.4 Averaging lemma

In the sequel, we shall need a priori compactness on the average of f_+. Let us write the Vlasov equation as

$$\partial_t f_+ + v \cdot \nabla_x f_+ = -\nabla_v \cdot (E f_+).$$

Here, from the apriori estimates, $E \in L^\infty$ and $f \in L^1 \cap L^\infty$. Hence,

$$\|f_+\|^2_{L^2(0,T;L^2(\Omega\times\mathbb{R}^3))} \le \|f_+\|_{L^\infty} \|f_+\|_{L^1(0,T;L^1(\Omega\times\mathbb{R}^3))} \le \|f_+\|_{L^\infty} \|n_I\|_{L^1((0,T)\times\Omega)}$$

and

$$\|E f_+\|_{L^2(0,T;L^2(\Omega\times\mathbb{R}^3))} \le \|E\|_{L^\infty} \|f_+\|_{L^2(0,T;L^2(\Omega\times\mathbb{R}^3))}.$$

It then follows from the classical averaging lemma (for instance, see [13, Theorem 7.2.1] or [12, Theorem 5]) that

$$\int_{\mathbb{R}^3} f_+(x,v,t)\varphi(v)\, dv \in H^{1/4}((0,T)\times\Omega)$$

together with the uniform bound

$$\Big\| \int_{\mathbb{R}^3} f_+(\cdot,\cdot,v)\varphi(v)\, dv \Big\|_{H^{1/4}((0,T)\times\Omega)} \le C_\varphi \|E\|_{L^\infty} \|f_+\|_{L^2(0,T;L^2(\Omega\times\mathbb{R}^3))}$$

for any v-compactly supported test function $\varphi(v)$ in $C^\infty(\mathbb{R}^3)$.

4.1.5 Local well-posedness

The existence of local solutions to the ion problem (4.1) now follows with minor modifications the standard iteration procedure. Indeed, we construct (β_n, ϕ_n, f_n) as follows. Let $f_{0,+} \in (L^\infty \cap L^1)(\Omega \times \mathbb{R}^d)$ be any initial data that have finite moments $M_k[f_{0,+}]$ and satisfy

$$\iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_{0,+}(x,v)\, dvdx \le a\mathcal{E}_0, \qquad \iint_{\Omega\times\mathbb{R}^d} f_{0,+}(x,v)\, dvdx = m_0$$

for some $a < 1$ (cf. (4.4)). Set $f_0(x,v,t) = f_{0,+}(x,v)$. We start the iteration with $n = 0$. We denote in the sequel $\rho_n(x,t) = \langle f_n(x,.,t)\rangle$.

- We will construct the unique solution (β_n, ϕ_n) to the elliptic problem

$$\begin{gathered} -\Delta\phi_n + e^{\beta_n\phi_n} = \rho_n, \qquad \int_\Omega e^{\beta_n\phi_n}\, dx = m_0, \\ \frac{m_0 d}{2\beta_n} + \frac{1}{2}\int_\Omega |\nabla\phi_n|^2\, dx = \mathcal{E}_0 - \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_n(x,v,t)\, dvdx. \end{gathered} \tag{4.22}$$

- Then we will construct f_{n+1} by solving the linearized Vlasov equation

$$\partial_t f_{n+1} + v\cdot\nabla_x f_{n+1} - \nabla_x\phi_n \cdot \nabla_v f_{n+1} = 0 \tag{4.23}$$

  with the same initial data $f_{n+1}(x,v,0) = f_{0,+}(x,v)$.

However to solve the elliptic problem (4.22) one needs to ensure that the quantity

$$\mathcal{E}_n(t) = \mathcal{E}_0 - \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_n(x,v,t)\, dvdx. \tag{4.24}$$

remains strictly positive. For a genuine solution this follows obviously from the energy conservation (4.2) and on the uniform bound (4.7), but for a iterative solution, this requires some extra argument. By iteration a sequence of decreasing positive times $0 < T_n$ is introduced. They are characterized by the fact that $\mathcal{E}_n(t)$ is strictly positive for $0 < t < T_n$. Hence on such interval the solution of (4.22) is well defined. On any such interval, bounds for (f_n, ϕ_n, β_n) are derived uniformly in n. Hence, it is shown (cf. Lemma 4.8, below) that

$$T_- = \inf T_n \tag{4.25}$$

is a strictly positive number which depends only on the properties of the data at $t = 0$.

For the n-uniform bound, using the bound (4.20) and Corollary 4.7 to the above iterative scheme, we obtain

$$\begin{aligned} M_k[f_{n+1}](t) &\le C_0 + C_0\Big(\int_0^t \|E_n(\cdot,s)\|_{L^\infty}\, ds\Big)^k \\ &\le C_0 + C_0\Big(\int_0^t M_k[f_n](s)^{\frac{d-1}{k}}\, ds\Big)^k \end{aligned} \tag{4.26}$$

for all $n \ge 0$. By iteration and the previous estimates, this proves that

$$M_k[f_n](t) \le C(t), \qquad \|E_n(\cdot,t)\|_{L^\infty} \le C(t), \qquad \beta_n(t) \le C(t), \tag{4.27}$$

uniformly in n, for all positive time t $(d = 2)$, and for $t \in [0,T]$ for some positive time T $(d \ge 3)$. Here, $C(t)$ denotes some continuous function in t.

Eventually, with $C_T = \sup_{0<t<T} C(t)$, the above estimates can be used to prove the following.

Lemma 4.8. *1. For any $f_{n+1}(x,v,t)$ one has, for $0 < t < T$, the estimate:*

$$\int_\Omega \Big\langle \frac{|v|^2}{2} f_{n+1}(t)\Big\rangle dx \le \left(2C_T^{\frac{3}{2}} t + \left(\int_\Omega \Big\langle \frac{|v|^2}{2} f_{n+1}(0)\Big\rangle dx\right)^{\frac{1}{2}}\right)^2. \tag{4.28}$$

2. As long as t is small enough to satisfy the relation

$$\left(2C_T^{\frac{3}{2}} t + (a\mathcal{E}_0)^{\frac{1}{2}}\right)^2 < \mathcal{E}_0 \tag{4.29}$$

in which $a > 1$ is given by (4.4), the expression:

$$\mathcal{E}_0 - \int_\Omega \Big\langle \frac{|v|^2}{2} f_{n+1}(t)\Big\rangle dx$$

remains strictly positive.

Proof. From the equation (4.23), one deduces the following usual relation:

$$\frac{d}{dt}\int_\Omega \left\langle \frac{|v|^2}{2} f_{n+1} \right\rangle dx = \int_\Omega \nabla_x \phi_n \cdot \langle v f_{n+1} \rangle dx \,. \tag{4.30}$$

Therefore, together with the Cauchy-Schwarz's inequality and (4.27), one has the following estimate:

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \left\langle \frac{|v|^2}{2} f_{n+1} \right\rangle dx \le \int_{\Omega\times\mathbb{R}^d} |\nabla_x \phi_n(x)||v f_{n+1}| dxdv \\ &\le \left(\int_{\Omega\times\mathbb{R}^d} |\nabla_x \phi_n(x)|^2 f_{n+1} dxdv\right)^{\frac12} \left(\int_\Omega \left\langle \frac{|v|^2}{2} f_{n+1} \right\rangle dx\right)^{\frac12} \\ &\le (C(t)\int_\Omega |\nabla_x \phi_n(x)|^2 dx)^{\frac12} (\int_\Omega \langle \frac{|v|^2}{2} f_{n+1} \rangle dx)^{\frac12} \\ &\le C_T (\int_\Omega \langle \frac{|v|^2}{2} f_{n+1} \rangle dx)^{\frac12} \text{ for } t\in[0,T]\,. \end{aligned} \tag{4.31}$$

Hence, (4.28) follows by integration. The second statement is a direct consequence of the first. It is important to observe that the estimates involve only the quantity C_T, which has been globally evaluated. □

Now we can consider the convergence of the sequence (f_n, ϕ_n, β_n). Up to a subtraction of subsequences, $f_n \rightharpoonup f$ in $L^p(\Omega\times\mathbb{R}^d)$, $E_n \rightharpoonup E$ in $L^p(\Omega)$, and $\beta_n(t) \to \beta(t)$ for almost every where $t \in [0,T]$. By view of the elliptic problem for ϕ_n, we in fact have $E_n = -\nabla\phi_n \in L^\infty(0,T;W^{1,p}(\Omega))$ for all $p \ge 1$.

To gain regularity in time, we use the averaging lemma and we get

$$\Big\| \int_{\mathbb{R}^3} f_n(\cdot,\cdot,v)\varphi(v)\, dv \Big\|_{H^{1/4}((0,T)\times\Omega)} \le C_\psi \|E_n\|_{L^\infty} \|f_n\|_{L^2(0,T;L^2(\Omega\times\mathbb{R}^3))} \le C_T C_\varphi \tag{4.32}$$

for any test function $\varphi \in C_c^\infty(R^3)$, the set of C^∞ functions of v which are compactly supported in v. Now we can pass to the limit as $n\to\infty$. By fixing a test function of the form $\theta(t,x)\psi(v)$ with $\psi \in C_c^\infty(R^3)$, we get

$$\begin{aligned} &\int_0^T \iint_{\Omega\times\mathbb{R}^3} \nabla_v \cdot ((\nabla_x\phi_n) f_{n+1}(t,x,v))\theta(t,x)\psi(v)\, dxdvdt \\ &\quad = -\int_0^T \int_\Omega \nabla_x \phi_n \theta(t,x) \cdot \Big(\int_{\mathbb{R}^3} f_{n+1}(t,x,v)\nabla_v\psi\, dv\Big)\, dxdt \\ &\quad \to -\int_0^T \int_\Omega \nabla_x \phi \theta(t,x) \cdot \Big(\int_{\mathbb{R}^3} f(t,x,v)\nabla_v\psi\, dv\Big)\, dxdt \end{aligned}$$

as $n\to\infty$.

Similarly for the transport operator $\partial_t f_n + v\cdot\nabla_x f_n$, we obtain in the limit as $n\to\infty$ the Vlasov equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = 0$$

in the weak sense. Now, we consider the elliptic problem

$$-\Delta\phi_n + e^{\beta_n\phi_n} = \rho_n(t)$$

together with the energy conservation

$$\frac{m_0 d}{2\beta_n} + \frac{1}{2}\|\nabla\phi_n\|^2 = \mathcal{E}_n = \mathcal{E}_0 - \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_n(x,v,t)\, dvdx.$$

By (4.27), the moments $M_k[f_n(t)]$ are uniformely bounded for $k = 1, 2, 3$. Hence, there exists a constant $\overline{M}_T$ such that for $t \le T$, we get

$$\iint \langle f_n(t,x)|v|\rangle\, dxdv \le \overline{M}_T, \qquad \iint \left\langle f_n(t,x)|v|^3\right\rangle dxdv \le \overline{M}_T.$$

Denote $\zeta_R \in C_c^\infty(R^3)$ a function equal to 1 in the ball $\{|v| \le R\}$ and χ_R the indicatrix of the complementary of this ball. We can see that

$$\|\langle f_n\zeta_R\rangle - \rho_n\|_{L^1_{t,x}} \le \iiint f_n(t,x,v)\chi_R dvdxdt \le \frac{1}{R}\iint \langle f_n(t,x)|v|\rangle dxdt \le \frac{T}{R}\overline{M}_T.$$

Now for each R fixed, we can apply the compactness property (4.32) for $\langle f_n\zeta_R\rangle$ in $L^2_{t,x}$, then up to a subtraction of a subsequence $\langle f_n\zeta_R\rangle$ converges in $L^2_{t,x}$. In particular, the subsequence $\langle f_n\zeta_R\rangle$ is a Cauchy sequence in $L^2_{t,x}$ and in $L^1_{t,x}$, that is to say $\|\langle f_n\zeta_R\rangle - \langle f_m\zeta_R\rangle\|_{L^1_{t,x}}$ is small enough if n, m are large enough. Next, we check that ρ_n is also a Cauchy sequence in $L^1_{t,x}$. Indeed, for a given ϵ, we have

$$\|\rho_n - \rho_m\|_{L^1_{t,x}} \le \|\langle f_n\zeta_R\rangle - \langle f_m\zeta_R\rangle\|_{L^1_{t,x}} + \epsilon$$

for $R = 2T\overline{M}_T/\epsilon$. In addition, by taking m, n sufficiently large, $\|\langle f_n\zeta_R\rangle - \langle f_m\zeta_R\rangle\|_{L^1_{t,x}} \le \epsilon$. Thus, ρ_n is a Cauchy sequence in $L^1_{t,x}$, and there exists some function ρ in $L^1_{t,x}$ so that

$$\rho_n \to \rho \qquad \text{in } L^1_{t,x} \qquad \text{and almost everywhere in } \ t,x.$$

In particular, for almost every t, we see that $\rho_n(t,.)$ converges pointwise to $\rho(t,.)$. But by Lemma 4.6 and the estimate (4.27), we know that there exists a constant C such that $\|\rho_n(t,.)\|_{L^2_x} \le C$. Then $\rho(t,.)$ is also in L^2_x and for almost all t, we have

$$\rho_n(t,.) \to \rho(t,.) \qquad \text{in } L^2_x$$

By the same argument for functions $\langle f_n(t,x)|v|^2\rangle$, we can check that there exist a function $Z(t,x)$ such that

$$\langle f_n(t,x)|v|^2\rangle \to Z \qquad \text{in } L^1_{t,x} \qquad \text{and almost everywhere in } t,x\,.$$

We can check also that for almost all time $t \in [0,T]$ we have

$$\rho(t,x) = \langle f(x)\rangle \ \text{ and } Z(t,x) = \left\langle f(t,x)|v|^2\right\rangle\,.$$

Then for almost all time t, we have

$$\int \langle f_n(t)|v|^2\rangle dx \to \int \langle f(t)|v|^2\rangle dx$$

Therefore $\rho_n(t)$ and $\mathcal{E}_n(t)$ converges pointwise in time to $\rho(t)$ and $\mathcal{E}(t) = \mathcal{E}_0 - \frac{1}{2}\|Z(t)\|_{L^1_x}$.

Now, the above elliptic problem has data $(\rho_n(t), \mathcal{E}_n)$ which converges for each fixed time t. Therefore the corresponding solution ϕ_n and β_n are bounded in $H^2(\Omega)$ and $\mathbb{R}_+$. Then, up to a subtraction of subsequences, they converge strongly to $\phi(t,x)$ and $\beta(t)$ in $H^1(\Omega)$ and $\mathbb{R}$, respectively.

Finally, for each time t, $(\beta(t), \phi(x,t))$ solves

$$-\Delta\phi + e^{\beta\phi} = \rho(t)$$
$$\frac{m_0 d}{2\beta} + \frac{1}{2}\int_\Omega |\nabla\phi|^2\, dx = \mathcal{E}(t) = \mathcal{E}_0 - \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f(x,v,t)\, dvdx.$$

Now by uniqueness of the above elliptic problem, (β, ϕ) is thus a solution to the reduced ion problem. This yields a local solution.

Remark 4.9. The use of the averaging lemma in the present proof seems to be an "overkill", since usually time regularity in a "weak space" is deduced from the equations and the Aubin-Lions theorem can be used. However in the present case the time regularity is obtained for $\rho_n(t)$ and $\langle \frac{|v|^2}{2} f_n(x,v,t)\rangle$, which is sufficient for the almost everywhere point wise convergence of $(\beta_n(t), \phi_n(x,t))$. Since the mapping $(\rho_n(t), \langle \frac{|v|^2}{2} f_n(x,v,t)\rangle) \to (\beta_n(t), \phi_n(x,t))$ is non linear and not explicit, the use of the above averaging lemma to obtain the almost everywhere convergence seems to be the simpler approach.

4.2 Global weak solutions

Theorem 4.10. *The local weak solution to the reduced ion problem* (4.1)-(4.2) *can be extended globally in time, provided any of the followings:*

- $d = 2$*; that is,* $\Omega \subset \mathbb{R}^2$.
- $\Omega = \mathbb{T}^3$*; that is,* Ω *is a periodic box.*

Proof. In the two dimensional case, since the linear Gronwall inequality yields the uniform bound (4.27) for all time t, the theorem follows at once in this case.

Next, we consider the case when $\Omega = \mathbb{T}^3$. In view of (4.19), it suffices to prove

$$\int_0^t \|E(X(s),s)\|_{L^\infty}\, ds \le C_0\Big(M_k[f_+](t)\Big)^{\frac{\alpha}{k}} + C(t) \tag{4.33}$$

for some positive constant $\alpha \le 1$. The boundedness of $M_k[f_+](t)$ and hence $E(x,t)$ then follow. To this end, let us write the Poisson equation as

$$-\Delta\phi = n_I - e^{\beta\phi}$$

and hence,

$$E(x,t) = -\int_\Omega \nabla_x K(x,y)\Big[n_I(y,t) - e^{\beta(t)\phi(y,t)}\Big]\,dy$$
$$=: \widetilde{E}(x,t) + \widehat{E}(x,t).$$

Since $e^{\beta(t)\phi(y,t)}$ is bounded, $\widehat{E}(t)$ is uniformly bounded. As for $\widetilde{E}(x,t)$, using (4.15), we bound

$$|\widetilde{E}(x,t)| \le C_0 \iint_{\mathbb{T}^3\times\mathbb{R}^3} \frac{f_+(y,w,t)\,dydw}{|x-y|^2}.$$

Now following word by word the proof of Theorem 5 in [27], which uses only the boundedness of f_+ and of the total kinetic energy of f_+, we obtain (4.33) for $\widetilde{E}(x,t)$, with $k > 14/3$.

This completes the proof of Theorem 4.1. □

Remark 4.11. We note that since the spatial domain is either bounded or periodic, no dispersion effect can be used (as done in [24]). As a consequence, Lemma 5 in [27] is proved to be crucial in the periodic setting.

4.3 Uniqueness

In this section, we establish the uniqueness of solutions of the ion problem, under an additional regularity assumption. Precisely, we prove the following theorem.

Theorem 4.12 (Uniqueness)**.** *For $T > 0$, there exists at most one weak solution (f_+, ϕ, β) defined on the time interval $[0,T]$ in the sense of Theorem 4.1 to the reduced ion problem with v-compactly supported initial data $f_{0,+}$, provided that*

$$\sup_{t\in[0,T]}\sup_{x\in\Omega} \|\nabla_v f_+\|_{L^2(\mathbb{R}^d)} < \infty. \tag{4.34}$$

The regularity assumption (4.34) will be verified in the next section in the case when $\Omega = \mathbb{T}^d$, whereas the compact support assumption on the initial data can easily be replaced by the finite moment assumption as done in Theorem 4.1. To proceed with the proof of Theorem 4.12, let (β_1, ϕ_1, f_1) and (β_2, ϕ_2, f_2) be the two solutions to (4.1) and (4.2), with the same compactly supported initial data f_0. By construction of weak solutions in Theorem 4.1, we have

$$\int_0^T \|E_j(s,\cdot)\|_{L^\infty}\,ds < \infty \tag{4.35}$$

and

$$\frac{m_0 d}{2\beta_j(t)} + \frac{1}{2}\int_\Omega |\nabla\phi_j|^2\,dx + \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_j(x,v,t)\,dvdx = \mathcal{E}_0 \tag{4.36}$$

for $j = 1,2$, and for the same energy constant $\mathcal{E}_0$. In addition, from the identity (4.10), $\beta_j(t)$ remains bounded. As a consequence of (4.18) and (4.35), the velocity support of $f_j(x,v,t)$ is bounded, for $j = 1,2$.

For convenience, let us denote

$$\beta = \beta_1 - \beta_2, \qquad \phi = \phi_1 - \phi_2, \qquad f = f_1 - f_2,$$

and set

$$y(t) = \iint_{\Omega\times\mathbb{R}^d} |f(x,v,t)|^2 \, dxdv.$$

We shall prove that $f = 0$. The uniqueness theorem then follows, upon using the uniqueness of (β,ϕ) to the elliptic problem, given a fixed ions density; see Theorem 1.4. In the other words, it suffices to prove the following proposition.

Proposition 4.13. *There holds*

$$\frac{d}{dt}y(t) \le C_0\Big(y(t) + y(t)^2\Big).$$

Proof. First, the difference $f = f_1 - f_2$ solves the following Vlasov equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x(\phi_1 + \phi)\cdot\nabla_v f = \nabla_x\phi\cdot\nabla_v f_1.$$

By assumption that $\sup_{x,t}\|\nabla_v f_1\|_{L^2(\mathbb{R}^d)} < \infty$, the standard energy estimate yields

$$\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2 \le C_0\Big(\|f\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\Big), \tag{4.37}$$

for some universal constant C_0 that depends on $\sup_{x,t}\|\nabla_v f_1\|_{L^2(\mathbb{R}^d)}$.

Next, we use the Poisson equation for ϕ, which now reads

$$-\Delta_x\phi + e^{\beta_1\phi_1} - e^{\beta_2\phi_2} = \rho = \int_{\mathbb{R}^d} f(x,v,t)\, dv. \tag{4.38}$$

We write

$$e^{\beta_1\phi_1} - e^{\beta_2\phi_2} = e^{\beta_1\phi_1} - e^{\beta_1\phi_2} + e^{\beta_1\phi_2} - e^{\beta_2\phi_2}$$

and use the fact that $|x-y|^{p-2}(e^x - e^y)(x-y) \ge \theta_0|x-y|^p$, for all x,y in a compact set and all $p>1$. Noting that β_j, ϕ_j are uniformly bounded and multiplying the elliptic equation by $|\phi|^{p-2}\phi$, we easily obtain

$$\|\phi\|_{L^p} \le C_0\Big(\beta + \|\rho\|_{L^p}\Big), \qquad \forall\ p > 1. \tag{4.39}$$

To obtain a better estimate, we write

$$\begin{aligned} e^{\beta_1\phi_1} - e^{\beta_2\phi_2} &= e^{\beta_1\phi_1}\Big(1 - e^{\beta_2\phi_2 - \beta_1\phi_1}\Big) \\ &= e^{\beta_1\phi_1}\Big(\beta_1\phi_1 - \beta_2\phi_2 + R_{\beta,\phi}\Big) \end{aligned}$$

in which $R_{\beta,\phi} = \mathcal{O}(|\beta_1-\beta_2|^2 + |\phi_1-\phi_2|^2)$. We further write

$$e^{\beta_1\phi_1} - e^{\beta_2\phi_2} = \frac{1}{2}e^{\beta_1\phi_1}\Big((\beta_1+\beta_2)(\phi_1-\phi_2) + (\beta_1-\beta_2)(\phi_1+\phi_2) + 2R_{\beta,\phi}\Big)$$

We next multiply the elliptic equation (4.38) by $-2e^{-\beta_1\phi_1}\Delta\phi$, upon using the above identity and recalling that $\phi = \phi_1 - \phi_2$ and $\beta = \beta_1 - \beta_2$, we obtain

$$\int_\Omega \Big[2e^{-\beta_1\phi_1}|\Delta\phi|^2 + (\beta_1+\beta_2)|\nabla\phi|^2 - \beta(\phi_1+\phi_2)\Delta\phi - R_{\beta,\phi}\Delta\phi\Big] = -2\int_\Omega \rho e^{-\beta_1\phi_1}\Delta\phi.$$

Together with the Young's inequality, this yields

$$\begin{aligned}\int_\Omega &\Big[e^{-\beta_1\phi_1}|\Delta\phi|^2 + (\beta_1+\beta_2)|\nabla\phi|^2 - \beta(\phi_1+\phi_2)\Delta\phi\Big] \\ &\le C_0\Big(|\beta|^4 + \int_\Omega (|\phi|^4 + |\rho|^2)\Big)\end{aligned} \tag{4.40}$$

in which the bound on remainder $R_{\beta,\phi} = \mathcal{O}(|\beta|^2 + |\phi|^2)$ was used.

We now use the fact that the energy for the two solutions are the same; see (4.36). Subtracting one to another, we get the conservation of the energy

$$\frac{m_0 d(\beta_2-\beta_1)}{2\beta_1\beta_2} + \frac{1}{2}\int_\Omega (|\nabla\phi_1|^2 - |\nabla\phi_2|^2)\,dx + \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2}(f_1 - f_2)\,dvdx = 0.$$

Recalling $\phi = \phi_1 - \phi_2$ and $\beta = \beta_1 - \beta_2$, we multiply the above by -2β and note that the middle term can be written as

$$\frac{1}{2}\int_\Omega (|\nabla\phi_1|^2 - |\nabla\phi_2|^2) = -\frac{1}{2}\int_\Omega (\phi_1+\phi_2)\Delta(\phi_1-\phi_2) = -\frac{1}{2}\int_\Omega (\phi_1+\phi_2)\Delta\phi.$$

We get

$$\frac{2m_0 d\beta^2}{2\beta_1\beta_2} + \beta\int_\Omega (\phi_1+\phi_2)\Delta\phi = 2\beta\iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f\,dvdx. \tag{4.41}$$

Here in (4.41) we note that the kinetic energy is bounded by $\|f\|_{L^2}$, since f is compactly supported in v. Adding (4.40) and (4.41) together and recalling that β_j are bounded below away from zero, we obtain at once

$$|\beta|^2 + \|\nabla\phi\|_{L^2}^2 + \|\Delta\phi\|_{L^2}^2 \le C_0\Big(|\beta|^4 + \|\phi\|_{L^4}^4 + \|f\|_{L^2}^2\Big). \tag{4.42}$$

Now using the L^p bound (4.39), with $p = 4$, and recalling that f is compactly supported, we obtain from the previous estimate

$$|\beta|^2 + \|\nabla\phi\|_{L^2}^2 + \|\Delta\phi\|_{L^2}^2 \le C_0\Big(|\beta|^4 + \|f\|_{L^2}^2 + \|f\|_{L^2}^4\Big). \tag{4.43}$$

It remains to take care of $|\beta|^4$ on the right-hand side. To this end, we shall prove that $\beta_j(t)$ is continuous in time. It suffices to show the continuity of β_1. Indeed, we note that f_1 is continuous in time, since f_1 is a C^1 function with respect to x, v, and

$$\partial_t f_1 = -v\cdot\nabla_x f_1 + \nabla_x\phi_1\cdot\nabla_v f_1.$$

Now we fix f_1, and study the elliptic problem

$$-\Delta\phi_1 + e^{\beta_1\phi_1} = \rho_1(t), \qquad \mathcal{E}(\beta_1) = \mathcal{E}_0(t) := \mathcal{E}_0 - \iint_{\Omega\times\mathbb{R}^d} \frac{|v|^2}{2} f_1(x,v,t)\, dxdv$$

in which $\mathcal{E}(\beta_1) := \frac{m_0 d}{\beta_1} + \frac{1}{2}\int_\Omega |\nabla\phi_1|^2$. Here, $\rho_1(t)$ and $\mathcal{E}_0(t)$ are two continuous functions. Fix a t and let t_n be a sequence so that $t_n \to t$. Then, there are unique solutions $(\beta_1(t_n), \phi_1(t_n))$ and $(\beta_1(t), \phi_1(t))$ to the elliptic problems, corresponding to $(\rho(t_n), \mathcal{E}_0(t_n))$ and $(\rho(t), \mathcal{E}_0(t))$, respectively. In addition, we have $\beta_1(t_n)$ and $\phi_1(t_n)$ are uniformly bounded in $\mathbb{R}$ and H^2, respectively. Hence, there is a subsequence t_{n_k} so that $(\beta_1(t_{n_k}), \phi_1(t_{n_k}))$ converges, and by uniqueness, the whole series converges to the same limit $(\beta_1(t), \phi_1(t))$. In particular, this yields the continuity of $\beta_1(t)$.

Finally, by the continuity, the term $|\beta|^4$ on the right-hand side of (4.43) can be absorbed into the left-hand side, for small t, since $\beta(0) = 0$, yielding

$$|\beta|^2 + \|\nabla\phi\|^2_{L^2} + \|\Delta\phi\|^2_{L^2} \le C_0\Big(\|f\|^2_{L^2} + \|f\|^4_{L^2}\Big).$$

Putting this into (4.37) finishes the proof of the proposition, and hence the proof of the uniqueness of the solutions to the ion problem (4.1)-(4.2) on a short time interval. Finally, one observes that the estimate (4.43) is uniform in time and $\beta(t)$ is continuous in $\mathbb{R}_+$. The uniqueness follows for all time $t \in [0,T]$. □

4.4 Propagation of regularity

In this short section, we prove the regularity assumption needed in Theorem 4.12 for the uniqueness of weak solutions in the case when $\Omega = \mathbb{T}^d$.

Proposition 4.14. *Let $\Omega = \mathbb{T}^d$ and (β, ϕ, f_+) be a solution to (4.1) and (4.2) with compactly v-supported and bounded initial data $f_{+,0}$. Assume in addition that*

$$\|\nabla_x f_{+,0}\|_{L^\infty_{x,v}} + \|\nabla_v f_{+,0}\|_{L^\infty_{x,v}} < \infty.$$

Then, for small positive time T, there holds

$$\sup_{t\in[0,T]} \Big(\|\nabla_x f_+(t)\|_{L^\infty_{x,v}} + \|\nabla_v f_+(t)\|_{L^\infty_{x,v}}\Big) < \infty.$$

Proof. The proof is straightforward. Indeed, $\nabla_x f_+$ and $\nabla_v f_+$ satisfy

$$\Big(\partial_t + v\cdot\nabla_x - E\cdot\nabla_v\Big)\nabla_x f_+ = \sum_k \nabla_x E_k \partial_{v_k} f_+$$
$$\Big(\partial_t + v\cdot\nabla_x - E\cdot\nabla_v\Big)\nabla_v f_+ = -\nabla_x f_+.$$

This yields

$$\|\nabla_v f_+(t)\|_{L^\infty} \le \int_0^t \|\nabla_x f_+(s)\|_{L^\infty}\, ds$$

and

$$\|\nabla_x f_+(t)\|_{L^\infty} \leq \int_0^t \|D_x^2\phi\|_{L^\infty} \|\nabla_v f_+(s)\|_{L^\infty} \, ds.$$

Here, ϕ solves the elliptic problem $-\Delta\phi = n_I - e^{\beta\phi}$ and hence

$$-\Delta D_x\phi = D_x n_I - D_x e^{\beta\phi}.$$

Hence, applying Lemma 4.5, for $D_x\phi$, together with the fact that Ω is bounded, yields at once

$$\begin{aligned}\|D_x^2\phi\|_{L^\infty} &\leq C_0\|D_x n_I\|_{L^\infty} + C_0\|D_x e^{\beta\phi}\|_{L^\infty} \\ &\leq C_0\|D_x f_+\|_{L^\infty} + C_0\|e^{\beta\phi}\|_{L^\infty}\|D_x\phi\|_{L^\infty}\end{aligned}$$

in which we noted that f_+ is compactly supported in v. Recall that $\|e^{\beta\phi}\|_{L^\infty} \leq \|n_I\|_{L^\infty} \leq C_0$ and $\|D_x\phi\|_{L^\infty} \leq C_0\|n_I\|_{L^\infty} \leq C_1$, since $f_+ \in L^\infty$. Hence,

$$\|\nabla_x f_+(t)\|_{L^\infty} \leq C_0 \int_0^t (1 + \|\nabla_x f_+(s)\|_{L^\infty})\|\nabla_v f_+(s)\|_{L^\infty} \, ds.$$

The proposition follows at once from the standard nonlinear Gronwall's lemma. □

5 Conclusion

Some remarks in conclusion:

- For the interaction for the evolution of a plasma involving ions and electrons an approximation of the density of electrons is often used and it is referred as the Boltzmann-Maxwell relation. The aim of the present contribution was to fully justify this approach assuming a kinetic description for the electrons where the characteristic interaction time is faster than rate of relaxation to equilibrium. This seems the most natural way to obtain a proof. On the other hand, as indicated by Theorem 1.2, considering a macroscopic equation for the ions seems compatible with the present approach. And eventually one should observe that in some case the counterpart of the Boltzmann -Maxwell relation can be derived for some well adapted macroscopic description; cf. [2, 16] for an example and references.

- One may wonder at getting a electrons temperature which is constant with respect to the space variable. But recall we deal here with a modelling at the scale of the Debye length (for instance some tens or hundreds of Debye lengths) and at this scale it is natural that the electrons temperature is constant even if it is not the case at a much larger scale.

- The main difficulty towards a complete proof of Theorem 1.1, without the regularity assumption, seems to come from the fact that the conservation of energy for large time for the solution of the Vlasov-Poisson-Boltzmann equation, even formally true and expected in general at the level of mathematical rigor, remains an open problem. This difficulty persists in the presence of a electromagnetic interaction. This is the reason why some uniform regularity hypothesis is assumed in Theorem 1.1.

- In the present contribution the coupling between the ions and electrons is described through the effect of the electric field, magnetic effect and collisions between ions and electrons are ignored, such issues may be the object of future works.

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