LONG-TIME STABILITY OF LARGE-AMPLITUDE NONCHARACTERISTIC BOUNDARY LAYERS FOR HYPERBOLIC–PARABOLIC SYSTEMS

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Abstract. Extending investigations of Yarahmadian and Zumbrun in the strictly parabolic case, we study time-asymptotic stability of arbitrary (possibly large) amplitude noncharacteristic boundary layers of a class of hyperbolic-parabolic systems including the Navier–Stokes equations of compressible gas- and magnetohydrodynamics, establishing that linear and nonlinear stability are both equivalent to an Evans function, or generalized spectral stability, condition. The latter is readily checkable numerically, and analytically verifiable in certain favorable cases; in particular, it has been shown by Costanzino, Humpherys, Nguyen, and Zumbrun to hold for sufficiently large-amplitude layers for isentropic ideal gas dynamics, with general adiabiatic index $\gamma \geq 1$. Together with these previous results, our results thus give nonlinear stability of large-amplitude isentropic boundary layers, the first such result for compressive ("shock-type") layers in other than the nearly-constant case. The analysis, as in the strictly parabolic case, proceeds by derivation of detailed pointwise Green function bounds, with substantial new technical difficulties associated with the more singular, hyperbolic behavior in the high-frequency/short time regime.

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Date: Last Updated: April 22, 2008.

This work was supported in part by the National Science Foundation award number DMS-0300487.

1. Introduction

In this paper, we study the stability of boundary layers assuming that the boundary layer solution is *noncharacteristic*, which means, roughly, that signals are transmitted into or out of but not along the boundary. In the context of gas dynamics or magnetohydrodynamics (MHD), this corresponds to the situation of a porous boundary with prescribed inflow or outflow conditions accomplished by suction or blowing, a scenario that has been suggested as a means to reduce drag along an airfoil by stabilizing laminar flow; see Example 1.1 below.

We consider a boundary layer, or stationary solution,

$$\tilde U = \bar U(x), \quad \lim_{z\to+\infty} \bar U(z) = U_+, \quad \bar U(0) = \bar U_0 \tag{1.1}$$

of a system of conservation laws on the quarter-plane

$$\tilde U_t + F(\tilde U)_x = (B(\tilde U)\tilde U_x)_x, \quad x,t>0, \tag{1.2}$$

$\tilde U, F \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$, with initial data $\tilde U(x,0) = \tilde U_0(x)$ and Dirichlet type boundary conditions specified in (1.5), (1.6) below. A fundamental question connected to the physical motivations from aerodynamics is whether or not such boundary layer solutions are *stable* in the sense of PDE, i.e., whether or not a sufficiently small perturbation of $\bar U$ remains close to $\bar U$, or converges time-asymptotically to $\bar U$, under the evolution of (1.2). That is the question we address here.

Our main result, in the general spirit of [ZH, MaZ3, MaZ4, Z3, HZ, YZ], is to reduce the questions of linear and nonlinear stability to verification of a simple and numerically well-posed *Evans function*, or generalized spectral stability, condition, which can then be checked either numerically or by the variety of methods available for study of eigenvalue ODE; see, for example, [Br1, Br2, BrZ, BDG, HuZ2, PZ, FS, BHRZ, HLZ, HLyZ1, HLyZ2, CHNZ]. Together with the results of [CHNZ], this yields in particular nonlinear stability of sufficiently large-amplitude boundary-layers of the compressible Navier–Stokes equations of isentropic ideal gas dynamics, with adiabatic index $\gamma \geq 1$, the first such result for a large compressive, or "shock-type", boundary layers. The main new difficulty beyond the strictly parabolic case of [YZ] is to treat the more singular, hyperbolic behavior in the high-frequency regime, both in obtaining pointwise Green function bounds, and in deriving energy estimates by which the nonlinear analysis is closed.

1.1. **Equations and assumptions.** We consider the general hyperbolic-parabolic system of conservation laws (1.2) in conserved variable $\tilde{U}$, with

$$\tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad \sigma(b_2) \geq \theta > 0,$$

$\tilde{u} \in \mathbb{R}$, and $\tilde{v} \in \mathbb{R}^{n-1}$, where, here and elsewhere, σ denotes spectrum of a linearized operator or matrix. Here for simplicity, we have restricted to the case (as in standard gas dynamics and MHD) that the hyperbolic part (equation for $\tilde{u}$) consists of a single scalar equation. As in [MaZ3], the results extend in straightforward fashion to the case $\tilde{u} \in \mathbb{R}^k$, $k > 1$, with $\sigma(A^{11})$ strictly positive or strictly negative.

Following [MaZ4, Z3], we assume that equations (1.2) can be written, alternatively, after a triangular change of coordinates

$$\tilde{W} := \tilde{W}(\tilde{U}) = \begin{pmatrix} \tilde{w}^I(\tilde{u}) \\ \tilde{w}^{II}(\tilde{u}, \tilde{v}) \end{pmatrix}, \tag{1.3}$$

in *the quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde{A}^0 \tilde{W}_t + \tilde{A} \tilde{W}_x = (\tilde{B} \tilde{W}_x)_x + \tilde{G}, \tag{1.4}$$

where, defining $\tilde{W}_+ := \tilde{W}(U_+)$,

(A1) $\tilde{A}(\tilde{W}_+), \tilde{A}^0, \tilde{A}^{11}$ are symmetric, $\tilde{A}^0$ block diagonal, $\tilde{A}^0 \geq \theta_0 > 0$,

(A2) no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(\tilde{W}_+)$ lies in the kernel of $\tilde{B}(\tilde{A}^0)^{-1}(\tilde{W}_+)$,

(A3) $\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}$, $\tilde{b} \geq \theta > 0$, and $\tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$ with $\tilde{g}(\tilde{W}_x, \tilde{W}_x) = \mathcal{O}(|\tilde{W}_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F, B, \tilde{A}^0, \tilde{A}, \tilde{B}, \tilde{W}(\cdot), \tilde{g}(\cdot,\cdot) \in C^4$.

(H1) $\tilde{A}^{11}$ (scalar) is either strictly positive or strictly negative, that is, either $\tilde{A}^{11} \geq \theta_1 > 0$, or $\tilde{A}^{11} \leq -\theta_1 < 0$. (We shall call these cases *the inflow case* or *the outflow case*, correspondingly.)

(H2) The eigenvalues of $dF(U_+)$ are real, distinct, and nonzero.

Condition (H1) corresponds to noncharacteristicity, while (H2) is the condition for the hyperbolicity of U_+. The assumptions (A1)-(A3) and (H0)-(H2) are satisfied for gas dynamics and MHD with van der Waals equation of state under inflow or outflow conditions; see discussions in [MaZ4, CHNZ, GMWZ5, GMWZ6].

We also assume:

(B) Dirichlet boundary conditions in $\tilde{W}$-coordinates:

$$(\tilde{w}^I, \tilde{w}^{II})(0,t) = \tilde{h}(t) := (\tilde{h}_1, \tilde{h}_2)(t) \tag{1.5}$$

for the inflow case, and

$$\tilde{w}^{II}(0,t) = \tilde{h}(t) \tag{1.6}$$

for the outflow case.

This is sufficient for the main physical applications; the situation of more general, Neumann- and mixed-type boundary conditions on the parabolic variable v can be treated as discussed in [GMWZ5, GMWZ6].

Example 1.1. The main example we have in mind consists of *laminar solutions* $(\rho, u, e)(x_1, t)$ of the compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u^t u) + \nabla p = \varepsilon\mu\Delta u + \varepsilon(\mu+\eta)\nabla \mathrm{div} u \\ \partial_t(\rho E) + \mathrm{div}\big((\rho E + p)u\big) = \varepsilon\kappa\Delta T + \varepsilon\mu \mathrm{div}\big((u\cdot\nabla)u\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \varepsilon(\mu+\eta)\nabla(u\cdot \mathrm{div} u), \end{cases} \tag{1.7}$$

$x \in \mathbb{R}^d$, on a half-space $x_1 > 0$, where ρ denotes density, $u \in \mathbb{R}^d$ velocity, e specific internal energy, $E = e + \frac{|u|^2}{2}$ specific total energy, $p = p(\rho, e)$ pressure, $T = T(\rho, e)$ temperature, $\mu > 0$ and $|\eta| \le \mu$ first and second coefficients of viscosity, $\kappa > 0$ the coefficient of heat conduction, and $\varepsilon > 0$ (typically small) the reciprocal of the Reynolds number, with no-slip *suction-type* boundary conditions on the velocity,

$$u_j(0, x_2, \dots, x_d) = 0,\ j \ne 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) < 0,$$

and prescribed temperature, $T(0, x_2, \dots, x_d) = T_{wall}(x)$. Under the standard assumptions p_ρ, $T_e > 0$, this can be seen to satisfy all of the hypotheses (A1)–(A3), (H0)–(H2); indeed these are satisfied also under much weaker van der Waals gas assumptions [MaZ4, Z3, CHNZ, GMWZ5, GMWZ6]. In particular, boundary-layer solutions are of noncharacteristic type, scaling as $(\rho, u, e) = (\bar\rho, \bar u, \bar e)(x_1/\varepsilon)$, with layer thickness $\sim \varepsilon$ as compared to the $\sim \sqrt{\varepsilon}$ thickness of the characteristic type found for an impermeable boundary.

This corresponds to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic surface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G.I. Taylor as a means to reduce drag by stabilizing laminar flow; see [S, Bra]. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds [Bra].[1] Possible mechanisms for this reduction are smaller thickness $\sim \varepsilon << \sqrt{\varepsilon}$ of noncharacteristic boundary layers as compared to characteristic type, and greater stability, delaying the transition from laminar to turbulent flow. In particular, stability properties appear to be quite important for the understanding of this phenomenon. For further discussion, including the related issues of matched asymptotic expansion, multi-dimensional effects, and more general boundary configurations, see [GMWZ5].

[1]See also NASA site http://www.dfrc.nasa.gov/Gallery/photo/F-16XL2/index.html

Example 1.2. For (1.7), or the general (1.2), a large class of boundary-layer solutions, sufficient for the present purposes, may be generated as truncations $\bar u^{x_0}(x) := \bar u(x - x_0)$ of *standing shock solutions*

$$u = \bar u(x), \quad \lim_{x\to\pm\infty} \bar u(x) = u_\pm \tag{1.8}$$

on the whole line $x \in \mathbb{R}$, with boundary conditions $\beta_h(t) \equiv \bar u(0)$ (inflow) or $\beta_h(t) \equiv \bar w^I(0)$ (outflow) chosen to match. However, there are also many other boundary-layer solutions not connected with any shock. For more general catalogs of boundary-layer solutions of (1.7), see, e.g., [MN, SZ, CHNZ, GMWZ5].

Lemma 1.3 ([MaZ3, Z3, GMWZ5])**.** *Given (A1)-(A3) and (H0)-(H2), a standing wave solution* (1.1) *of* (1.2)*, (B) satisfies*

$$\Big|(d/dx)^k(\bar U - U_+)\Big| \le Ce^{-\theta x}, \quad k = 0, ..., 4, \tag{1.9}$$

as $x \to +\infty$. Moreover, a solution, if it exists, is in the inflow or strictly parabolic case unique; in the outflow case it is locally unique.

Proof. As in the shock case [MaZ4, Z3], (1.9) follows by the observation that, under hypotheses (A1)-(A3) and (H0)-(H2), U_+ is a hyperbolic rest point of the layer profile ODE; see also [GMWZ5].

Uniqueness follows by the observation [MaZ3] that the standing-wave ODE may be integrated from x to $+\infty$ and rearranged to yield

$$\begin{aligned} F^1(U) &\equiv F^1(U_+),\\ (b_1, b_2)(U)U' &= C(U, U_+), \end{aligned} \tag{1.10}$$

and thereby the first-order ODE

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} F^1_u & F^1_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C(U, U_+) \end{pmatrix}. \tag{1.11}$$

In the strictly parabolic or inflow case, $U(0)$ is specified by the boundary conditions at $x = 0$, thus determining a unique solution for all $x \ge 0$ through (1.11). In the outflow case, we observe, comparing U and W equations, that (1.10) can be rewritten alternatively as

$$\begin{aligned} F^1(W) &\equiv F^1(W_+),\\ (w^{II})' &= D(w^I, w^{II}), \end{aligned} \tag{1.12}$$

where the first equation may by the Implicit Function Theorem be *locally* solved for w^I as a function of w^{II}. Substituting in the second equation, and noting that $w^{II}(0)$ is specified by the boundary conditions at $x = 0$, we again obtain uniqueness, this time only local, by uniqueness of solutions of the initial-value problem for ODE $(w^{II})' = D(w^I, w^{II})$. We omit the details. (Local uniqueness is here essentially a remark, as it is a consequence, by Rousset's Lemma [R2, MZ1, GMWZ5, GMWZ6], of our later assumption (D) of Evans stability.) □

1.2. **Main results.** Linearizing the equations (1.2), (B) about the boundary layer $\bar{U}$, we obtain the linearized equation

$$U_t = LU := -(\bar{A}U)_x + (\bar{B}U_x)_x, \tag{1.13}$$

where

$$\bar{B} := B(\bar{U}), \quad \bar{A}U := dF(\bar{U})U - (dB(\bar{U})U)\bar{U}_x,$$

with boundary conditions (now expressed in U-coordinates)

$$(\partial\tilde{W}/\partial\tilde{U})(\bar{U}_0)U(0,t) = h(t) := \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}(t) \tag{1.14}$$

for the inflow case, and

$$(\partial\tilde{w}^{II}/\partial\tilde{U})(\bar{U}_0)U(0,t) = h(t) \tag{1.15}$$

for the outflow case, where $(\partial\tilde{W}/\partial\tilde{U})(\bar{U}_0)$ is constant and invertible,

$$(\partial\tilde{w}^{II}/\partial\tilde{U})(\bar{U}_0) = m \begin{pmatrix} \bar{b}_1 & \bar{b}_2 \end{pmatrix} (\bar{U}_0), \tag{1.16}$$

(by (A1) and triangular structure (1.3)) is constant with $m \in \mathbb{R}^{(n-1)\times(n-1)}$ invertible, and $h := \tilde{h} - \bar{h}$.

Definition 1.4. *The boundary layer $\bar{U}$ is said to be linearly $X \to Y$ stable if, for some $C > 0$, the problem* (1.13) *with initial data U_0 in X and homogeneous boundary data $h \equiv 0$ has a unique global solution $U(\cdot,t)$ such that $|U(\cdot,t)|_Y \le C|U_0|_X$ for all t; it is said to be linearly asymptotically $X \to Y$ stable if also $|U(\cdot,t)|_Y \to 0$ as $t \to \infty$.*

We define the following *stability criterion*, where $D(\lambda)$ described below, denotes the Evans function associated with the linearized operator L about the layer, an analytic function analogous to the characteristic polynomial of a finite-dimensional operator, whose zeroes away from the essential spectrum agree in location and multiplicity with the eigenvalues of L:

(D) There exist no zeroes of $D(\cdot)$ in the nonstable half-plane $Re\lambda \ge 0$.

As discussed, e.g., in [R2, MZ1, GMWZ5, GMWZ6], under assumptions (H0)-(H2), this is equivalent to *strong spectral stability*, $\sigma(L) \subset \{Re\lambda < 0\}$, (ii) *transversality* of $\bar{U}$ as a solution of the connection problem in the associated standing-wave ODE, and *hyperbolic stability* of an associated boundary value problem obtained by formal matched asymptotics. See [GMWZ5, GMWZ6] for further discussions.

Definition 1.5. *The boundary layer $\bar{U}$ is said to be nonlinearly $X \to Y$ stable if, for each $\varepsilon > 0$, the problem* (1.2) *with initial data $\tilde{U}_0$ sufficiently close to the profile $\bar{U}$ in $|\cdot|_X$ has a unique global solution $\tilde{U}(\cdot,t)$ such that $|\tilde{U}(\cdot,t) - \bar{U}(\cdot)|_Y < \varepsilon$ for all t; it is said to be nonlinearly asymptotically $X \to Y$ stable if also $|\tilde{U}(\cdot,t) - \bar{U}(\cdot)|_Y \to 0$ as $t \to \infty$. We shall sometimes not explicitly define the norm X, speaking instead of stability or asymptotic stability in Y under perturbations satisfying specified smallness conditions.*

Our first main result is as follows.

Theorem 1.6 (Linearized stability)**.** *Assume (A1)-(A3), (H0)-(H2), and (B) with* $|h(t)| \le E_0(1+t)^{-1-\epsilon}$*,* $|h'(t)| \le E_0(1+t)^{-1}$*, for arbitrary fixed* $\epsilon > 0$*. Let* $\bar U$ *be a boundary layer. Then linearized* $L^1 \cap L^p \to L^1 \cap L^p$ *stability,* $1 \le p \le \infty$*, is equivalent to (D). In the case of stability, there holds also linearized asymptotic* $L_1 \cap L^p \to L^p$ *stability,* $p > 1$*, with rate*

$$|U(\cdot,t)|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/p)}|U_0|_{L^1\cap L^p} + CE_0(1+t)^{-\frac{1}{2}(1-1/p)}. \tag{1.17}$$

To state the pointwise nonlinear stability result, we need some notations. Denoting by

$$a_1^+ < a_2^+ < \cdots < a_n^+ \tag{1.18}$$

the eigenvalues of of the limiting convection matrix $A_+ := dF(U_+)$, define

$$\theta(x,t) := \sum_{a_j^+>0} (1+t)^{-1/2} e^{-|x-a_j^+ t|^2/Mt}, \tag{1.19}$$

$$\psi_1(x,t) := \chi(x,t) \sum_{a_j^+>0} (1+|x|+t)^{-1/2}(1+|x-a_j^+ t|)^{-1/2}, \tag{1.20}$$

and

$$\psi_2(x,t) := (1-\chi(x,t))(1+|x-a_n^+ t|+t^{1/2})^{-3/2}, \tag{1.21}$$

where $\chi(x,t) = 1$ for $x \in [0, a_n^+ t]$ and $\chi(x,t) = 0$ otherwise and $M > 0$ is a sufficiently large constant.

For simplicity, we measure the boundary data by function

$$\mathcal{B}_h(t) := \sum_{r=0}^{2} |(d/dt)^r h| \tag{1.22}$$

for the outflow case, and

$$\mathcal{B}_h(t) := \sum_{r=0}^{4} |(d/dt)^r h_1| + \sum_{r=0}^{2} |(d/dt)^r h_2| \tag{1.23}$$

for the inflow case.

Then, our next result is as follows.

Theorem 1.7 (Nonlinear stability)**.** *Assuming (A1)-(A3), (H0)-(H2), (B), and the linear stability condition (D), the profile* $\bar U$ *is nonlinearly asymptotically stable in* $L^p \cap H^4$*,* $p > 1$*, with respect to perturbations* $U_0 \in H^4$*,* $h \in C^4$ *in initial and boundary data satisfying:* $|h(t)| \le E_0(1+t)^{-1-\epsilon}$*,* $|h'(t)| \le E_0(1+t)^{-1}$*, for arbitrary fixed* $\epsilon > 0$*, and*

$$\|(1+|x|^2)^{3/4} U_0\|_{H^4} \le E_0 \quad \text{and} \quad |\mathcal{B}_h(t)| \le E_0(1+t)^{-1/4}$$

for E_0 sufficiently small. More precisely,

$$|\tilde U(x,t)-\bar U(x)|\le CE_0(\theta+\psi_1+\psi_2)(x,t),\qquad (1.24)$$
$$|\tilde U_x(x,t)-\bar U_x(x)|\le CE_0(\theta+\psi_1+\psi_2)(x,t),$$

where $\tilde U(x,t)$ denotes the solution of (1.2) with initial and boundary data $\tilde U(x,0)=\bar U(x)+U_0(x)$ and $\tilde U(0,t)=\bar U_0+h(t)$, yielding the sharp rates

$$\|\tilde U(x,t)-\bar U(x)\|_{L^p}\le CE_0(1+t)^{-\frac12(1-\frac1p)},\quad 1\le p\le\infty,\qquad (1.25)$$

$$\|\tilde U(x,t)-\bar U(x)\|_{H^4}\le CE_0(1+t)^{-\frac14}.\qquad (1.26)$$

Remark 1.8. *By the one dimensional Sobolev embedding, from the hypothesis on U_0, we automatically assume that*

$$\|U_0\|_{H^4}\le E_0,\quad |U_0(x)|+|U_0'(x)|\le E_0(1+|x|)^{-3/2}.$$

A crucial step in establishing Theorems 1.6 and 1.7 is to obtain pointwise bounds on the Green function $G(x,t;y)$ of the linearized evolution equations (1.13) (more properly speaking, a distribution), which we now describe. Let $a_j^+, j=1,....,n$ denote the eigenvalues of $A(+\infty)$, and l_j^+,r_j^+ associated left and right eigenvectors, respectively, normalized so that $l_j^+r_k^+=\delta_j^k$. Eigenvalues $a_j(x)$, and eigenvectors $l_j(x),r_j(x)$ correspond to large-time convection rates and modes of propagation of the linearized model (1.13).

Define time-asymptotic, scalar diffusion rates

$$\beta_j^+:=(l_jBr_j)_+,\quad j=1,...,n,\qquad (1.27)$$

and local dissipation coefficient

$$\eta_*:=-D_*(x)\qquad (1.28)$$

where

$$D_*(x):=A_{12}b_2^{-1}\Big[A_{21}-A_{22}b_2^{-1}b_1+b_2^{-1}b_1A_*+b_2\partial_x(b_2^{-1}b_1)\Big](x)$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman-Enskog expansion in the (dual) relaxation case,

$$A_*:=A_{11}-A_{12}b_2^{-1}b_1.$$

Note that as a consequence of dissipativity, (A2), we obtain

$$\eta_*^+>0,\quad \beta_j^+>0,\quad \text{for all } j.\qquad (1.29)$$

We also define modes of propagation for the reduced, hyperbolic part of system (1.13) as

$$L_*=\begin{pmatrix}1\\0_{n-1}\end{pmatrix},\quad R_*=\begin{pmatrix}1\\-b_2^{-1}b_1\end{pmatrix}\qquad (1.30)$$

We define the Green function $G(x,t;y)$ of the linearized evolution equations (1.13) with homogeneous boundary conditions (more properly speaking, a distribution), by

(i) $(\partial_t - L_x)G = 0$ in the distributional sense, for all $x, y, t > 0$;
(ii) $G(x,t;y) \to \delta(x-y)$ as $t \to 0$;
(iii) for all $y, t > 0$, $\begin{pmatrix} \bar{A}_* & 0 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix} G(0,t;y) = \begin{pmatrix} * \\ 0 \end{pmatrix}$ where $* = 0$ for the inflow case $\bar{A}_* > 0$ and $*$ is arbitrary for the outflow case $\bar{A}_* < 0$, noting that no boundary condition is needed to be prescribed on the hyperbolic part.

By standard arguments as in [MaZ3], we have the spectral resolution, or inverse Laplace transform formulae

$$e^{Lt} f = \frac{1}{2\pi i} P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} (\lambda - L)^{-1} f d\lambda \tag{1.31}$$

and

$$G(x,t;y) = \frac{1}{2\pi i} P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_\lambda(x,y)\, d\lambda \tag{1.32}$$

for any large positive η.

We prove the following pointwise bounds on the Green function $G(x,t;y)$.

Proposition 1.9. *Under assumptions (A1)-(A3), (H0)-(H2), (B), and (D), we obtain*

$$G(x,t;y) = H(x,t;y) + \tilde{G}(x,t;y), \tag{1.33}$$

where

$$\begin{aligned} H(x,t;y) &= \frac{1}{2\pi} A_*(x)^{-1} A_*(y) \delta_{x - \bar{a}_* t}(y) e^{-\int_y^x (\eta_*/A_*)(z) dz} R_* L_*^{tr} \\ &= \mathcal{O}(e^{-\eta_0 t}) \delta_{x - \bar{a}_* t}(y) R_* L_*^{tr}, \end{aligned} \tag{1.34}$$

and

$$\begin{aligned} |\partial_x^\gamma \partial_y^\alpha \tilde{G}(x,t;y)| &\le C e^{-\eta(|x-y|+t)} \\ &+ \quad C(t^{-(|\alpha|+|\gamma|)/2} + |\alpha| e^{-\eta|y|} + |\gamma| e^{-\eta|x|}) \Big(\sum_{k=1}^n t^{-1/2} e^{-(x-y-a_k^+ t)^2/Mt} \\ &+ \sum_{a_k^+ < 0,\, a_j^+ > 0} \chi_{\{|a_k^+ t| \ge |y|\}} t^{-1/2} e^{-(x - a_j^+ (t - |y/a_k^+|))^2/Mt} \Big), \end{aligned} \tag{1.35}$$

$0 \le |\alpha|, |\gamma| \le 1$, *for some* η, C, $M > 0$, *where indicator function* $\chi_{\{|a_k^+ t| \ge |y|\}}$ *is* 1 *for* $|a_k^+ t| \ge |y|$ *and* 0 *otherwise.*

Here, the averaged convection rate $\bar{a}_*(x,t)$ *in* (1.34) *denotes the time-averages over* $[0,t]$ *of* $A_*(z)$ *along backward characteristic paths* $z_* = z_*(x,t)$ *defined by*

$$\frac{dz_*}{dt} = A_*(z_*(x,t)), \quad z_*(t) = x. \tag{1.36}$$

In all equations, a_j^+, A_*, L_*, R_* *are as defined just above.*

1.3. **Discussion and open problems.** The stability of noncharacteristic boundary layers in gas dynamics has been treated using energy estimates in, e.g., [MN, KNZ, R3], for both "compressive" boundary layers including the truncated shock-solutions (1.8), and for "expansive" solutions analogous to rarefaction waves. However, in the case of compressive waves, these and most subsequent analyses were restricted to the *small-amplitude case*

$$\|\bar u - u_+\|_{L^1(\mathbb{R}^+)} \text{ sufficiently small.} \tag{1.37}$$

Examining this condition even for the special class (1.8) of truncated shock solutions, we find that it is extremely restrictive.

For, consider the one-parameter family $\bar u^{x_0}(x) = \bar u(x - x_0)$ of boundary-layers associated with a standing shock $\bar u$ of amplitude $\delta := |u_+ - u_-| << 1$. By center manifold analysis [Pe], $\bar u - u_+ \sim \delta e^{-c\delta x}$, hence

$$\|\bar u - u_+\|_{L^1(\mathbb{R}^+)} \sim e^{-c\delta x} \sim \frac{|u_+ - u(0)|}{|u_+ - u_-|}$$

in fact measures *relative amplitude* with respect to the amplitude $|u_+ - u_-|$ of the background shock solution $\bar u$. Thus, smallness condition (1.37) requires that the boundary layer consist of a small, nearly-constant piece of the original shock.

The present results, extending results of [YZ] in the strictly parabolic case, remove this restriction, allowing applications in principle to shocks of any amplitude. In particular, in combination with the spectral stability results obtained in [CHNZ] by asymptotic Evans function analysis, they yield stability of noncharacteristic isentropic gas-dynamical layers of sufficiently *large* amplitude. Together with further, numerical, investigations of [CHNZ] give strong evidence that in fact *all* noncharacteristic isentropic gas layers are spectrally stable, independent of amplitude, which would together with our results yield nonlinear stability.

Spectral stability of full (nonisentropic) gas layers may be investigated numerically as for shocks in [HLyZ1, HLyZ2], in both one- and multi-dimensions. However, analytical results of [SZ] show that in this case instability is possible, even for ideal gas equation of state. The numerical classification of stability for full gas dynamics, and the extension of our present nonlinear stability results to multi-dimensions, are two interesting direction for further investigation.

Finally, we comment briefly on the difference between our analysis and the earlier analysis [YZ] carried out by similar techniques based on the Evans function and stationary phase estimates on the inverse Laplace transform formula. Our analysis is in the same spirit as, and borrows heavily from this earlier work. The main new issues are technical ones connected with the more singular high-frequency/short-time behavior of hyperbolic-parabolic equations as compared to the strictly parabolic equations considered in [YZ]. In particular, linearized behavior in the u coordinate, $U = (u, v)$, is essentially hyperbolic, governed for short times approximately by the principle part

$$v_t + A_*(x)v_x = 0, \quad A_* := (A_0^{11})^{-1}A^{11} \tag{1.38}$$

Thus, we may expect as in the whole-line analysis of hyperbolic-parabolic equations in [MaZ3] that the associated Green function contain a delta-function component transported along the hyperbolic characteristic

$$dx/dt = A_*(x),$$

with the difference that now we must consider also a possibly-complicated interaction with the boundary.

A key point is that in fact this potential complication *does not occur*. For, in the special case occurring in continuum-mechanical systems [Z3] that all hyperbolic signals either enter or leave the boundary, there is no such boundary interaction and no reflected signal. For example, in the simple scalar example (1.38), the Green function on the half-line with either homogeneous inflow ($A^{11} > 0$) boundary condition $v(0) = 0$ or outflow ($A^{11} < 0$) condition $v(0)$ arbitrary, is by inspection exactly the whole-line Green function

$$g(x, t; y) = \delta_{x-\bar{a}t}(y)/A_*(x)$$

restricted to the half-line $x, y > 0$, where $\bar{a}$ is the average over $[0, t]$ of $A_*(z_*(t))$ along the backward characteristic path

$$\frac{dz_*}{dt} = A_*(z_*(x, t)), \quad z_*(t) = x.$$

Indeed, comparing the description of the homogeneous boundary-value Green function in Proposition 1.9 with that of the whole-line Green function in [MaZ3], we see that they are identical. However, to prove this simple observation costs us considerable care in the high-frequency analysis.

A further issue at the nonlinear level is to obtain nonlinear damping estimates using energy estimates as in [MaZ4], which are somewhat complicated by the presence of a boundary. This is necessary to prevent a loss of derivatives in the nonlinear iteration.

As in [YZ], we get stability also with respect to perturbations in boundary data, something that was not accounted for in earlier works on long-time stability. We mention, finally, the works [GR, MZ1, GMWZ5, GMWZ6] in one- and multi-dimensions of a similar spirit but somewhat different technical flavor on the related small viscosity problem– for example, $\varepsilon \to 0$ in (1.7)– which establish that the Evans condition (or its multi-dimensional analog) is also sufficient for existence and stability of matched asymptotic solution as viscosity goes to zero.

2. Pointwise bounds on resolvent kernel G_λ

In this section, we shall establish estimates on resolvent kernel $G_\lambda(x, y)$.

2.1. **Evans function framework.**

Before starting the analysis, we review the basic Evans function methods and gap/conjugation lemma.

2.1.1. *The gap/conjugation lemma.* Consider a family of first order ODE systems on the half-line:

$$
\begin{aligned}
W' = \mathbb{A}(x,\lambda)W, \quad \lambda \in \Omega \quad \text{and} \quad x > 0,\\
\mathbb{B}(\lambda)W = 0, \quad \lambda \in \Omega \quad \text{and} \quad x = 0.
\end{aligned}
\tag{2.1}
$$

These systems of ODEs should be considered as a generalized eigenvalue equation, with λ representing frequency. We assume that the boundary matrix $\mathbb{B}$ is analytic in λ and that the coefficient matrix $\mathbb{A}$ is analytic in λ as a function from Ω into $L^\infty(x)$, C^K in x, and approaches exponentially to a limit $\mathbb{A}_+(\lambda)$ as $x \to \infty$, with uniform exponentially decay estimates

$$
|(\partial/\partial x)^k (\mathbb{A} - \mathbb{A}_+)| \le C_1 e^{-\theta|x|/C_2}, \quad \text{for } x > 0,\ 0 \le k \le K,
\tag{2.2}
$$

C_j, $\theta > 0$, on compact subsets of Ω. Now we can state a refinement of the "Gap Lemma" of [GZ, KS], relating solutions of the variable-coefficient ODE to the solutions of its constant-coefficient limiting equations

$$
Z' = \mathbb{A}_+(\lambda)Z
\tag{2.3}
$$

as $x \to +\infty$.

Lemma 2.1 (Conjugation Lemma [MZ1])**.** *Under assumption* (2.2)*, there exists locally to any given* $\lambda_0 \in \Omega$ *a linear transformation* $P_+(x,\lambda) = I + \Theta_+(x,\lambda)$ *on* $x \ge 0$*,* Φ_+ *analytic in* λ *as functions from* Ω *to* $L^\infty[0,+\infty)$*, such that:*

(i) $|P_+|$ *and their inverses are uniformly bounded, with*

$$
|(\partial/\partial\lambda)^j (\partial/\partial x)^k \Theta_+| \le C(j)C_1C_2 e^{-\theta|x|/C_2} \quad \text{for } x > 0,\ 0 \le k \le K+1,
\tag{2.4}
$$

$j \ge 0$*, where* $0 < \theta < 1$ *is an arbitrary fixed parameter, and* $C > 0$ *and the size of the neighborhood of definition depend only on* θ*,* j*, the modulus of the entries of* $\mathbb{A}$ *at* λ_0*, and the modulus of continuity of* $\mathbb{A}$ *on some neighborhood of* $\lambda_0 \in \Omega$*.*

(ii) The change of coordinates $W := P_+Z$ *reduces* (2.1) *on* $x \ge 0$ *to the asymptotic constant-coefficient equations* (2.3)*. Equivalently, solutions of* (2.1) *may be conveniently factorized as*

$$
W = (I + \Theta_+)Z_+,
\tag{2.5}
$$

where Z_+ *are solutions of the constant-coefficient equations, and* Θ_+ *satisfy bounds.*

Proof. As described in [MaZ3], for $j = k = 0$ this is a straightforward corollary of the gap lemma as stated in [Z.3], applied to the "lifted" matrix-valued ODE

$$
P' = \mathbb{A}_+P - P\mathbb{A} + (\mathbb{A} - \mathbb{A}_+)P
$$

for the conjugating matrices P_+. The x-derivative bounds $0 < k \leq K+1$ then follow from the ODE and its first K derivatives. Finally, the λ-derivative bounds follow from standard interior estimates for analytic functions. □

Definition 2.2. *Following* [AGJ], *we define the domain of consistent splitting for the ODE system* $W' = \mathbb{A}(x,\lambda)W$ *as the (open) set of* λ *such that the limiting matrix* $\mathbb{A}_+$ *is hyperbolic (has no center subspace) and the boundary matrix* $\mathbb{B}$ *is full rank, with* $\dim S_+ = \operatorname{rank}\mathbb{B}$.

Lemma 2.3. *On any simply connected subset of the domain of consistent splitting, there exist analytic bases* $\{v_1, \dots, v_k\}^+$ *and* $\{v_{k+1}, \dots, v_N\}^+$ *for the subspaces* S_+ *and* U_+ *defined in Definition 2.2.*

Proof. By spectral separation of U_+, S_+, the associated (group) eigenprojections are analytic. The existence of analytic bases then follows by a standard result of Kato; see [Kat], pp. 99–102. □

Corollary 2.4. *By the Conjugation Lemma , on the domain of consistent splitting, the stable manifold of solutions decaying as* $x \to +\infty$ *of* (2.1) *is*

$$\mathcal{S}^+ := \operatorname{span}\{P_+v_1^+, \dots, P_+v_k^+\}, \tag{2.6}$$

where $W_+^j := P_+v_j^+$ *are analytic in* λ *and* C^{K+1} *in* x *for* $\mathbb{A} \in C^K$.

2.1.2. *Definition of the Evans Function.* On any simply connected subset of the domain of consistent splitting, let $W_1^+, \dots, W_k^+ = P_+v_1^+, \dots, P_+v_k^+$ be the analytic basis described in Corollary 2.4 of the subspace $\mathcal{S}^+$ of solutions W of (2.1) satisfying the boundary condition $W \to 0$ at $+\infty$. Then, the *Evans function* for the ODE systems $W' = \mathbb{A}(x,\lambda)W$ associated with this choice of limiting bases is defined as the $k \times k$ Gramian determinant

$$\begin{aligned} D(\lambda) &:= \det\Big(\mathbb{B}W_1^+, \dots, \mathbb{B}W_k^+\Big)_{|x=0,\lambda} \\ &= \det\Big(\mathbb{B}P_+v_1^+, \dots, \mathbb{B}P_+v_k^+\Big)_{|x=0,\lambda}. \end{aligned} \tag{2.7}$$

Remark 2.5. *Note that* D *is independent of the choice of* P_+ *as, by uniqueness of stable manifolds, the exterior products (minors)* $P_+v_1^+ \wedge \cdots \wedge P_+v_k^+$ *are uniquely determined by their behavior as* $x \to +\infty$.

Proposition 2.6. *Both the Evans function and the subspace* $\mathcal{S}^+$ *are analytic on the entire simply connected subset of the domain of consistent splitting on which they are defined. Moreover, for* λ *within this region, equation* (2.1) *admits a nontrivial solution* $W \in L^2(x > 0)$ *if and only if* $D(\lambda) = 0$.

Proof. Analyticity follows by uniqueness, and local analyticity of P_+, v_k^+. Noting that the first $P_+v_j^+$ are a basis for the stable manifold of (2.1) at $x \to +\infty$, we find that the determinant of $\mathbb{B}P_+v_j^+$ vanishes if and only if $\mathbb{B}(\lambda)$ has nontrivial kernel on $\mathcal{S}_+(\lambda, 0)$, whence the second assertion follows. □

Remark 2.7. *In the case (as here) that the ODE system describes an eigenvalue equation associated with an ordinary differential operator L, Proposition 2.6 implies that eigenvalues of L agree in location with zeroes of D. (Indeed, they agree also in multiplicity; see* [GJ1, GJ2]*; Lemma 6.1,* [ZH]*; or Proposition 6.15 of* [MaZ3]*.)*

When $\ker \mathbb{B}$ has an analytic basis $v^0_{k+1}, \dots, v^0_N$, for example, in the commonly occurring case, as here, that $\mathbb{B} \equiv$ constant, we have the following useful alternative formulation. This is the version that we will use in our analysis of the Green function and Resolvent kernel.

Proposition 2.8. *Let $v^0_{k+1}, \dots, v^0_N$ be an analytic basis of* $\ker \mathbb{B}$*, normalized so that* $\det\left(\mathbb{B}^*, v^0_{k+1}, \dots v^0_N\right) \equiv 1$. *Then, the solutions W^0_j of* (2.1) *determined by initial data $W^0_j(\lambda, 0) = v^0_j$ are analytic in λ and C^{K+1} in x, and*

$$D(\lambda) := \det\left(W^+_1, \dots, W^+_k, W^0_{k+1}, \dots, W^0_N\right)_{|x=0,\lambda}. \tag{2.8}$$

Proof. Analyticity/smoothness follow by analytic/smooth dependence on initial data/parameters. By the chosen normalization, and standard properties of Grammian determinants,

$$D(\lambda) = \det\left(W^+_1, \dots, W^+_k, v^0_{k+1}, \dots, v^0_N\right)_{|x=0,\lambda},$$

yielding (2.8). □

2.1.3. *The tracking/reduction lemma.* Next, consider a family of systems

$$\begin{aligned} W' = \mathbb{A}(x, p, \varepsilon)W, \quad p \in \mathcal{P},\, \varepsilon \in \mathbb{R}^+ \quad \text{and} \quad x > 0, \\ \mathbb{B}(p, \varepsilon)W = 0, \quad \lambda \in \Omega \quad \text{and} \quad x = 0 \end{aligned} \tag{2.9}$$

parametrized by p, ε, with $\varepsilon \to 0$. The main example we have in mind is (2.1) with $p = \lambda/|\lambda|$ and $\varepsilon := |\lambda|^{-1}$, in the high-frequency regime $|\lambda| \to \infty$. We assume further that by some coordinate change we can arrange that

$$\mathbb{A} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} + \Theta, \tag{2.10}$$

with

$$|\Theta| \le \delta(\varepsilon), \quad \Re(M_+ - M_-) \ge 2\eta(\varepsilon) + \alpha^\varepsilon(x), \tag{2.11}$$

$\|\alpha\|_{L^1(\mathbb{R}^+)}$ uniformly bounded for all ε sufficiently small, and

$$(\delta/\eta)(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{2.12}$$

where $\Re(Q) := (1/2)(Q + Q^*)$ denotes the symmetric part of a matrix Q.

Then, we have the following analog of Lemma 2.1.1, asserting that the approximately block-diagonalized equations (2.9) may be converted by a smooth coordinate transformation

$$\begin{pmatrix} I & \Theta^1 \\ \Theta^2 & I \end{pmatrix} \to I \quad \text{as} \quad \varepsilon \to 0$$

to exactly diagonalized form with the same leading part $\mathbb{M}$.

Lemma 2.9 ([MaZ3])**.** *Consider a system* (2.10)*, with* $\tilde{F} \equiv 0$ *and* $\delta/\eta \to 0$ *as* $\varepsilon \to 0$. *Then, (i) for all* $0 < \epsilon \le \epsilon_0$*, there exist (unique) linear transformations* $\Phi_1^\epsilon(z,p)$ *and* $\Phi_2^\epsilon(z,p)$*, possessing the same regularity with respect to the various parameters* z*,* p*,* ϵ *as do coefficients* $M_\pm$ *and* Θ*, for which the graphs* $\{(Z_1, \Phi_2^\epsilon Z_1)\}$ *and* $\{(\Phi_1^\epsilon Z_2, Z_2)\}$ *are invariant under the flow of* (2.10)*, and satisfying*

$$|\Phi_1^\epsilon|,\, |\Phi_2^\epsilon| \le C\delta(\epsilon)/\eta(\epsilon) \,\text{for all}\; z.$$

In particular, (ii) the subspace E_- *of data at* $z=0$ *for which the solution decays as* $z \to +\infty$*, given by* $\operatorname{span}\{(\Phi_1^\varepsilon(0,p)v, v)\}$*, converges as* $\varepsilon \to 0$ *to* $\tilde{E}_- := \operatorname{span}\{(0,v)\}$.

Proof. Standard contraction mapping argument carried out on the "lifted" equations governing the flow of the conjugating matrices Φ_j^ε; see Appendix C, [MaZ3].

Remark 2.10. In practice, we usually have $\alpha^\varepsilon \equiv 0$, as can be obtained in general by a change of coordinates multiplying the first coordinate by exponential weight $e^{\int \alpha^\varepsilon dx}$.

2.2. **Construction of the resolvent kernel.** In this section we construct the explicit form of the resolvent kernel, which is nothing more than the Green function $G_\lambda(x,y)$ associated with the elliptic operator $(L-\lambda I)$, where

$$(L-\lambda I)G_\lambda(\cdot,y) = \delta_y I, \quad \begin{pmatrix} \bar{A}_* & 0 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix} G_\lambda(0,y) \equiv \begin{pmatrix} * \\ 0 \end{pmatrix} \tag{2.13}$$

where $* = 0$ for the inflow case and is arbitrary for the outflow case.

Let Λ be the region of consistent splitting for L. It is a standard fact (see, e.g., [He]) that the resolvent $(L-\lambda I)^{-1}$ and the Green function $G_\lambda(x,y)$ are meromorphic in λ on Λ, with isolated poles of finite order.

Writing the associated eigenvalue equation $LU - \lambda U = 0$ in the form of a first-order system (2.1) as follows: $W := (u,v,z) \in \mathbb{C}^{2n-1}$ with $z := b_1 u' + b_2 v'$, and

$$\begin{aligned} u' &= A_*^{-1}(-A_{12}b_2^{-1}z - (A'_{11}+\lambda)u - A'_{12}v),\\ v' &= b_2^{-1}z - b_2^{-1}b_1u',\\ z' &= (A_{21} - A_{22}b_2^{-1}b_1)u' + A_{22}b_2^{-1}z + A'_{21}u + (A'_{22}+\lambda)v. \end{aligned} \tag{2.14}$$

2.2.1. *Domain of consistent splitting.* Define

$$\Lambda := \cap \Lambda_j^+, \quad j = 1,2,...,n \tag{2.15}$$

where Λ_j^+ denote the open sets bounded on the left by the algebraic curves $\lambda_j^+(\xi)$ determined by the eigenvalues of the symbols $-\xi^2 B_+ - i\xi A_+$ of the limiting constant-coefficient operators

$$L_+ w := B_+ w'' - A_+ w' \tag{2.16}$$

as ξ is varied along the real axis. The curves λ_j^+ comprise the essential spectrum of operators L_+.

Lemma 2.11 ([MaZ3]). *The set Λ is equal to the component containing real $+\infty$ of the domain of consistent splitting for* (2.14). *Moreover, under (A1)–(A3), (H0)–(H2),*

$$\Lambda \subset \{\lambda \;:\; \Re e\lambda > -\eta|\Im m\lambda|/(1+|\Im m\lambda|), \quad \eta > 0. \tag{2.17}$$

2.2.2. *Basic construction.* We first recall the following duality relation derived for the degenerate viscosity case in [MaZ3].

Lemma 2.12 ([ZH, MaZ3]). *The function $W = (U, Z)$ is a solution of* (2.14) *if and only if $\tilde W^* \tilde{\mathcal{S}} W \equiv$ constant for any solution $\tilde W = (\tilde U, \tilde Z)$ of the adjoint eigenvalue equation, where*

$$\tilde{\mathcal{S}} = \begin{pmatrix} -A_{11} & -A_{12} & 0 \\ -A_{21} & -A_{22} & I_r \\ -b_2^{-1}b_1 & -I_r & 0 \end{pmatrix} \tag{2.18}$$

and

$$Z = (b_1, b_2)U', \quad \tilde Z = (0, b_2^*)\tilde U'. \tag{2.19}$$

For future reference, we note the representation

$$\tilde{\mathcal{S}}^{-1} = \begin{pmatrix} -A_*^{-1} & 0 & A_*^{-1}A_{12} \\ b_2^{-1}b_1A_*^{-1} & 0 & -b_2^{-1}b_1A_*^{-1}A_{12} - I_r \\ -\tilde A A_*^{-1} & I_r & -A_{22} + \tilde A A_*^{-1} A_{12} \end{pmatrix} \tag{2.20}$$

where $\tilde A := A_{21} - A_{22}b_{-1}b_1$, $A_* := A_{11} - A_{12}b_2^{-1}b_1$, obtained by direct computation in [MaZ3].

Denote by

$$\Phi^0 = (\phi^0_{k+1}(x;\lambda), \cdots, \phi^0_{n+r}(x;\lambda)), \tag{2.21}$$

$$\Phi^+ = (\phi^+_1(x;\lambda), \cdots, \phi^+_k(x;\lambda) = (P_+v_1^+, \cdots, P_+v_k^+), \tag{2.22}$$

and

$$\Phi = (\Phi^+, \Phi^0), \tag{2.23}$$

the matrices whose columns span the subspaces of solutions of (2.1) that, respectively, decay at $x = +\infty$, and satisfy the prescribed boundary conditions at $x = 0$, denoting (analytically chosen) complementary subspaces by

$$\Psi^0 = (\psi^0_1(x;\lambda), \cdots, \psi^0_k(x;\lambda)), \tag{2.24}$$

$$\Psi^+ = (\psi^+_{k+1}(x;\lambda), \cdots, \psi^+_{n+r}(x;\lambda)) \tag{2.25}$$

and

$$\Psi = (\Psi^0, \Psi^+). \tag{2.26}$$

As described in the previous subsection, eigenfunctions decaying at $+\infty$ and satisfying the prescribed boundary conditions at 0 occur precisely when the subspaces span Φ^0 and span Φ^+ intersect, i.e., at zeros of the Evans function defined in (2.8):

$$D_L(\lambda) := \det(\Phi^0, \Phi^+)_{|x=0}. \tag{2.27}$$

Define the solution operator from y to x of $(L-\lambda)U = 0$, denoted by $\mathcal{F}^{y\to x}$, as

$$\mathcal{F}^{y\to x} = \Phi(x,\lambda)\Phi^{-1}(y,\lambda)$$

and the projections Π^0_y, Π^+_y on the stable manifolds at $0, +\infty$ as

$$\Pi^+_y = \begin{pmatrix}\Phi^+(y) & 0\end{pmatrix}\Phi^{-1}(y), \quad \Pi^0_y = \begin{pmatrix}0 & \Phi^0(y)\end{pmatrix}\Phi^{-1}(y).$$

With these preparations, the construction of the Resolvent kernel goes exactly as in the construction performed in [ZH, MaZ3] on the whole line.

Lemma 2.13. *We have the the representation*

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\mathcal{F}^{y\to x}\Pi^+_y\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x>y,\\ -(I_n,0)\mathcal{F}^{y\to x}\Pi^0_y\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x<y.\end{cases} \tag{2.28}$$

Moreover, on any compact subset K of $\rho(L)\cap\Lambda$,

$$|G_\lambda(x,y)| \le Ce^{\eta|x-y|}, \tag{2.29}$$

where $C>0$ and $\eta>0$ depend only on K, L.

We define also the dual subspaces of solutions of $(L^*-\lambda^*)\tilde{W} = 0$. We denote growing solutions

$$\tilde{\Phi}^0 = \begin{pmatrix}\tilde{\phi}^0_1(x;\lambda) & \cdots & \tilde{\phi}^0_k(x;\lambda)\end{pmatrix}, \tag{2.30}$$

$$\tilde{\Phi}^+ = \begin{pmatrix}\tilde{\phi}^+_{k+1}(x;\lambda) & \cdots & \tilde{\phi}^+_{n+r}(x;\lambda)\end{pmatrix}, \tag{2.31}$$

$\tilde{\Phi} := (\tilde{\Phi}^0, \tilde{\Phi}^+)$ and decaying solutions

$$\tilde{\Psi}^0 = \begin{pmatrix}\tilde{\psi}^0_1(x;\lambda) & \cdots & \tilde{\psi}^+_k(x;\lambda)\end{pmatrix}, \tag{2.32}$$

$$\tilde{\Psi}^+ = \begin{pmatrix}\tilde{\psi}^+_{k+1}(x;\lambda) & \cdots & \tilde{\psi}^+_{n+r}(x;\lambda),\end{pmatrix} \tag{2.33}$$

and $\tilde{\Psi} := (\tilde{\Psi}^0, \tilde{\Psi}^+)$, satisfying the relations

$$(\tilde{\Psi}\tilde{\Phi})^*_{0,+}\tilde{S}(\Psi\Phi)_{0,+} \equiv I.$$

Then, we have

Proposition 2.14. *The resolvent kernel may alternatively be expressed as*

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\Phi^+(x;\lambda)M^+(\lambda)\tilde{\Psi}^{0*}(y;\lambda)(I_n,0)^{tr} & x>y,\\ -(I_n,0)\Phi^0(x;\lambda)M^0(\lambda)\tilde{\Psi}^{+*}(y;\lambda)(I_n,0)^{tr} & x<y,\end{cases} \tag{2.34}$$

where

$$M(\lambda) := \operatorname{diag}(M^+(\lambda), M^0(\lambda)) = \Phi^{-1}(z;\lambda)\bar{\mathcal{S}}^{-1}(z)\tilde{\Psi}^{-1*}(z;\lambda). \tag{2.35}$$

From Proposition 2.14, we obtain the following scattering decomposition, generalizing the Fourier transform representation in the constant-coefficient case

Corollary 2.15. *On* $\Lambda \cap \rho(L)$,

$$G_\lambda(x,y) = \sum_{j,k} d^+_{jk}\phi^+_j(x;\lambda)\tilde\psi^+_k(y;\lambda)^* + \sum_k \phi^+_k(x;\lambda)\tilde\phi^+_k(y;\lambda)^* \tag{2.36}$$

for $0 \le y \le x$, *and*

$$G_\lambda(x,y) = \sum_{j,k} d^0_{jk}(\lambda)\phi^+_j(x;\lambda)\tilde\psi^+_k(y;\lambda)^* + \sum_k \psi^+_k(x;\lambda)\tilde\psi^+_k(y;\lambda)^* \tag{2.37}$$

for $0 \le x \le y$, *where* $d^{0,+}_{jk}(\lambda) = \mathcal{O}(\lambda^{-K})$ *are scalar meromorphic functions with pole of order* K *less than or equal to the order to which the Evans function* $D(\lambda)$ *vanishes at* $\lambda = 0$ *(note that* $K = 0$ *under assumption (D)).*

Proof. Matrix manipulation of expression (2.35), Kramer's rule, and the definition of the Evans function; see [MaZ3]. □

Remark 2.16. *In the constant-coefficient case, with a choice of common bases* $\Psi^{0,+} = \Phi^{+,0}$ *at* $0, +\infty$, *the above representation* (2.15) *reduces to the simple formula*

$$G_\lambda(x,y) = \begin{cases} \sum_{j=k+1}^N \phi^+_j(x;\lambda)\tilde\phi^{+*}_j(y;\lambda) & x > y, \\ -\sum_{j=1}^k \psi^+_j(x;\lambda)\tilde\psi^{+*}_j(y;\lambda) & x < y. \end{cases} \tag{2.38}$$

2.3. **High frequency estimates.** We now turn to the crucial estimation of the resolvent kernel in the high-frequency regime $|\lambda| \to +\infty$, following the general approach of [MaZ3]. Define sectors

$$\Omega_P := \{\lambda \ : \ \Re e\lambda \ge -\theta_1|\Im m\lambda| + \theta_2\}, \quad \theta_j > 0. \tag{2.39}$$

and

$$\Omega := \{\lambda \ : \ -\eta_1 \le \Re e\lambda\} \tag{2.40}$$

with η_1 sufficiently small such that $\Omega \setminus B(0,r)$ is compactly contained in the set of consistent splitting Λ, for some small r to be chosen later. Then, we have the following crucial result analogous to the estimates on the whole line performed in [MaZ3].

Proposition 2.17. *Assume that (A1)-(A3), (H0)-(H2), and (B) hold. Then for any* $r > 0$ *and* $\eta_1 = \eta_1(r) > 0$ *chosen sufficiently small such that* $\Omega \setminus B(0,r) \subset \Lambda \cap \rho(L)$. *Moreover for* $R > 0$ *sufficiently large, the following decomposition holds on* $\Omega \setminus B(0,R)$*:*

$$G_\lambda(x,y) = H_\lambda(x,y) + P_\lambda(x,y) + \Theta^H_\lambda(x,y) + \Theta^P_\lambda(x,y), \tag{2.41}$$

where

$$H_\lambda(x,y) = \begin{cases} \chi_{\{A_*>0\}}A_*(x)^{-1}e^{\int_y^x(-\lambda/A_* - \eta_*/A_*)(z)dz}R_*L^{tr}_* & x > y, \\ \chi_{\{A_*<0\}}A_*(x)^{-1}e^{\int_y^x(-\lambda/A_* - \eta_*/A_*)(z)dz}R_*L^{tr}_* & x < y, \end{cases} \tag{2.42}$$

and

$$\begin{aligned}\Theta^H_\lambda(x,y) &= \lambda^{-1}B_\lambda(x,y;\lambda)+\lambda^{-1}(x-y)C_\lambda(x,y;\lambda),\\ \Theta^P_\lambda(x,y) &= \lambda^{-2}D_\lambda(x,y;\lambda)\end{aligned} \tag{2.43}$$

where

$$B_\lambda(x,y)=C_\lambda(x,y)=\begin{cases}\chi_{\{A_*>0\}}e^{-\int_y^x \lambda/A_*(z)dz}b_*(x,y) & x>y,\\ \chi_{\{A_*<0\}}e^{-\int_y^x \lambda/A_*(z)dz}b_*(x,y) & x<y,\end{cases} \tag{2.44}$$

with

$$b_* := e^{\int_y^x(-\eta_*/A_*)(z)dz} = \mathcal{O}(e^{-\theta|x-y|}), \tag{2.45}$$

due to (1.29)*, and*

$$D_\lambda(x,y;\lambda) = \mathcal{O}(e^{-\theta(1+Re\lambda)|x-y|}+e^{-\theta|\lambda|^{1/2}|x-y|}), \tag{2.46}$$

for some uniform $\theta>0$ independent of x,y,z, each described term separately analytic in λ, and P_λ is analytic in λ on a (larger) sector Ω_P as in (2.39)*, with θ_1 sufficiently small, and θ_2 sufficiently large, satisfying uniform bounds*

$$(\partial/\partial x)^\alpha(\partial/\partial y)^\beta P_\lambda(x,y) = \mathcal{O}(|\lambda|^{(|\alpha|+|\beta|-1)/2})e^{-\theta|\lambda|^{1/2}|x-y|}, \quad \theta>0, \tag{2.47}$$

for $|\alpha|+|\beta|\le 2$ and $0\le|\alpha|,|\beta|\le 1$.

Likewise, the following derivative bounds also hold:

$$\begin{aligned}(\partial/\partial x)\Theta_\lambda(x,y) =& \Big(B^0_x(x,y;\lambda)+(x-y)C^0_x(x,y;\lambda)\Big)+\lambda^{-1}\Big(B^1_x(x,y;\lambda)\\ &+(x-y)C^1_x(x,y;\lambda)+(x-y)^2D^1_x(x,y;\lambda)\Big)\\ &+\lambda^{-3/2}E_x(x,y;\lambda)\end{aligned}$$

and

$$\begin{aligned}(\partial/\partial y)\Theta_\lambda(x,y) =& \Big(B^0_y(x,y;\lambda)+(x-y)C^0_y(x,y;\lambda)\Big)+\lambda^{-1}\Big(B^1_y(x,y;\lambda)\\ &+(x-y)C^1_y(x,y;\lambda)+(x-y)^2D^1_y(x,y;\lambda)\Big)\\ &+\lambda^{-3/2}E_y(x,y;\lambda)\end{aligned}$$

where B^α_β, C^α_β, and D^1_β satisfy bounds of the form (2.44)*, and E_β satisfies a bound of the form* (2.46)*.*

Proof. We shall follow closely the argument in [MaZ3], with the new feature of boundary treatments, or estimates of Φ^0,Ψ^0. Writing the associated eigenvalue equation $LU-\lambda U=0$ in the form of a first-order system as follows: $W:=(u,v,z)\in\mathbb{C}^{2n-1}$ with $z:=b_1u'+b_2v'$, and

$$\begin{aligned}u' &= A_*^{-1}(-A_{12}b_2^{-1}z-(A'_{11}+\lambda)u-A'_{12}v),\\ v' &= b_2^{-1}z-b_2^{-1}b_1u',\\ z' &= (A_{21}-A_{22}b_2^{-1}b_1)u'+A_{22}b_2^{-1}z+A'_{21}u+(A'_{22}+\lambda)v\end{aligned} \tag{2.48}$$

or

$$W' = AW. \tag{2.49}$$

Recall from Lemma 2.13 that we have the the representation

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\mathcal{F}_W^{y\to x}\Pi_W^+(y)\tilde S^{-1}(y)(I_n,0)^{tr}, & for \quad x>y,\\ -(I_n,0)\mathcal{F}_W^{y\to x}\Pi_W^0(y)\tilde S^{-1}(y)(I_n,0)^{tr}, & for \quad x<y. \end{cases} \tag{2.50}$$

We shall find it more convenient to use the "local" coordinates $\tilde u := A_* u, \tilde v := b_1 u + b_2 v$. yielding from (2.14):

$$\begin{aligned} \tilde u_x &= -\lambda A_*^{-1}\tilde u - (A_{12}b_2^{-1}\tilde v)_x\\ (\tilde v_x)_x &= \Big[((A_{21} - A_{22}b_2^{-1}b_1 + b_2\partial_x(b_2^{-1}b_1))A_*^{-1}\tilde u)_x\\ &\quad + ((A_{22} + \partial_x(b_2)b_2^{-1})\tilde v)_x + \lambda b_2^{-1}b_1A_*^{-1}\tilde u + \lambda b_2^{-1}\tilde v\Big]. \end{aligned} \tag{2.51}$$

Following standard procedure (e.g., [AGJ, GZ, ZH, MaZ3]), performing the rescaling

$$\tilde x := |\lambda| x, \quad \tilde\lambda := \lambda/|\lambda|, \tag{2.52}$$

and changing coordinates $W \mapsto Y = \mathcal{Q}W$, where

$$Y = (\tilde u, \tilde v, \tilde v_x)^{tr} = (A_*u, b_1u + b_2v, (b_1u+b_2v)_x)^{tr}, \tag{2.53}$$

$$\mathcal{Q} = \begin{pmatrix} A_* & 0 & 0\\ b_1 & b_2 & 0\\ |\lambda|^{-1}\partial_x b_1 & |\lambda|^{-1}\partial_x b_2 & |\lambda|^{-1}I_r, \end{pmatrix} \tag{2.54}$$

and

$$\mathcal{Q}^{-1} = \begin{pmatrix} A_*^{-1} & 0 & 0\\ -b_2^{-1}b_1A_*^{-1} & b_2^{-1} & 0\\ -|\lambda|b_2\partial_x(b_2^{-1}b_1)A_*^{-1} & -|\lambda|\partial_x(b_2)b_2^{-1} & |\lambda|I_r, \end{pmatrix} \tag{2.55}$$

we obtain the first order equations

$$Y' = A(\tilde x, |\lambda|^{-1})Y, \quad Y := (\tilde u, \tilde v, \tilde v')^{tr}, \quad ' := \partial_{\tilde x} \tag{2.56}$$

where

$$A(\tilde x, |\lambda|^{-1}) = A_0(\tilde x) + |\lambda|^{-1}A_1(\tilde x) + \mathcal{O}(|\lambda|^{-2}), \tag{2.57}$$

with

$$\begin{aligned} A_0(\tilde x) &= \begin{pmatrix} -\tilde\lambda A_*^{-1} & 0 & -A_{12}b_2^{-1}\\ 0 & 0 & I_r\\ 0 & 0 & 0 \end{pmatrix}\\ A_1(\tilde x) &= \begin{pmatrix} 0 & -\partial_x(A_{12}b_2^{-1}) & 0\\ 0 & 0 & 0\\ -\tilde\lambda d_* A_*^{-2} & \tilde\lambda b_2^{-1} & e_* b_2^{-1} \end{pmatrix} \end{aligned} \tag{2.58}$$

$$d_* := A_{21} - A_{22}b_2^{-1}b_1 - b_2^{-1}b_1A_* + b_2\partial_x(b_2^{-1}b_1),$$
$$e_* := A_{22} + d_*A_*^{-1}A_{12} + \partial_x(b_2). \tag{2.59}$$

We will carry out the details of the lower-order estimates in Proposition 2.17, leaving high-order estimates and derivative bounds as brief remarks at the end. First, observe that the representation (2.50) becomes

$$G_\lambda(x,y) = \begin{cases} (I_n,0)\mathcal{Q}^{-1}\mathcal{F}_Y^{y\to x}\Pi_Y^+(y)\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x>y,\\ -(I_n,0)\mathcal{Q}^{-1}\mathcal{F}_Y^{y\to x}\Pi_Y^0(y)\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x<y \end{cases} \tag{2.60}$$

where $\Pi_Y^{0,+}$ and $\mathcal{F}_Y^{y\to x}$ denote projections and flows in $Y-$coordinates.

2.3.1. *Initial diagonalization.* Applying the formal iterative diagonalization procedure described in [MaZ3, Proposition 3.12], one obtains the approximately block-diagonalized system

$$Z' = D(\tilde{x},|\lambda|^{-1})Z, \quad TZ := Y, \quad D := T^{-1}AT, \tag{2.61}$$

$$T(\tilde{x},|\lambda|^{-1}) = T_0(\tilde{x}) + |\lambda|^{-1}T_1(\tilde{x}) + \cdots + |\lambda|^{-3}T_3(\tilde{x}) \tag{2.62}$$

$$D(\tilde{x},|\lambda|^{-1}) = D_0(\tilde{x}) + |\lambda|^{-1}D_1(\tilde{x}) + \cdots + D_3(\tilde{x})|\lambda|^{-3} + \mathcal{O}(|\lambda|^{-4}), \tag{2.63}$$

where without loss of generality (since T_0 is uniquely determined up to a constant linear coordinate change)

$$T_0 := \begin{pmatrix} 1 & 0 & -\tilde{\lambda}^{-1}A_*A_{12}b_2^{-1}\\ 0 & I_r & 0\\ 0 & 0 & I_r \end{pmatrix}, \quad T_0^{-1} = \begin{pmatrix} 1 & 0 & \tilde{\lambda}^{-1}A_*A_{12}b_2^{-1}\\ 0 & I_r & 0\\ 0 & 0 & I_r \end{pmatrix} \tag{2.64}$$

and

$$D_0 := \begin{pmatrix} -\tilde{\lambda}A_*^{-1} & 0 & 0\\ 0 & 0 & I_r\\ 0 & 0 & 0 \end{pmatrix}, \quad D_1 := \begin{pmatrix} -\eta_*A_*^{-1} & 0 & 0\\ 0 & 0 & 0\\ 0 & \tilde{\lambda}b_2^{-1} & * \end{pmatrix} \tag{2.65}$$

with η_* as defined in (1.28); see Proposition 3.12 [MaZ3]. (Here, the simple block upper-triangular form of A_0 has been used to deduce the above simple form of D_0, D_1.)

2.3.2. *The parabolic block.* At this point, we have approximately diagonalized our system into a 1×1 hyperbolic block with eigenvalue $\tilde{\mu} = -\tilde{\lambda}/A_*$ of A_0, and a $2r\times 2r$ parabolic block

$$Z_p' = NZ_p \tag{2.66}$$

with

$$N := \begin{pmatrix} 0 & I_r\\ 0 & 0 \end{pmatrix} + |\lambda|^{-1}\begin{pmatrix} 0 & 0\\ \tilde{\lambda}b_2^{-1} & * \end{pmatrix} + \mathcal{O}(|\lambda|^{-2}). \tag{2.67}$$

Balancing this matrix N by transformations $\mathcal{B} := diag\{I_r, |\lambda|^{-1/2} I_r\}$ we get

$$\tilde{M} := \mathcal{B}^{-1} N \mathcal{B} = |\lambda|^{-1/2} \tilde{M}_1 + \mathcal{O}(|\lambda|^{-1}), \quad \tilde{M}_1 := \begin{pmatrix} 0 & I_r \\ \tilde{\lambda} b_2^{-1} & 0 \end{pmatrix} \tag{2.68}$$

Observe that $\sigma(\tilde{M}_1) = \pm\sqrt{\sigma(\tilde{\lambda} b_2^{-1})}$ has a uniform spectral gap of order one. Thus, there is a well-conditioned transformation $S = S(\tilde{M}_1)$ depending continuously on $\tilde{M}_1$ such that

$$\hat{M}_1 := S^{-1} \tilde{M}_1 S = \text{diag}\{\hat{M}^-, \hat{M}^+\} \tag{2.69}$$

with $\hat{M}_1^\pm$ uniformly positive/negative definite, respectively. Applying this coordinate change, and noting that the "dynamic error" $S^{-1}\partial_{\tilde{x}} S$ is of order $\partial_{\tilde{x}} \tilde{M}_1 = \mathcal{O}(|\lambda|^{-1})$, we obtain the formal expansion

$$\hat{M}(\tilde{x}, |\lambda|^{-1}) = |\lambda|^{-1/2} \text{diag}\{\hat{M}_1^-, \hat{M}_1^+\} + \mathcal{O}(|\lambda|^{-1}). \tag{2.70}$$

Finally, on sector Ω_P, blocks $|\lambda|^{-1/2} \hat{M}_1^\pm$ are exponentially separated to order $|\lambda|^{-1/2}$. Thus, by the *reduction lemma*, Lemma 2.9, there is a further transformation $\hat{S} := I_{2r} + \mathcal{O}(|\lambda|^{-1/2})$ converting $\hat{M}$ to the fully diagonalized form

$$\begin{aligned} M(\tilde{x}, |\lambda|^{-1}) &:= |\lambda|^{-1/2} \hat{S}^{-1} \Big(\hat{M}_1 + \mathcal{O}(|\lambda|^{-1/2}) \Big) \hat{S} \\ &= \mathcal{O}(|\lambda|^{-1/2}) \text{diag}\{M_1^-, M_1^+\} \end{aligned}$$

where $M_1^\pm = \hat{M}_1^\pm + \mathcal{O}(|\lambda|^{-1/2})$ are still uniformly positive/negative definite.

In summary, changing coordinates

$$\mathcal{B} S \hat{S} \hat{Z}_p = Z_p, \tag{2.71}$$

(2.66) yields

$$\hat{Z}_p' = \mathcal{O}(|\lambda|^{-1/2}) \begin{pmatrix} M_1^- & 0 \\ 0 & M_1^+ \end{pmatrix} \hat{Z}_p + \mathcal{O}(|\lambda|^{-3/2}) \tag{2.72}$$

Therefore the transformation

$$\mathcal{T} := (T_0 + |\lambda|^{-1} T_1) \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{B} S \hat{S} \end{pmatrix} \tag{2.73}$$

converts equations (2.56) to the following:

$$\begin{aligned} \zeta' &= -(\tilde{\lambda} A_*^{-1} + |\lambda|^{-1} \eta_* A_*^{-1}) \zeta + \mathcal{O}(|\lambda|^{-2}) \\ \rho_\pm' &= |\lambda|^{-1/2} M_1^\pm \rho_\pm + \mathcal{O}(|\lambda|^{-3/2}) \end{aligned} \tag{2.74}$$

by relation

$$\mathcal{T} \mathcal{Z} = Y, \quad \mathcal{Z} = (\zeta, \rho_-, \rho_+)^{tr}. \tag{2.75}$$

Then, we have the the representation

(2.76)
$$G_\lambda(x,y)=\begin{cases}(I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi^+_{\mathcal{Z}}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x>y,\\ -(I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi^0_{\mathcal{Z}}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x<y,\end{cases}$$

thanks to the fact that

$$\mathcal{F}_Y^{y\to x}=\mathcal{T}\mathcal{F}_{\mathcal{Z}}^{y\to x}\mathcal{T}^{-1},\quad \Pi^+_Y=\mathcal{T}\Pi^+_{\mathcal{Z}}\mathcal{T}^{-1}. \tag{2.77}$$

Computing, we have

$$\mathcal{T}=\begin{pmatrix}1 & |\lambda|^{-1/2} & |\lambda|^{-1/2}\\ 0 & \mathcal{O}(1) & \mathcal{O}(1)\\ 0 & |\lambda|^{-1/2} & |\lambda|^{-1/2}\end{pmatrix}\qquad \mathcal{T}^{-1}=\begin{pmatrix}1 & 0 & \tilde{\lambda}^{-1}A_*A_{12}b_2^{-1}\\ 0 & \mathcal{O}(1) & |\lambda|^{1/2}\\ 0 & \mathcal{O}(1) & |\lambda|^{1/2}\end{pmatrix}$$

and

$$(I_n,0)\mathcal{Q}^{-1}=\begin{pmatrix}A_*^{-1} & 0 & 0\\ -b_2^{-1}b_1A_*^{-1} & b_2^{-1} & 0\end{pmatrix} \tag{2.78}$$

$$(I_n,0)\mathcal{Q}^{-1}\mathcal{T}=\begin{pmatrix}A_*^{-1} & \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2})\\ -b_2^{-1}b_1A_*^{-1} & \mathcal{O}(1) & \mathcal{O}(1)\end{pmatrix} \tag{2.79}$$

and

$$\mathcal{Q}\tilde{S}^{-1}(I_n,0)^{tr}=\begin{pmatrix}-1 & 0\\ 0 & 0\\ |\lambda|^{-1} & |\lambda|^{-1}I_r\end{pmatrix} \tag{2.80}$$

$$\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(I_n,0)^{tr}=\begin{pmatrix}-1+|\lambda|^{-1} & \mathcal{O}(|\lambda|^{-1})\\ \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2})\\ \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2})\end{pmatrix} \tag{2.81}$$

Therefore now we are ready to estimate $\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi^+_{\mathcal{Z}}$ and $\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi^+_{\mathcal{Z}}$.

2.3.3. *Estimates on projections and solution operators.* We shall give estimates on the projections:

$$\Pi^+_{\mathcal{Z}}=(\Phi^+,0)(\Phi^+,\Phi^0)^{-1},\quad \Pi^0_{\mathcal{Z}}=(0,\Phi^0)(\Phi^+,\Phi^0)^{-1} \tag{2.82}$$

and the solution operators:

$$\mathcal{F}_{\mathcal{Z}}^{y\to x}=(\Phi^+(x),\Phi^0(x))(\Phi^+(y),\Phi^0(y))^{-1}. \tag{2.83}$$

First, let Φ^{p+}/Ψ^{p+} be the decaying/growing basis solutions of

$$\rho_-'=|\lambda|^{-1/2}M_1^-\rho_-\quad\text{and}\quad \rho_+'=|\lambda|^{-1/2}M_1^+\rho_+ \tag{2.84}$$

and ϕ^{h+}/ψ^{h+} be the decaying/growing basis solutions of

$$\zeta'=-(\tilde{\lambda}A_*^{-1}+|\lambda|^{-1}\eta_*A_*^{-1})\zeta. \tag{2.85}$$

Lemma 2.18. [Inflow case] *For the inflow case $A_* > 0$, we obtain*

$$\Pi_{\mathcal{Z}}^{+} = \begin{pmatrix} 1 & 0 & -|\lambda|^{-1/2}\phi^{h+}e(\lambda)\Psi^{p+^{-1}} \\ 0 & I_r & -\Phi^{p+}E(\lambda)\Psi^{p+^{-1}} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.86}$$

$$\Pi_{\mathcal{Z}}^{0} = \begin{pmatrix} 0 & 0 & |\lambda|^{-1/2}\phi^{h+}e(\lambda)\Psi^{p+^{-1}} \\ 0 & 0 & \Phi^{p+}E(\lambda)\Psi^{p+^{-1}} \\ 0 & 0 & I_r \end{pmatrix} \tag{2.87}$$

with bounded functions $e(\lambda), E(\lambda)$, and

$$\mathcal{F}_{\mathcal{Z}}^{y\to x} = \begin{pmatrix} \phi^{h+}(x)\phi^{h+}(y)^{-1} & 0 & 0 \\ 0 & \Phi^{p+}(x)\Phi^{p+}(y)^{-1} & 0 \\ 0 & 0 & \Psi^{p+}(x)\Psi^{p+}(y)^{-1} \end{pmatrix} \tag{2.88}$$

Proof. We have the decaying basis solution in $\mathcal{Z}$-coordinates of the first order equations (2.74)

$$\Phi^{+} = \begin{pmatrix} \phi^{h+} & 0 \\ 0 & \Phi^{p+} \\ 0 & 0 \end{pmatrix} + \mathcal{O}(|\lambda|^{-1}). \tag{2.89}$$

Since Φ^+ and Ψ^+ (exactly Ψ^{p+}) form a basis solution, we can write

$$\Phi^{0}(x) = e(\lambda)\begin{pmatrix} \phi^{h+} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \Phi^{p+}(x)E(\lambda) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Psi^{p+}(x)F(\lambda) \end{pmatrix} \tag{2.90}$$

Now since $\{\psi_j^{p+}\}_j$ forms a basis, we can take $\{\psi_j^{p+}(0)\}$ to be the analytic basis for Y at $x = 0$. Also as we recall that $\mathcal{Z} = \mathcal{T}^{-1}Y$, we compute

$$\phi^{0}_{j|_{x=0}} = \mathcal{T}^{-1}\begin{pmatrix} 0 \\ 0 \\ \psi_j^{p+}(0) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ |\lambda|^{1/2}\psi_j^{p+}(0) \\ |\lambda|^{1/2}\psi_j^{p+}(0) \end{pmatrix} \tag{2.91}$$

This and (2.90) yield

$$\Phi^{0}(x) = \begin{pmatrix} e(\lambda)\phi^{h+}(x) \\ |\lambda|^{1/2}\Phi^{p+}(x)E(\lambda) \\ |\lambda|^{1/2}\Psi^{p+}(x) \end{pmatrix} + \mathcal{O}(|\lambda|^{-1/2}) \tag{2.92}$$

where

$$E(\lambda) = (E_1(\lambda), \dots, E_r(\lambda))^{tr}, \quad E_j(\lambda) = \psi_j^{p+}(0,\lambda)\Phi^{p+}(0,\lambda)^{-1} \tag{2.93}$$

and $e(\lambda), E_j(\lambda) \in \mathbb{R}^r$ are bounded functions in λ. Therefore computing, we get

$$(\Phi^{+}, \Phi^{0})^{-1} = \begin{pmatrix} \phi^{h+^{-1}} & 0 & -|\lambda|^{-1/2}e(\lambda)\Psi^{p+^{-1}} \\ 0 & \Phi^{p+^{-1}} & -E(\lambda)\Psi^{p+^{-1}} \\ 0 & 0 & |\lambda|^{-1/2}\Psi^{p+^{-1}} \end{pmatrix} \tag{2.94}$$

and hence straightforward computations give the lemma. □

Lemma 2.19. [Outflow case] *For the outflow case $A_* < 0$, we obtain*

$$\Pi^+_{\mathcal{Z}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_r & -\Phi^{p+}E\Psi^{p+-1} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi^0_{\mathcal{Z}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \Phi^{p+}E\Psi^{p+-1} \\ 0 & 0 & I_r \end{pmatrix}, \tag{2.95}$$

where $E(\lambda)$ is a bounded function in λ determined below. Moreover,

$$\mathcal{F}^{y\to x}_{\mathcal{Z}} = \begin{pmatrix} \psi^{h+}(x)\psi^{h+}(y)^{-1} & 0 & 0 \\ 0 & \Phi^{p+}(x)\Phi^{p+}(y)^{-1} & 0 \\ 0 & 0 & \Psi^{p+}(x)\Psi^{p+}(y)^{-1} \end{pmatrix} \tag{2.96}$$

Proof. Similarly, we have $\Phi^+ = \Phi^{p+}$ and $\Phi^0 = (\phi^{h0}, \Phi^{p0})$ where we can write

$$\Phi^0(x) = \begin{pmatrix} 0 \\ \Phi^{p+}(x)E(\lambda) \\ 0 \end{pmatrix} + e(\lambda)\begin{pmatrix} \psi^{h+} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Psi^{p+}(x)F(\lambda) \end{pmatrix}. \tag{2.97}$$

As before, using the form of the linearized boundary conditions (1.15), we can take

$$\phi^{p0}_{j|_{x=0}} = \mathcal{T}^{-1}\begin{pmatrix} 0 \\ 0 \\ \psi^{p+}_j(0) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ |\lambda|^{1/2}\psi^{p+}_j(0) \\ |\lambda|^{1/2}\psi^{p+}_j(0) \end{pmatrix} \tag{2.98}$$

and thus

$$\Phi^{p0}(x) = \begin{pmatrix} e(\lambda)\psi^{h+}(x) \\ |\lambda|^{1/2}\Phi^{p+}(x)E(\lambda) \\ |\lambda|^{1/2}\Psi^{p+}(x) \end{pmatrix} \tag{2.99}$$

with bounded functions $e(\lambda)$ and $E_j(\lambda) = \psi^{p+}_j(0,\lambda)\Phi^{p+}(0,\lambda)^{-1}$.

Similarly, we take

$$\phi^{h0}_{|_{x=0}} = \mathcal{T}^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and thus

$$\phi^{h0}(x) = \begin{pmatrix} \psi^{h+}(x) \\ 0 \\ 0 \end{pmatrix} \tag{2.100}$$

Putting together and computing, we obtain

$$(\Phi^+, \Phi^0) = \begin{pmatrix} 0 & \psi^{h+} & e(\lambda)\psi^{h+} \\ \Phi^{p+} & 0 & |\lambda|^{1/2}\Phi^{p+}E(\lambda) \\ 0 & 0 & |\lambda|^{1/2}\Psi^{p+} \end{pmatrix} \tag{2.101}$$

and

$$(\Phi^+,\Phi^0)^{-1}=\begin{pmatrix}0 & \Phi^{p+^{-1}} & -E(\lambda)\Psi^{p+^{-1}}\\ \psi^{h+^{-1}} & 0 & -|\lambda|^{-1/2}e(\lambda)\Psi^{p+^{-1}}\\ 0 & 0 & |\lambda|^{-1/2}\Psi^{p+^{-1}}\end{pmatrix} \tag{2.102}$$

Direct computations yield the lemma. □

2.3.4. *Estimates on G_λ: Inflow case $A_*>0$.* Now we are ready to combine all above estimates to give the bounds on resolvent kernel G_λ. We shall work in detail for the case $x>y$. Similar estimates can be easily obtained for $x<y$. First decompose the projection as $\Pi^+_{\mathcal{Z}}=\Pi^{h+}_{\mathcal{Z}}+\Pi^{p+}_{\mathcal{Z}}$ where

$$\begin{aligned}\Pi^{h+}_{\mathcal{Z}}&=\begin{pmatrix}1 & 0 & -|\lambda|^{-1/2}\phi^{h+}e(\lambda)\Psi^{p+^{-1}}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ \Pi^{p+}_{\mathcal{Z}}&=\begin{pmatrix}0 & 0 & 0\\ 0 & I_r & -\Phi^{p+}E(\lambda)\Psi^{p+^{-1}}\\ 0 & 0 & I_r\end{pmatrix}\end{aligned} \tag{2.103}$$

Hence

$$\begin{aligned}H_\lambda(x,y)&=(I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}^{y\to x}_{\mathcal{Z}}\Pi^{h+}_{\mathcal{Z}}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde S^{-1}(y)(I_n,0)^{tr}\\ &=\phi^{h+}(x)\phi^{h+}(y)^{-1}\begin{pmatrix}(-1+\mathcal{O}(|\lambda|^{-1}))A_*^{-1} & \mathcal{O}(|\lambda|^{-1})A_*^{-1}\\ (1+\mathcal{O}(|\lambda|^{-1}))b_2^{-1}b_1A_*^{-1} & \mathcal{O}(|\lambda|^{-1})b_2^{-1}b_1A_*^{-1}\end{pmatrix}\\ &=\phi^{h+}(x)\phi^{h+}(y)^{-1}\begin{pmatrix}-A_*^{-1}(x) & 0\\ b_2^{-1}b_1A_*^{-1}(x) & 0\end{pmatrix}+\mathcal{O}(|\lambda|^{-1})\phi^{h+}(x)\phi^{h+}(y)^{-1},\\ &=\phi^{h+}(x)\phi^{h+}(y)^{-1}R_*L_*^{tr}+\mathcal{O}(|\lambda|^{-1})\phi^{h+}(x)\phi^{h+}(y)^{-1},\end{aligned}$$

recalling that $\phi^{h+}(x)\phi^{h+}(y)^{-1}$ is the solution operator of hyperbolic equation in (2.85) and thus satisfies

$$\phi^{h+}(x)\phi^{h+}(y)^{-1}=e^{\int_{\tilde y}^{\tilde x}(-1/A_*-|\lambda|^{-1}\eta_*/A_*)(z)dz}=e^{\int_y^x(-\lambda/A_*-\eta_*/A_*)(z)dz}. \tag{2.104}$$

At the same time, computing $P_\lambda(x,y)$, we obtain

$$\begin{aligned}P_\lambda(x,y)&=(I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}^{y\to x}_{\mathcal{Z}}\Pi^{p+}_{\mathcal{Z}}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde S^{-1}(y)(I_n,0)^{tr}\\ &=\mathcal{O}(|\lambda|^{-1/2})\Phi^{p+}(x)\Phi^{p+}(y)^{-1}\end{aligned}$$

recalling that $\Phi^{p+}(x)\Phi^{p+}(y)^{-1}$ is the (stable) solution operator of parabolic equation (2.84), with M_1^- uniformly negative definite, and thus we have an obvious estimate

$$|\Phi^{p+}(x)\Phi^{p+}(y)^{-1}|\le Ce^{-\theta|\lambda|^{-1/2}(\tilde x-\tilde y)}\le Ce^{-\theta|\lambda|^{1/2}(x-y)}. \tag{2.105}$$

We therefore obtain

$$P_\lambda(x,y)=\mathcal{O}(|\lambda|^{-1/2})e^{-\theta|\lambda|^{1/2}(x-y)}. \tag{2.106}$$

2.3.5. *Estimates on G_λ: Outflow case $A_* < 0$.* Again as above, we shall work in detail for the case $x > y$. Similar estimates can be easily obtained for $x < y$. Estimates in Lemma 2.19 yield

$$\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi_{\mathcal{Z}}^{+}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Phi^{p+}(x)\Phi^{p+}(y)^{-1} & -\Phi^{p+}(x)E(\lambda)\Psi^{p+}(y)^{-1} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.107}$$

where $\Phi^{p+}(x)E(\lambda)\Psi^{p+}(y)^{-1} \le C\Phi^{p+}(x)\Phi^{p+}(y)^{-1}$. Observe that $\Pi_{\mathcal{Z}}^{h+} \equiv 0$. Therefore, $H_\lambda(x, y) = 0$ and

$$\begin{aligned} P_\lambda(x, y) &= (I_n, 0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi_{\mathcal{Z}}^{p+}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(y)(I_n, 0)^{tr} \\ &= \Phi^{p+}(x)\Phi^{p+}(y)^{-1}\begin{pmatrix} \mathcal{O}(|\lambda|^{-1}) & \mathcal{O}(|\lambda|^{-1}) \\ \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2}) \end{pmatrix} \\ &\le C|\lambda|^{-1/2}e^{-\theta|\lambda|^{1/2}(x-y)} \end{aligned}$$

We thus complete the proof of estimates H_λ and of P_λ appearing in Proposition 2.17.

2.3.6. *Derivative estimates.* Derivative estimates now follow in a straightforward fashion, by differentiation of (2.76), noting from the approximately decoupled equations that differentiation of the flow brings down a factor (to absorbable error) of λ in hyperbolic modes, $\lambda^{1/2}$ in parabolic modes. This completes the proof of Proposition 2.17. □

2.4. **Low frequency estimates.** Our goal in this section is the estimation of the resolvent kernel in the critical regime $|\lambda| \to 0$, i.e., the large time behavior of the Green function G, or global behavior in space and time. We are basically following the same treatment as that carried out for viscous shock waves of strictly parabolic conservation laws in [ZH, MaZ3]; we refer to those references for details. In the low frequency case the behavior is essentially governed by the limiting far-field equation

$$U_t = L_+U := -A_+U_x + B_+U_{xx} \tag{2.108}$$

Lemma 2.20 ([MaZ3])**.** *Assuming (A1)–(A3), (H0)-(H2), for $|\lambda|$ sufficiently small, the eigenvalue equation $(L_+ - \lambda)W = 0$ associated with the limiting, constant-coefficient operator L_+, considered as a first-order system $W' = \mathbb{A}_+W$, $W = (u, v, v')$, has a basis of $2n - 1$ solutions $\bar{W}_j^+ = e^{\mathbb{A}_+(\lambda)x}V_j(\lambda)$, consisting of $n - 1$ "fast" modes (not necessarily eigenmodes)*

$$|e^{\mathbb{A}_+(\lambda)x}V_j| \le Ce^{-\theta|x|}, \quad \theta > 0, \tag{2.109}$$

and n analytic "slow" (eigen-)modes

$$\begin{aligned} &e^{\mathbb{A}_+(\lambda)x}V_j = e^{\mu_j(\lambda)x}V_j, \\ &\mu_{n-1+j}^+(\lambda) := -\lambda/a_j^+ + \lambda^2\beta_j^+/{a_j^+}^3 + \mathcal{O}(\lambda^3), \\ &V_{n-1+j}^+(\lambda) := r_j^+ + \mathcal{O}(\lambda), \end{aligned} \tag{2.110}$$

where a_j^+, l_j^+, r_j^+, β_j^+ *are defined as in Proposition 1.9. The same is true for the adjoint eigenvalue equation*

$$(L_+ - \lambda)^* Z = 0,$$

i.e, it has a basis of solutions $\bar{\tilde{W}}_j^+ = e^{-\mathbb{A}_+^*(\lambda)x}\tilde{V}_j(\lambda)$ *with* $n-1$ *analytic "fast" modes*

$$|e^{-\mathbb{A}_+^*(\lambda)x}\tilde{V}_j| \le Ce^{-\theta|x|}, \quad \theta > 0, \tag{2.111}$$

and n *analytic "slow" (eigen-)modes*

$$\tilde{V}_{n-1+j}^+(\lambda) = l_j^+ + \mathcal{O}(\lambda). \tag{2.112}$$

Proof. Standard matrix perturbation theory; see [MaZ3], Appendix B. □

Also we recall from the representation of G_λ in Corollary 2.15:

Proposition 2.21. *Assuming (A1)–(A3), (H0)-(H2), let* K *be the order of the pole of* G_λ *at* $\lambda = 0$ *and* r *be sufficiently small that there are no other poles in* $B(0,r)$*. Then for* $\lambda \in \Omega_\theta$ *such that* $|\lambda| \le r$ *and we have*

$$G_\lambda(x,y) = \sum_{j,k} d_{jk}^+(\lambda)\phi_j^+(x)\tilde{\psi}_k^+(y) + \sum_k \phi_k^+(x)\tilde{\phi}_k^+(y), \tag{2.113}$$

for $x > y > 0$*, and*

$$G_\lambda(x,y) = \sum_{j,k} d_{jk}^0(\lambda)\phi_j^+(x)\tilde{\psi}_k^+(y) + \sum_k \psi_k^+(x)\tilde{\psi}_k^+(y), \tag{2.114}$$

for $0 < x < y$*, where* $d_{jk}^{0,+}(\lambda) = \mathcal{O}(\lambda^{-K})$ *are scalar meromorphic functions, moreover* $K \le$ *order of vanishing of the Evans function* $D(\lambda)$ *at* $\lambda = 0$.

Proof. See [ZH, Proposition 7.1] for the first statement and theorem 6.3 for the second statement linking order K of the pole to multiplicity of the zero of the Evans Function. □

Our main result of this section is then:

Proposition 2.22. *Assume (A1)–(A3), (H0)-(H2), and (D). Then, for* $r > 0$ *sufficiently small, the resolvent kernel* G_λ *associated with the linearized evolution equation* (2.108) *satisfies, for* $0 \le y \le x$:

$$\begin{aligned} &|\partial_x^\gamma \partial_y^\alpha G_\lambda(x,t;y)| \\ &\quad \le C(|\lambda|^\gamma + e^{-\theta|x|})(|\lambda|^\alpha + e^{-\theta|y|})\Big(\sum_{a_k^+>0} \big|e^{(-\lambda/a_k^+ + \lambda^2\beta_k^+/{a_k^+}^3)(x-y)}\big| \\ &\qquad + \sum_{a_k^+<0,\, a_j^+>0} \big|e^{(-\lambda/a_j^+ + \lambda^2\beta_j^+/{a_j^+}^3)x + (\lambda/a_k^+ - \lambda^2\beta_k^+/{a_k^+}^3)y}\big|\Big), \end{aligned} \tag{2.115}$$

$0 \le |\alpha|, |\gamma| \le 1$, $\theta > 0$, *with similar bounds for* $0 \le x \le y$*. Moreover, each term in the summation on the righthand side of* (2.115) *bounds a separately analytic function.*

Proof. By condition (D), $D(\lambda)$ does not vanish on $Re(\lambda) \geq 0$, hence, by continuity, on $|\lambda| \leq r$. Thus, according to Proposition 2.21, all $|d_{jk}(\lambda)|$ are uniformly bounded on $|\lambda| \leq r$, and thus it is enough to find estimates for fast and slow modes ϕ_j^+, $\tilde{\phi}_j^+$, ψ_j^+ and $\tilde{\psi}_j^+$. By applying Lemma 2.20 and using (2.22) we find:

$$\begin{pmatrix} \phi_j^+ \\ \partial_x \phi_j^+ \end{pmatrix} = e^{\mathbb{A}_+(\lambda)x} P^+ \begin{pmatrix} v_j \\ \mu_j v_j \end{pmatrix} = e^{\mathbb{A}_+(\lambda)x}(I + \Theta) \begin{pmatrix} v_j \\ \mu_j v_j \end{pmatrix} \tag{2.116}$$

and similarly for $\tilde{\phi}_j^+$, ψ_j^+ and $\tilde{\psi}_j^+$. Now using (2.4) and the fact, by Lemma 2.20, that $e^{\mu_j(\lambda)x}$ is of order $e^{-(\lambda/a_j^+ + \lambda^2 \beta_j^+/{a_j^+}^3 + \mathcal{O}(\lambda^3))x}$ for slow modes and order $e^{-\theta|x|}$ for fast modes, so by substituting this and corresponding dual estimates in (2.116) and grouping terms, we obtain the result.

□

3. Pointwise bounds on Green function $G(x,t;y)$

In this section, we prove the pointwise bounds on the Green function G following the general approach of [MaZ3] in the whole-line, shock, case. Our starting point is the representation

$$G(x,t;y) = \frac{1}{2\pi i} P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_\lambda(x,y)\, d\lambda \tag{3.1}$$

where η is any sufficiently large positive real number.

Case I. $|x-y|/t$ **large.** We first treat the simple case that $|x-y|/t \geq S$, S sufficiently large. Fixing x, y, t, set $\lambda = \eta + i\xi$, for $\eta > 0$ sufficiently large. Applying Proposition 2.17, we obtain the decomposition

$$\begin{aligned} G(x,t;y) &= \frac{1}{2\pi i} P.V \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} \Big[H_\lambda + \Theta_\lambda^H + P_\lambda + \Theta_\lambda^P \Big](x,y) d\lambda \\ &=: I + II + III + IV. \end{aligned}$$

For definiteness considering the inflow case $A_* > 0$ and taking $x > y$, we estimate each term in turn.

Term I. Computing,

$$\begin{aligned} I &= \frac{1}{2\pi i} P.V \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} H_\lambda(x,y) d\lambda \\ &= \frac{1}{2\pi} A_*(x)^{-1} e^{\eta(t - \int_y^x 1/A_*(z)dz)} e^{-\int_y^x (\eta_*/A_*)(z)dz} P.V \int_{-\infty}^{+\infty} e^{i\xi(t - \int_y^x 1/A_*(z)dz)} d\xi \\ &= \frac{1}{2\pi} A_*(x)^{-1} \delta(t - \int_y^x 1/A_*(z)dz) e^{-\int_y^x (\eta_*/A_*)(z)dz} \\ &= \frac{1}{2\pi} A_*(x)^{-1} A_*(y) \delta_{x - \bar{a}_* t}(y) e^{-\int_y^x (\eta_*/A_*)(z)dz} \end{aligned}$$

where $\bar{a}_*$ is defined as in Proposition 1.9. Noting that $\bar{a}_* \geq \inf_x A_*(x) > 0$ and $\eta_*^+ > 0$, we get $e^{-\int_y^x (\eta_*/A_*)(z)dz} = \mathcal{O}(e^{-\theta(x-y)})$ and thus

$$I = \mathcal{O}(e^{-\theta t})\delta_{x-\bar{a}_* t}(y), \tag{3.2}$$

vanishing for $|x-y|/t$ large.

Term II. Similar calculations show that the "hyperbolic error term" II also vanishes. For example, the term $e^{\lambda t}\lambda^{-1}B(x,y;\lambda)$ contributes

$$\frac{1}{2\pi}e^{\eta(t-\int_y^x 1/A_*(z)dz)}e^{-\int_y^x(\eta_*/A_*)(z)dz}P.V\int_{-\infty}^{+\infty}(\eta+i\xi)^{-1}e^{i\xi(t-\int_y^x 1/A_*(z)dz)}d\xi.$$

The integral though not absolutely convergent, is integrable and uniformly bounded as a principal value integral, for all real η bounded away from zero, by explicit computation. On the other hand

$$e^{\eta(t-\int_y^x 1/A_*(z)dz)} \leq e^{\eta(t-|x-y|/\min_z A_*(z))} \leq e^{\eta t(1-S/\min_z A_*(z))} \to 0,$$

as $\eta \to +\infty$, for S sufficiently large. Thus, we find that the above integral term goes to zero. Likewise, the result applies for the term of $e^{\lambda t}C(x,y;\lambda)$, since $(x-y)e^{-\int_y^x(\eta_*/A_*)(z)dz} \leq C(x-y)e^{-\theta(x-y)}$ is also bounded. Thus, each term of II vanishes as $\eta \to +\infty$.

Term III. The parabolic term III may be treated exactly as in the strictly parabolic case [ZH]. Precisely, we may first deform the contour in the principle value integral to

$$\int_{\Gamma_1\cup\Gamma_2} e^{\lambda t}P_\lambda(x,y)\,d\lambda, \tag{3.3}$$

where $\Gamma_1 := \partial B(0,R)\cap\bar{\Omega}_P$ and $\Gamma_2 := \partial\Omega_P \setminus B(0,R)$, recalling the parabolic sector Ω_P defined in (2.39). Choose

$$\bar{\alpha} := \frac{|x-y|}{2\theta t}, \quad R := \theta\bar{\alpha}^2, \tag{3.4}$$

where θ is as in (2.47). Note that the intersection of Γ with the real axis is $\lambda_{min} = R = \theta\bar{\alpha}^2$. By the large $|\lambda|$ estimates of Proposition 2.17, we have for all $\lambda \in \Gamma_1 \cup \Gamma_2$ that

$$|P_\lambda(x,y)| \leq C|\lambda|^{-1/2}e^{-\theta|\lambda|^{1/2}|x-y|}.$$

Further, we have

$$\begin{aligned} Re\lambda &\leq R(1-\eta\omega^2), \quad \lambda\in\Gamma_1,\\ Re\lambda &\leq Re\lambda_0 - \eta(|Im\lambda|-|Im\lambda_0|), \quad \lambda\in\Gamma_2 \end{aligned} \tag{3.5}$$

for R sufficiently large, where ω is the argument of λ and λ_0 and λ_0^* are the two points of intersection of Γ_1 and Γ_2, for some $\eta > 0$ independent of $\bar{\alpha}$.

Combining these estimates, we obtain

$$
\begin{aligned}
\Big|\int_{\Gamma_1} e^{\lambda t} P_\lambda d\lambda\Big| &\le C\int_{\Gamma_1} |\lambda|^{-1/2}\, e^{Re\lambda t-\theta|\lambda|^{1/2}|x-y|}\, d\lambda \\
&\le Ce^{-\theta\bar\alpha^2 t}\int_{-arg(\lambda_0)}^{+arg(\lambda_0)} R^{-1/2}e^{-\theta R\eta\omega^2 t}\, Rd\omega \\
&\le Ct^{-1/2}e^{-\theta\bar\alpha^2 t}.
\end{aligned} \tag{3.6}
$$

Likewise,

$$
\begin{aligned}
|\int_{\Gamma_2} e^{\lambda t} P_\lambda d\lambda| &\le \int_{\Gamma_2} C|\lambda|^{-1/2}\, Ce^{Re\lambda t-\theta|\lambda|^{1/2}|x-y|} d\lambda \\
&\le Ce^{Re(\lambda_0)t-\theta|\lambda_0|^{1/2}|x-y|}\int_{\Gamma_2} |\lambda|^{-1/2}e^{(Re\lambda-Re\lambda_0)t}\; |d\lambda| \\
&\le Ce^{-\theta\bar\alpha^2 t}\int_{\Gamma_2} |Im\,\lambda|^{-1/2}e^{-\eta|Im\,\lambda-Im\,\lambda_0|t}\; |d\,Im\lambda| \\
&\le Ct^{-1/2}e^{-\theta\bar\alpha^2 t}.
\end{aligned} \tag{3.7}
$$

Combining these last two estimates, we have

$$
III \le Ct^{-1/2}e^{-\theta\bar\alpha^2 t/2}e^{-(x-y)^2/8\theta t} \le Ct^{-1/2}e^{-\eta t}e^{-(x-y)^2/8\theta t}, \tag{3.8}
$$

for $\eta > 0$ independent of $\bar\alpha$. Observing that $|x-at|/2t \le |x-y|/t \le 2|x-at|/t$ for any bounded a, for $|x-y|/t$ sufficiently large, we find that III may be absorbed in any summand $t^{-1/2}e^{-(x-y-a_k^+t)^2/Mt}$.

Term IV. Similarly, as in the treatment of the term III, the principle value integral for the "parabolic error term IV may be shifted to $\eta = R = \theta\bar\alpha^2$, $\bar\alpha$ as above. This yields an estimate

$$
|IV| \le Ce^{-\theta\bar\alpha^2 t}\int_{-\infty}^{+\infty} |\eta_0 + i\xi|^{-2}d\xi \le Ce^{-\theta\bar\alpha^2 t},
$$

absorbed in $\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})$ for all t.

Case II. $|x-y|/t$ **bounded.** We now turn to the critical case where $|x-y|/t \le S$, for some fixed S.

Decomposition of the contour: We begin by converting the contour integral (3.1) into a more convenient form decomposing high, intermediate, and low frequency contributions.

We first observe that L has no spectrum on the portion of Ω lying outside the rectangle

$$
\mathcal{R} := \{\lambda \;:\; -\eta_1 \le \Re\lambda \le \eta, -R \le \Im\lambda \le R\} \tag{3.9}
$$

for $\eta > 0$, $R > 0$ sufficiently large, hence G_λ is analytic on this region. Since, also, H_λ is analytic on the whole complex plane, contours involving either one of these contributions may be arbitrarily deformed within $\Omega \setminus \mathcal{R}$

without affecting the result, by Cauchy's theorem. Likewise, P_λ is analytic on $\Omega_P \setminus \mathcal{R}$, and so contours involving this contribution may be arbitrarily deformed within this region. Thus, we obtain

Observation 3.1 ([MaZ3]). *Assume (A1)–(A3), (H0)-(H2), and (D). Then, the principle value integral* (3.1) *may be replaced by*

$$G(x,t;y) = I_a + I_b + I_c + II_a + II_b + III \tag{3.10}$$

where

$$\begin{aligned}
I_a &:= P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} H_\lambda(x,y) d\lambda \\
I_b &:= P.V\Big(\int_{-\eta_1 - i\infty}^{-\eta_1 - iR} + \int_{-\eta_1 + iR}^{-\eta_1 + i\infty} \Big) e^{\lambda t}(G_\lambda - H_\lambda - P_\lambda)(x,y) d\lambda \\
I_c &:= \int_{\Gamma_2} e^{\lambda t} P_\lambda(x,y) d\lambda \\
II_a &:= \Big(\int_{-\eta_1 - iR}^{-\eta_1 - ir/2} + \int_{-\eta_1 + ir/2}^{-\eta_1 + iR} \Big) e^{\lambda t} G_\lambda(x,y) d\lambda \\
II_b &:= - \int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} H_\lambda(x,y) d\lambda \\
III &:= \int_{\Gamma_1} e^{\lambda t} G_\lambda(x,y) d\lambda
\end{aligned}$$

with

$$\begin{aligned}
\Gamma_1 &:= [-\eta_1 - ir/2, \eta - ir/2] \cup [\eta - ir/2, \eta + ir/2] \cup [\eta + ir/2, -\eta_1 + ir/2] \\
\Gamma_2 &:= \partial\Omega_P \setminus \Omega,
\end{aligned}$$

for any $\eta, r > 0$, R sufficiently large, and η_1 sufficiently small with respect to r.

Using the above decomposition (3.10), we shall estimate in turn the high-frequency contributions I_a, I_b, and I_c, the intermediate-frequency contributions II_a and II_b, and the low-frequency contributions III.

High-frequency contribution. We first carry out the straightforward estimation of the high-frequency terms I_a, I_b, and I_c. The principal term I_a has already been computed in (3.2) to be $H(x,t;y)$. Likewise, calculations similar to those of *Term II* show that the term I_b is time-exponentially small. For example, the term $e^{\lambda t}\lambda^{-1}B(x,y;\lambda)$ contributes

$$\begin{aligned}
&P.V.\Big(\int_{-\infty}^{-R} + \int_{R}^{+\infty} \Big) (-\eta_1 + i\xi)^{-1} e^{i\xi(t - \int_y^x 1/A_*(z)dz)} d\xi \\
&\quad \times e^{-\eta_1(t - \int_y^x 1/A_*(z)dz)} e^{-\int_y^x (\eta_*/A_*)(z)dz}
\end{aligned} \tag{3.11}$$

where

$$(3.12)\qquad P.V.\Big(\int_{-\infty}^{-R}+\int_{R}^{+\infty}\Big)(-\eta_1+i\xi)^{-1}e^{i\xi(t-\int_y^x 1/A_*(z)dz)}d\xi<\infty$$

and

$$(3.13)\quad e^{\eta_1\int_y^x 1/A_*(z)dz}e^{-\int_y^x(\eta_*/A_*)(z)dz}\le Ce^{\frac{\eta_1|x-y|}{\min_z A_*(z)}}e^{-\theta|x-y|}\le Ce^{-\theta|x-y|/2},$$

for η_1 sufficiently small. This contributes in the term $\mathcal{O}(e^{-\eta_1(t+|x-y|)})$ of R. Likewise, the contributions of terms $e^{\lambda t}\lambda^{-1}(x-y)C(x,y;\lambda)$ and $e^{\lambda t}\lambda^{-2}D(x,y;\lambda)$ split into the product of a convergent, uniformly bounded integral in ξ, a bounded factor analogous to (3.13), and the factor $e^{-\eta_1 t}$, giving the result.

The term I_c may be estimated exactly as was term III in the large $|x-y|/t$ case, to obtain contribution $\mathcal{O}(t^{-1/2}e^{-\eta_1 t})$ absorbable again in the residual term $\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})$ for $t\ge\epsilon$, any $\epsilon>0$, and by any summand $\mathcal{O}(t^{-1/2}(1+t)^{-1/2}e^{-(x-y-a_k^+)^2/Mt})e^{-\eta(x+y)}$ for t small.

Intermediate-frequency contribution. Error term II_b is time-exponentially small for η_1 sufficiently small, by the same calculation as in (3.11)-(3.13), hence negligible. Likewise, term II_a by the basic estimate (2.29) is seen to be time-exponentially small of order $\mathcal{O}(e^{-\eta_1 t})$ for any $\eta_1>0$ sufficiently small that the associated contour lies in the resolvent set of L.

Low-frequency contribution. It remains to estimate the low-frequency term III, which is of essentially the same form as the low-frequency contribution analyzed in [ZH, YZ] in the strictly parabolic case, in that the contour is the same and the resolvent kernel G_λ satisfies same bounds (with no E_λ term) in this regime. Thus, we may conclude from these previous analyses that III gives contribution as claimed, exactly as in the strictly parabolic case. For completeness, we indicate the main features of the argument here.

Bounded time. For t bounded, we can use the medium-λ bounds $|G_\lambda|$, $|G_{\lambda_x}|$, $|G_{\lambda_y}|\le C$ to obtain $|\int_{\Gamma_1}e^{\lambda t}G_\lambda d\lambda|\le C_2|\Gamma_1|$. This contribution is order $Ce^{-\eta t}$ for bounded time, hence can be absorbed.

Large time. For t large, we must instead estimate $\int_{\Gamma_1}e^{\lambda t}G_\lambda d\lambda$ using the small-$|\lambda|$ expansions. First, observe that, all coefficient functions $d_{jk}(\lambda)$ are uniformly bounded (since $|\lambda|$ is bounded in this case).

Case II(i). $(0<y<x)$. By our low-frequency estimates in Proposition

2.21, we have

$$\int_{\Gamma_1} e^{\lambda t} G_\lambda(x,y)\, d\lambda = \int_{\Gamma_1} \sum_{j,k} e^{\lambda t} d_{jk}\phi_j^+(x)\tilde\psi_k^+(y) d\lambda + \int_{\Gamma_1} \sum_{k} e^{\lambda t} \phi_k^+(x)\tilde\phi_k^+(y) d\lambda, \tag{3.14}$$

where each d_{jk} is analytic, hence bounded. We estimate separately each of the terms

$$\int_{\Gamma_1} e^{\lambda t} d_{jk}\phi_j^+(x)\tilde\psi_k^+(y) d\lambda$$

on the righthand side of (3.14). Estimates for terms

$$\int_{\Gamma_1} e^{\lambda t} \phi_k^+(x)\tilde\phi_k^+(y) d\lambda$$

go similarly.

Case II(ia). First, consider the critical case $a_j^+ > 0$, $a_k^+ < 0$. For this case,

$$|d_{jk}\phi_j^+(x)\tilde\psi_k^+(y)| \le Ce^{Re(\rho_j^+ x - \nu_k^+ y)},$$

where

$$\begin{cases} \nu_k^+(\lambda) = -\lambda/a_k^+ + \lambda^2\beta_k^+/(a_k^+)^3 + \mathcal{O}(\lambda^3) \\ \rho_j^+(\lambda) = -\lambda/a_j^+ + \lambda^2\beta_j^+/(a_j^+)^3 + \mathcal{O}(\lambda^3). \end{cases}$$

Set

$$\bar\alpha = \frac{a_k^+ x/a_j^+ - y - a_k^+ t}{2t}, \quad p := \frac{\beta_j^+ a_k^+ x/(a_j^+)^3 - \beta_k^+ y/(a_k^+)^2}{t} < 0.$$

Define Γ'_{1a} to be the portion contained in Ω_θ of the hyperbola

$$\begin{aligned} &Re(\rho_j^+ x - \nu_k^+ y) + \mathcal{O}(\lambda^3)(|x|+|y|) \\ &\quad = (1/a_k^+)Re[\lambda(-a_k^+ x/a_j^+ + y) + \lambda^2(x\beta_j^+ a_k^+/(a_j^+)^3 - y\beta_k^+/(a_k^+)^2)] \\ &\quad \equiv \text{constant} \\ &\quad = (1/a_k^+)[(\lambda_{min}(-a_k^+ x/a_j^+ + y) + \lambda_{min}^2(x\beta_j^+ a_k^+/(a_j^+)^3 - y\beta_k^+/(a_k^+)^2)], \end{aligned} \tag{3.15}$$

where

$$\lambda_{min} := \begin{cases} \frac{\bar\alpha}{p} & if \quad |\frac{\bar\alpha}{p}| \le \epsilon \\ \pm\epsilon & if \quad \frac{\bar\alpha}{p} \gtrless \epsilon \end{cases} \tag{3.16}$$

Denoting by λ_1, λ_1^*, the intersections of this hyperbola with $\partial\Omega_\theta$, define Γ'_{1_b} to be the union of $\lambda_1\lambda_0$ and $\lambda_0^*\lambda_1^*$, and define $\Gamma'_1 = \Gamma'_{1_a} \cup \Gamma'_{1_b}$. Note that $\lambda = \bar\alpha/p$ minimizes the left hand side of (3.15) for λ real. Note also that that p is bounded for $\bar\alpha$ sufficiently small, since $\bar\alpha \le \epsilon$ implies that

$$(|a_k^+ x/a_j^+| + |y|)/t \le 2|a_k^+| + 2\epsilon$$

i.e. $(|x|+|y|)/t$ is controlled by $\bar\alpha$.

With these definitions, we readily obtain that

$$\begin{aligned} Re(\lambda t+\rho_j^+ x-\nu_k^+ y) &\le -(t/a_k^-)(\bar\alpha^2/4p)-\eta Im(\lambda)^2 t\\ &\le -\bar\alpha^2 t/M-\eta Im(\lambda)^2 t, \end{aligned} \tag{3.17}$$

for $\lambda\in\Gamma'_{1_a}$ (note: here, we have used the crucial fact that $\bar\alpha$ controls $(|x|+|y|)/t$, in bounding the error term $\mathcal{O}(\lambda^3)(|x|+|y|)/t$ arising from expansion Likewise, we obtain for any q that

$$\int_{\Gamma'_{1_a}} |\lambda|^q e^{Re(\lambda t+\rho_j^+ x-\nu_k^- y)}d\lambda \le Ct^{-\frac12-\frac q2}e^{-\bar\alpha^2 t/M}, \tag{3.18}$$

for suitably large C, $M>0$ (depending on q). Observing that

$$\bar\alpha=(a_k^+/a_j^+)(x-a_j^+(t-|y/a_k^+|))/2t,$$

we find that the contribution of (3.18) can be absorbed in the described bounds for $t\ge|y/a_k^-|$. At the same time, we find that $\bar\alpha\ge x>0$ for $t\le|y/a_k^+|$, whence

$$\bar\alpha\ge(x-y-a_j^+t)/Mt+|x|/M,$$

for some $\epsilon>0$ sufficiently small and $M>0$ sufficiently large.

This gives

$$e^{-\bar\alpha^2/|p|}\le e^{-(x-y-a_k^+t)^2/Mt}e^{-\eta|x|}$$

provided $|x|/t>a_j^+$, a contribution which can again be absorbed. On the other hand, if $t\le|x/a_j^+|$, we can use the dual estimate

$$\begin{aligned} \bar\alpha&=(-y-a_k^+(t-|x/a_j^+|))/2t\\ &\ge(x-y-a_k^+t)/Mt+|y|/M, \end{aligned} \tag{3.19}$$

together with $|y|\ge|a_k^-t|$, to obtain

$$e^{-\bar\alpha^2/|p|}\le e^{-(x-y-a_j^+t)^2/Mt}e^{-\eta|y|},$$

a contribution that can likewise be absorbed.

Case II(ib). In case $a_j^+<0$ or $a_k^+>0$, terms $|\varphi_j^+|\le Ce^{-\eta|x|}$ and $|\tilde\psi_j^+|\le Ce^{-\eta|y|}$ are strictly smaller than those already treated in Case II(ia), so may be absorbed in previous terms.

Case II(ii) $(0<x<y)$**.** The case $0<x<y$ can be treated very similarly to the previous one; see [ZH] for details. This completes the proof of Case II, and the theorem.

4. Energy estimates

4.1. **Energy estimate I.** We shall require the following energy estimate adapted from [MaZ4, Z2]. Define the nonlinear perturbation variables $U = (u, v)$ by

$$U(x,t) := \tilde{U}(x,t) - \bar{U}(x). \tag{4.1}$$

Proposition 4.1. *Under the hypotheses of Theorem 1.7, let $U_0 \in H^4$ and $U = (u,v)^T$ be a solution of (1.2) and (4.1). Suppose that, for $0 \le t \le T$, the $W_x^{2,\infty}$ norm of the solution U remains bounded by a sufficiently small constant $\zeta > 0$. Then*

$$\|U(t)\|_{H^4}^2 \le Ce^{-\theta t}\|U_0\|_{H^4}^2 + C\int_0^t e^{-\theta(t-\tau)}\Big(\|U(\tau)\|_{L^2}^2 + \mathcal{B}_h(\tau)^2\Big)d\tau \tag{4.2}$$

for all $0 \le t \le T$, where the boundary operator $\mathcal{B}_h$ is defined in Theorem 1.7.

Proof. Observe that a straightforward calculation shows that $|U|_{H^r} \sim |W|_{H^r}$,

$$W = \tilde{W} - \bar{W} := W(\tilde{U}) - W(\bar{U}), \tag{4.3}$$

for $0 \le r \le 4$, provided $|U|_{W^{2,\infty}}$ remains bounded, hence it is sufficient to prove a corresponding bound in the special variable W. We first carry out a complete proof in the more straightforward case with conditions (A1)-(A3) replaced by the following global versions, indicating afterward by a few brief remarks the changes needed to carry out the proof in the general case.

(A1') $\tilde{A}(\tilde{W}), \tilde{A}^0, \tilde{A}^{11}$ are symmetric, $\tilde{A}^0 \ge \theta_0 > 0$,

(A2') no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(\tilde{W})$ lies in the kernel of $\tilde{B}(\tilde{A}^0)^{-1}(\tilde{W})$,

(A3') $\tilde{W} = \begin{pmatrix} \tilde{w}^I \\ \tilde{w}^{II} \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}, \tilde{b} \ge \theta > 0$, and $\tilde{G} \equiv 0$.

Substituting (4.3) into (1.4), we obtain the quasilinear perturbation equation

$$A^0 W_t + AW_x = (BW_x)_x + M_1\bar{W}_x + (M_2\bar{W}_x)_x \tag{4.4}$$

where $A^0 := A^0(W + \bar{W})$ is positive definite symmetric, $A := A(W + \bar{W})$ is symmetric,

$$M_1 = A(W+\bar{W}) - A(\bar{W}) = \Big(\int_0^1 dA(\bar{W} + \theta W)d\theta\Big)W,$$
$$M_2 = B(W+\bar{W}) - B(\bar{W}) = \begin{pmatrix} 0 & 0 \\ 0 & (\int_0^1 db(\bar{W} + \theta W)d\theta)W \end{pmatrix}.$$

As shown in [MaZ4], we have bounds

$$|A^0| \le C, \quad |A^0_t| \le C|W_t| \le C(|W_x| + |w^{II}_{xx}|) \le C\zeta, \tag{4.5}$$

$$|\partial_x A^0| + |\partial_x^2 A^0| \le C(\sum_{k=1}^2 |\partial_x^k W| + |\bar{W}_x|) \le C(\zeta + |\bar{W}_x|). \tag{4.6}$$

We have the same bounds for A, B, K, and also due to the form of M_1, M_2,

$$|M_1|, |M_2| \leq C(\zeta + |\bar{W}_x|)|W|. \tag{4.7}$$

Note that thanks to Lemma 1.3 we have the bound on the profile: $|\bar{W}_x| \leq Ce^{-\theta|x|}$, as $x \to +\infty$.

The following results assert that hyperbolic effects can compensate for degenerate viscosity B, as revealed by the existence of a compensating matrix K.

Lemma 4.2 ([KSh]). *Assuming (A1'), condition (A2') is equivalent to the following:*

(K1) There exists a smooth skew-symmetric matrix $K(W)$ such that

$$\Re e(K(A^0)^{-1}A + B)(W) \geq \theta_2 > 0. \tag{4.8}$$

Define α by the ODE

$$\alpha_x = -\text{sign}(A^{11})c_*|\bar{W}_x|\alpha, \quad \alpha(0) = 1 \tag{4.9}$$

where $c_* > 0$ is a large constant to be chosen later. Note that we have

$$(\alpha_x/\alpha)A^{11} \leq -c_*\theta_1|\bar{W}_x| =: -\omega(x) \tag{4.10}$$

and

$$|\alpha_x/\alpha| \leq c_*|\bar{W}_x| = \theta_1^{-1}\omega(x). \tag{4.11}$$

In what follows, we shall use $\langle \cdot, \cdot \rangle$ as the α-weighted L^2 inner product defined as

$$\langle f, g \rangle = \langle \alpha f, g \rangle_{L^2}$$

and $\|f\|_s = \sum_{i=0}^s \langle \frac{d^{(i)}}{dx^i} f, \frac{d^{(i)}}{dx^i} f \rangle^{1/2}$ as the norm in weighted H^s space. Note that for any symmetric operator S,

$$\langle Sf_x, f \rangle = -\frac{1}{2}\langle (S_x + (\alpha_x/\alpha)S)f, f \rangle - \frac{1}{2}S_0 f_0.f_0.$$

Note that in what follows, we shall pay attention to keeping track of c_*. For constants independent of c_*, we simply write them as C.

4.1.1. *Zeroth order "Friedrichs-type" estimate.* First employing integration by parts yields, and using estimates (4.5), (4.6), and then (4.10), we obtain

$$\begin{aligned}
&-\langle AW_x, W \rangle \\
&\quad = \frac{1}{2}\langle (A_x + (\alpha_x/\alpha)A)W, W \rangle + \frac{1}{2}A_0 W(0) \cdot W(0) \\
&\quad \leq \frac{1}{2}\langle (\alpha_x/\alpha)A^{11}w^I, w^I \rangle + C\langle (\zeta + |\bar{W}_x|)|W| + \omega(x)|w^{II}|, |W| \rangle + J_b^0 \\
&\quad \leq -\frac{1}{2}\langle \omega(x)w^I, w^I \rangle + C(\zeta\|w^I\|_0^2 + \langle |\bar{W}_x|w^I, w^I \rangle) + C(c_*)\|w^{II}\|_0^2 + J_b^0
\end{aligned}$$

where J_b^0 denotes the boundary term $\frac{1}{2}A_0W(0)\cdot W(0)$. The term $\langle|\bar W_x|w^I,w^I\rangle$ may be easily absorbed into the first term of the right-hand side, since for c_* sufficiently large,

$$\langle|\bar W_x|w^I,w^I\rangle \le (c_*\theta_1)^{-1}\langle\omega(x)w^I,w^I\rangle \le \frac{1}{4C}\langle\omega(x)w^I,w^I\rangle. \tag{4.12}$$

Also, integration by parts yields

$$\begin{aligned}\langle(BW_x)_x,W\rangle &= -\langle BW_x,W_x\rangle - \langle(\alpha_x/\alpha)BW_x,W\rangle - B_0W_x(0)\cdot W(0)\\ &\le -\theta\|w_x^{II}\|_0^2 + C\langle\omega(x)w_x^{II},w^{II}\rangle - b_0w_x^{II}(0)\cdot w^{II}(0)\\ &\le -\theta\|w_x^{II}\|_0^2 + C(c_*)\|w^{II}\|_0^2 - b_0w_x^{II}(0)\cdot w^{II}(0).\end{aligned}$$

where we used the fact that $BW_x\cdot W = bw_x^{II}\cdot w^{II}$, noting that B has block-diagonal form with the first block identical to zero. Similarly, recalling that $M_2 = B(W+\bar W) - B(\bar W)$, we have

$$\begin{aligned}&\langle(M_2\bar W_x)_x,W\rangle\\ &\quad= -\langle M_2\bar W_x,W_x\rangle - \langle(\alpha_x/\alpha)M_2\bar W_x,W\rangle - M_2(0)\bar W_x(0)\cdot W(0)\\ &\quad\le C\langle|\bar W_x||W|,|w_x^{II}|\rangle + C\langle\omega(x)|W|,w^{II}\rangle - m_2(0)\bar W_x(0)\cdot w^{II}(0)\\ &\quad\le \xi\|w_x^{II}\|_0^2 + C\Big(\epsilon\langle\omega(x)w^I,w^I\rangle + C(c_*)\|w^{II}\|_0^2\Big) - m_2(0)\bar W_x(0)\cdot w^{II}(0)\end{aligned}$$

for any small ξ,ϵ. Note that C is independent of c_*. Therefore, for $\xi=\theta/2$ and c_* sufficiently large, combining all above estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle A^0W,W\rangle\\ &\quad= \langle A^0W_t,W\rangle + \frac{1}{2}\langle A_t^0W,W\rangle\\ &\quad= \langle -AW_x + (BW_x)_x + M_1\bar W_x + (M_2\bar W_x)_x, W\rangle + \frac{1}{2}\langle A_t^0W,W\rangle\\ &\quad\le -\frac{1}{4}[\langle\omega(x)w^I,w^I\rangle + \theta\|w_x^{II}\|_0^2] + C\zeta\|w^I\|_0^2 + C(c_*)\|w^{II}\|_0^2 + I_b^0\end{aligned} \tag{4.13}$$

where the boundary term

$$I_b^0 := \frac{1}{2}A_0W(0)\cdot W(0) - b_0w_x^{II}(0)w^{II}(0) - M_2(0)\bar W_x(0)\cdot W(0) \tag{4.14}$$

which, in the outflow case (thanks to the negative definiteness of A_{11}), is estimated as

$$I_b^0 \le -\frac{\theta_1}{2}|w^I(0)|^2 + C(|w^{II}(0)|^2 + |w_x^{II}(0)||w^{II}(0)|), \tag{4.15}$$

and similarly in the inflow case, estimated as

$$I_b^0 \le C(|W(0)|^2 + |w_x^{II}(0)||w^{II}(0)|). \tag{4.16}$$

Therefore together with these boundary treatments, (4.13) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle A^0W,W\rangle\\ &\le -\frac{1}{4}[\langle\omega(x)w^I,w^I\rangle+\theta\|w_x^{II}\|_0^2]+C\zeta\|w^I\|_0^2+C(c_*)\|w^{II}\|_0^2+I_b^0.\end{aligned}\tag{4.17}$$

4.1.2. *First order "Friedrichs-type" estimate.* Similarly as above, we need the following key estimate, computing by the use of integration by parts, (4.12), and c_* being sufficiently large,

$$\begin{aligned}-\langle W_x,AW_{xx}\rangle &= \frac{1}{2}\langle W_x,(A_x+(\alpha_x/\alpha)A)W_x\rangle+\frac{1}{2}A_0W_x(0)\cdot W_x(0)\\ &\le -\frac{1}{4}\langle\omega(x)w_x^I,w_x^I\rangle+C\zeta\|w_x^I\|_0^2+Cc_*^2\|w_x^{II}\|_0^2\\ &\quad+\frac{1}{2}A_0W_x(0)\cdot W_x(0).\end{aligned}\tag{4.18}$$

We deal with the boundary term later. Now let us compute

$$\frac{1}{2}\frac{d}{dt}\langle A^0W_x,W_x\rangle=\langle W_x,(A^0W_t)_x\rangle-\langle W_x,A_x^0W_t\rangle+\frac{1}{2}\langle A_t^0W_x,W_x\rangle.\tag{4.19}$$

We control each term in turn. By (4.5) and (4.6), we first have

$$\langle A_t^0W_x,W_x\rangle\le C\zeta\|W_x\|_0^2$$

and by multiplying $(A^0)^{-1}$ into (4.4),

$$\begin{aligned}|\langle W_x,A_x^0W_t\rangle| &\le C\langle(\zeta+|\bar W_x|)|W_x|,(|W_x|+|w_{xx}^{II}|+|W|)\rangle\\ &\le \xi\|w_{xx}^{II}\|_0^2+C\langle(\zeta+|\bar W_x|)w_x^I,w_x^I\rangle\\ &\quad+C\langle(\zeta+|\bar W_x|)w^I,w^I\rangle+C\|w^{II}\|_1^2,\end{aligned}$$

where the term $\langle|\bar W_x|w_x^I,w_x^I\rangle$ may be treated in the same way as was $\langle|\bar W_x|w^I,w^I\rangle$ in (4.12). Using (4.4), we write the first term in the right-hand side of (4.19) as

$$\begin{aligned}\langle W_x,(A^0W_t)_x\rangle &= \langle W_x,[-AW_x+(BW_x)_x+M_1\bar W_x+(M_2\bar W_x)_x]_x\rangle\\ &= -\langle W_x,AW_{xx}\rangle+\langle W_x,-A_xW_x+(M_1\bar W_x)_x\rangle\\ &\quad-\langle W_{xx}+(\alpha_x/\alpha)W_x,[(BW_x)_x+(M_2\bar W_x)_x]\rangle\\ &\quad-W_x(0).[(BW_x)_x+(M_2\bar W_x)_x](0)\\ &\le -\frac{1}{4}\Big[\langle\omega(x)w_x^I,w_x^I\rangle+\theta\|w_{xx}^{II}\|_0^2\Big]\\ &\quad+C\Big[\zeta\|w^I\|_1^2+C(c_*)\|w_x^{II}\|_0^2+\langle|\bar W_x|w^I,w^I\rangle\Big]+I_b^1\end{aligned}$$

where I_b^1 denotes the boundary terms

$$I_b^1:=\frac{1}{2}A_0W_x(0)\cdot W_x(0)-W_x(0)\cdot[(BW_x)_x+(M_2\bar W_x)_x](0),\tag{4.20}$$

and we have used estimates (4.18),(4.12) for w^I, w^I_x, and Young's inequality to obtain:

$$\langle W_x, -A_xW_x + (M_1\bar W_x)_x\rangle \le C\langle(\zeta + |\bar W_x|)|W_x|, |W_x| + |W|\rangle.$$

$$-\langle W_{xx} + (\alpha_x/\alpha)W_x, (BW_x)_x\rangle \le$$
$$-\theta\|w^{II}_{xx}\|_0^2 + C\langle|w^{II}_{xx}| + \omega(x)|w^{II}_x|, (\zeta + |\bar W_x|)|w^{II}_x|\rangle$$

$$-\langle W_{xx} + (\alpha_x/\alpha)W_x, (M_2\bar W_x)_x\rangle \le$$
$$C\langle|w^{II}_{xx}| + \omega(x)|w^{II}_x|, (\zeta + |\bar W_x|)(|W_x| + |W|)\rangle.$$

Putting these estimates together into (4.19), we have obtained

$$\frac{1}{2}\frac{d}{dt}\langle A^0W_x, W_x\rangle + \frac{1}{4}\theta\|w^{II}_{xx}\|_0^2 + \frac{1}{4}\langle\omega(x)w^I_x, w^I_x\rangle$$
$$\le C\Big[\zeta\|w^I\|_1^2 + \langle|\bar W_x|w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big] + I^1_b. \tag{4.21}$$

Let us now treat the boundary term. First observe that using the parabolic equations, noting that A^0 is the diagonal-block form, we can estimate

$$\Big|W_x(0)\cdot[(BW_x)_x + (M_2\bar W_x)_x](0)\Big|$$
$$= \Big|w^{II}_x(0)\cdot[(bw^{II}_x)_x + (M_2^{22}\bar W_x)_x](0)\Big|$$
$$= \Big|w^{II}_x(0)\cdot[A^0_2w^{II}_t + A_{21}w^I_x + A_{22}w^{II}_x - M_1\bar W_x](0)\Big|$$
$$\le \epsilon|w^{II}_x(0)|^2 + C(|W(0)|^2 + |w^{II}_x(0)|^2 + |w^{II}_t(0)|^2).$$

For the first term in I_b, we consider each inflow/outflow case separately. For the outflow case, since $A^{11} \le -\theta_1 < 0$, we get

$$A_0W_x(0)\cdot W_x(0) \le -\frac{\theta_1}{2}|w^I_x(0)|^2 + C|w^{II}_x(0)|^2.$$

Therefore

$$I^1_b \le -\frac{\theta_1}{2}|w^I_x(0)|^2 + C(|W(0)|^2 + |w^{II}_x(0)|^2 + |w^{II}_t(0)|^2). \tag{4.22}$$

Meanwhile, for the inflow case, since $A^{11} \ge \theta_1 > 0$, we have

$$|A_0W_x(0)\cdot W_x(0)| \le C|W_x(0)|^2.$$

In this case, the invertibility of A^{11} allows us to use the hyperbolic equation to derive

$$|w^I_x(0)| \le C(|w^I_t(0)| + |w^{II}_x(0)|).$$

Therefore we get

$$I^1_b \le C(|W(0)|^2 + |W_t(0)|^2 + |w^{II}_x(0)|^2). \tag{4.23}$$

Now applying the standard Sobolev inequality (applies for $\alpha-$weighted norms as long as $|\alpha_x/\alpha|$ is uniformly bounded):

$$|w(0)|^2 \le C\|w\|_{L^2}(\|w_x\|_{L^2} + \|w\|_{L^2}) \tag{4.24}$$

to control the term $|w_x^{II}(0)|^2$ in I_b^1 in both cases. We get

$$|w_x^{II}(0)|^2 \le \epsilon' \|w_{xx}^{II}\|_0^2 + C\|w_x^{II}\|_0^2. \tag{4.25}$$

Using this with $\epsilon' = \theta/8$, (4.20), and (4.22), the estimate (4.21) reads

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\langle A^0 W_x, W_x\rangle + \frac{\theta}{8}\|w_{xx}^{II}\|_0^2 + \frac{1}{4}\langle \omega(x) w_x^I, w_x^I\rangle \\ &\le C\Big(\zeta\|w^I\|_1^2 + \langle |\bar{W}_x| w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big) + I_b^1 \end{aligned} \tag{4.26}$$

where the boundary term I_b^1 is estimated as

$$I_b^1 \le -\frac{\theta_1}{2}|w_x^I(0)|^2 + C(|W(0)|^2 + |w_t^{II}(0)|^2) \tag{4.27}$$

for the outflow case, and similarly

$$I_b^1 \le C(|W(0)|^2 + |W_t(0)|^2) \tag{4.28}$$

for the inflow case.

4.1.3. *Higher order "Friedrichs-type" estimate.* Similarly as above, we shall now derive an estimate for $\langle A^0\partial_x^k W, \partial_x^k W\rangle$, $k = 2, 3, 4$. We need the following key estimate. Integration by parts and (4.10) give

$$\begin{aligned} -\langle \partial_x^k W, A\partial_x^{k+1} W\rangle =& \frac{1}{2}\langle \partial_x^k W, (A_x + (\alpha_x/\alpha)A)\partial_x^k W\rangle + \frac{1}{2}A_0 \partial_x^k W(0)\cdot \partial_x^k W(0) \\ \le& -\frac{1}{4}\langle \omega(x)\partial_x^k w^I, \partial_x^k w^I\rangle + C\zeta\|\partial_x^k w^I\|_0^2 \\ &+ Cc_*^2\|\partial_x^k w^{II}\|_0^2 + \frac{1}{2}A_0\partial_x^k W(0)\cdot\partial_x^k W(0). \end{aligned}$$

We compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\langle A^0\partial_x^k W, \partial_x^k W\rangle =& \frac{1}{2}\langle A_t^0 \partial_x^k W, \partial_x^k W\rangle + \langle A^0\partial_x^k W, \partial_x^k W_t\rangle \\ =& \frac{1}{2}\langle A_t^0 \partial_x^k W, \partial_x^k W\rangle + \langle A^0\partial_x^k W, \partial_x^k[(A^0)^{-1} \\ & (-AW_x + (BW_x)_x) + M_1\bar{W}_x + (M_2\bar{W}_x)_x]\rangle. \end{aligned} \tag{4.29}$$

We shall estimate each term in turn. First, $|\langle A_t^0\partial_x^k W, \partial_x^k W\rangle| \le C\zeta\|\partial_x^k W\|_0^2$, and

$$\begin{aligned} &\langle A^0\partial_x^k W, \partial_x^k[-(A^0)^{-1}AW_x]\rangle \\ &= \langle A^0\partial_x^k W, \sum_{i=0}^k \partial_x^i[-(A^0)^{-1}A]\partial_x^{k-i+1}W\rangle \\ &= -\langle \partial_x^k W, A\partial_x^{k+1}W\rangle + \sum_{i=1}^k \langle A^0\partial_x^k W, \partial_x^i[-(A^0)^{-1}A]\partial_x^{k-i+1}W\rangle \end{aligned}$$

where we have

$$\Big|\partial_x^i[-(A^0)^{-1}A]\Big| \le C\sum_{\sum \alpha_j = i}\prod_{1\le j\le i}|\partial_x^{\alpha_j}W|. \tag{4.30}$$

Using the hypothesis on the boundedness of solutions in $W^{2,\infty}$ and weak Moser inequality [Z4, Lemma 1.5], we get

$$|\langle A^0\partial_x^k W, \partial_x^i[-(A^0)^{-1}A]\partial_x^{k-i+1}W\rangle| \le C\Big(\|w^{II}\|_k^2 + \zeta\|w^I\|_k^2 + \sum_{i=1}^k \langle|\bar W_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big).$$

This, (4.29), similar treatment (4.12) for $\langle|\bar W_x|\partial_x^k w^I, \partial_x^k w^I\rangle$ with c_* being sufficiently large give

$$\langle A^0\partial_x^k W, \partial_x^k[-(A^0)^{-1}AW_x]\rangle \le -\frac{1}{4}\langle\omega\partial_x^k w^I, \partial_x^k w^I\rangle + \frac{1}{2}A_0\partial_x^k W(0)\cdot\partial_x^k W(0) + C\Big(\|w^{II}\|_k^2 + \zeta\|w^I\|_k^2 + \sum_{i=1}^{k-1} \langle|\bar W_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big) \tag{4.31}$$

Next, similarly, we obtain

$$|\langle A^0\partial_x^k W, \partial_x^k[(A^0)^{-1}M_1\bar W_x]\rangle| \le C\Big(\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^k \langle|\bar W_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big).$$

Finally, we compute and

$$\begin{aligned}
\langle A^0\partial_x^k W,&\partial_x^k[(A^0)^{-1}(BW_x + M_2\bar W_x)_x]\rangle\\
&=\sum_{i=0}^k \langle A^0\partial_x^k W, \partial_x^i[(A^0)^{-1}]\partial_x^{k-i+1}(BW_x + M_2\bar W_x)\rangle\\
&=\langle\partial_x^k W, \partial_x^{k+1}(BW_x + M_2\bar W_x)\rangle\\
&\quad+\sum_{i=1}^k \langle A^0\partial_x^k W, \partial_x^i[(A^0)^{-1}]\partial_x^{k-i+1}(BW_x + M_2\bar W_x)\rangle\\
&\le-\langle\partial_x^{k+1}W + (\alpha_x/\alpha)\partial_x^k W, \partial_x^k(BW_x + M_2\bar W_x)\rangle\\
&\quad-\partial_x^k[b\partial_x w^{II} + M_2^{22}\bar W_x](0)\partial_x^k w^{II}(0)\\
&\quad+\xi\|\partial_x^{k+1}w^{II}\|_0^2 + C\Big(c_*^2\|w^{II}\|_k^2 + \zeta\|w^I\|_k^2 + \sum_{i=1}^k \langle|\bar W_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big)\\
&\le-\frac{\theta}{2}\|\partial_x^{k+1}w^{II}\|_0^2 - \partial_x^k[b\partial_x w^{II} + M_2^{22}\bar W_x](0)\partial_x^k w^{II}(0)\\
&\quad+C\Big(c_*^2\|w^{II}\|_k^2 + \zeta\|w^I\|_k^2 + \sum_{i=1}^k \langle|\bar W_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big)
\end{aligned}$$

where in the last inequality we used the special form of B and M_2 to get

$$\begin{aligned}&\langle\partial_x^{k+1}W+(\alpha_x/\alpha)\partial_x^kW,\partial_x^k(BW_x+M_2\bar W_x)\rangle\\&\le\langle|\partial_x^{k+1}w^{II}|+\omega(x)|\partial_x^kw^{II}|,|\partial_x^k(bw_x^{II}+\Pi_2M_2\bar W_x)|\rangle\\&\le-\theta\|\partial_x^{k+1}w^{II}\|_0^2+C\Big(C(c_*)\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^k\langle|\bar W_x|\partial_x^iw^I,\partial_x^iw^I\rangle\Big).\end{aligned}$$

Note that in the last inequality, there is no term of $\langle\omega(x)\partial_x^iw^I,\partial_x^iw^I\rangle$ because of the presence of $|\bar W_x|$ in term of Π_2M_2.

Put all these estimates into (4.29) together, we have obtained

$$\begin{aligned}&\frac12\frac{d}{dt}\langle A^0\partial_x^kW,\partial_x^kW\rangle+\frac14\theta\|\partial_x^{k+1}w^{II}\|_0^2+\frac14\langle\omega(x)\partial_x^kw^I,\partial_x^kw^I\rangle\\&\le C\Big(C(c_*)\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^{k-1}\langle|\bar W_x|\partial_x^iw^I,\partial_x^iw^I\rangle\Big)+I_b\end{aligned}\tag{4.32}$$

where the boundary term

$$I_b:=\frac12A_0\partial_x^kW(0)\cdot\partial_x^kW(0)-\partial_x^k[b\partial_xw^{II}+M_2^{22}\bar W_x](0)\partial_x^kw^{II}(0).\tag{4.33}$$

For this boundary term, we shall treat the same as we did before. First using the parabolic equations with noting that A^0 is the diagonal-block matrix $\mathrm{diag}(A_1^0,A_2^0)$, we can write

$$\begin{aligned}&\partial_x^k[b\partial_xw^{II}+M_2^{22}\bar W_x](0)\\&=\partial_x^{k-1}[A_2^0(0)w_t^{II}(0,t)+A_{21}w_x^I(0)+A_{22}w_x^{II}(0)-\Pi_2M_1(0)\bar W_x(0)].\end{aligned}\tag{4.34}$$

Therefore we get

$$\begin{aligned}&|\partial_x^k[b\partial_xw^{II}+M_2^{22}\bar W_x](0)\partial_x^kw^{II}(0)|\\&\qquad\le C|\partial_x^kw^{II}(0)|\Big[|\partial_x^{k-1}w_t^{II}(0)|+\sum_{i=0}^k(|\partial_x^iw^{II}(0)|+|\partial_x^iw^I(0)|)\Big]\end{aligned}$$

$$\le\epsilon\sum_{i=0}^k|\partial_x^iw^I(0)|^2+C\sum_{i=1}^k|\partial_x^iw^{II}(0)|^2\tag{4.35}$$

$$+C|\partial_x^kw^{II}(0)||\partial_x^{k-1}w_t^{II}(0)|\tag{4.36}$$

for any ϵ small. To deal with the term of w_t^{II}, for simplicity, assume $k=3$. By solving the parabolic-part equations and using the invertibility of b, we obtain

$$\begin{aligned}|\partial_x^2w_t^{II}|=|\partial_tw_{xx}^{II}|&\le C(|w_{tt}^{II}|+|W_t|+|W_x|+|W_{xt}|)\\|W_{xt}|&\le C(|W|+|W_x|+|W_{xx}|+|w_{xxx}^{II}|).\end{aligned}\tag{4.37}$$

Since for case $k=3$ we have a good term $\|\partial_x^4w^{II}\|_0$ (see (4.32)), the term $|w_{xxx}^{II}(0)|$ resulting from the boundary treatment is easily treated via

Sobolev embedding inequality. Hence all terms in a form $\partial_x^r w^{II}(0)$ are easily estimated. Meanwhile, using the hyperbolic-part equations, we have

$$|w_t^I| \le C(|W| + |W_x|). \tag{4.38}$$

Employing Young's inequality to the last term in (4.35), we obtain

$$\begin{aligned} &|\partial_x^k[b\partial_x w^{II} + M_2^{22}\bar W_x](0)\partial_x^k w^{II}(0)| \\ &\le \epsilon \sum_{i=0}^k |\partial_x^i w^I(0)|^2 + C(\sum_{i=0}^k |\partial_x^i w^{II}(0)|^2 + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2) \end{aligned} \tag{4.39}$$

To deal with the term of w^I, we need to consider two cases separately. When $A^{11} \le -\theta_1 < 0$, we get

$$A_0\partial_x^k W(0)\cdot\partial_x^k W(0) \le -\frac{\theta_1}{2}|\partial_x^k w^I(0)|^2 + C|\partial_x^k w^{II}(0)|^2.$$

Therefore

$$\begin{aligned} I_b^k \le &-\frac{\theta_1}{2}|\partial_x^k w^I(0)|^2 + C(\sum_{i=0}^{k-1}|\partial_x^i w^I(0)|^2 \\ &+ \sum_{i=0}^k |\partial_x^i w^{II}(0)|^2 + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2). \end{aligned} \tag{4.40}$$

Meanwhile, for the case $A^{11} \ge \theta_1 > 0$, we have

$$|A_0\partial_x^k W(0)\cdot\partial_x^k W(0)| \le C(|\partial_x^k w^I(0)|^2 + |\partial_x^k w^{II}(0)|^2).$$

The invertibility of A^{11} allows us to use the hyperbolic equation to derive

$$|\partial_x^k w^I(0)| \le C(\sum_{i=0}^k (|\partial_x^i w^{II}(0)|^2 + |\partial_t^i w^I(0)|^2) + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2).$$

Therefore in the case of $A^{11} \ge \theta_1 > 0$, we get

$$I_b^k \le C(\sum_{i=0}^k (|\partial_x^i w^{II}(0)|^2 + |\partial_t^i w^I(0)|^2) + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2). \tag{4.41}$$

Employing the boundary estimates into (4.32), we have obtained

$$\begin{aligned} &\frac{d}{dt}\langle A^0\partial_x^k W, \partial_x^k W\rangle + \theta\|\partial_x^{k+1} w^{II}\|_0^2 + c_*\theta_1\langle|\bar W_x|\partial_x^k w^I, \partial_x^k w^I\rangle \\ &\le C\Big(\zeta\|w^I\|_k^2 + c_*^2\|w^{II}\|_k^2 + \sum_{j=0}^{k-1}\langle|\bar W_x|\partial_x^j w^I, \partial_x^j w^I\rangle\Big) + I_b^k \end{aligned} \tag{4.42}$$

where, after absorbing the terms of $|\partial_x^r w^{II}(0)|$ via Sobolev embedding, the boundary term I_b^k satisfies

$$I_b^k \le -\frac{\theta_1}{2}|\partial_x^k w^I(0)|^2 + C(\sum_{i=0}^{k-1}|\partial_x^i w^I(0)|^2 + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2) \tag{4.43}$$

for outflow case, and

$$I_b^k \le C(\sum_{i=0}^{k} |\partial_t^i w^I(0)|^2 + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2) \tag{4.44}$$

for the inflow case.

We shall establish an Kawashima-type estimate to bound the term $\|w^I\|_k^2$ appearing on the left hand side of the above.

4.1.4. *"Kawashima-type" estimate.* Let K be the skew-symmetry in (4.8). Integration by parts and skew-symmetry property of K yield

$$\begin{aligned}\langle KW_{xt}, W\rangle &= -\langle KW_t, W_x\rangle - \langle (K_x + (\alpha_x/\alpha)K)W_t, W\rangle - K_0 W_0 \cdot (W_0)_t \\ &= \langle KW_x, W_t\rangle + \langle (K_x + (\alpha_x/\alpha)K)W, W_t\rangle - K_0 W_0 \cdot (W_0)_t.\end{aligned}$$

Using this, we compute

$$\begin{aligned}&\frac{d}{dt}\langle KW_x, W\rangle = \\ &\quad \langle K_t W_x + KW_{xt}, W\rangle + \langle KW_x, W_t\rangle \\ &= \langle K_t W_x, W\rangle + \langle 2KW_x + (K_x + (\alpha_x/\alpha)K)W, W_t\rangle \\ &\quad - K_0 W_0 \cdot (W_0)_t \\ &= \langle K_t W_x, W\rangle + \langle 2KW_x + (K_x + (\alpha_x/\alpha)K)W, -(A^0)^{-1} AW_x\rangle \\ &\quad + \langle 2KW_x + (K_x + (\alpha_x/\alpha)K)W, (A^0)^{-1}(BW_x)_x \\ &\quad + M_1 \bar{W}_x + (M_2 \bar{W}_x)_x\rangle - K_0 W_0 \cdot (W_0)_t \\ &\le -2\langle K(A^0)^{-1} AW_x, W_x\rangle + \xi \|w_x^I\|_0^2 - K_0 W_0 \cdot (W_0)_t \\ &\quad + C\Big(C(c_*)\|w^{II}\|_2^2 + \zeta \|w^I\|_0^2 + \langle \omega(x) w^I, w^I\rangle + \langle \omega(x) w_x^I, w_x^I\rangle\Big).\end{aligned}$$

Using (4.8), we get

$$\langle K(A^0)^{-1} AW_x, W_x\rangle \ge \theta_2 \|w_x^I\|_0^2 - C(c_0)\|w_x^{II}\|_0^2,$$

and thus obtain from the above estimate with $\xi = \theta_2/2$

$$\begin{aligned}\frac{d}{dt}\langle KW_x, W\rangle \le &-\frac{\theta_2}{2}\|w_x^I\|_0^2 + C\Big(C(c_*)\|w^{II}\|_2^2 + \zeta\|w^I\|_0^2 \\ &+ \langle \omega(x) w^I, w^I\rangle + \langle \omega(x) w_x^I, w_x^I\rangle\Big) - K_0 W_0 \cdot (W_0)_t.\end{aligned} \tag{4.45}$$

4.1.5. *Higher order "Kawashima-type" estimate.* With similar calculations, we shall obtain an estimate for $\frac{d}{dt}\langle K\partial_x^k W, \partial_x^{k-1} W\rangle, k \ge 1$. We compute

$$\begin{aligned}\langle K\partial_x^k W_t, \partial_x^{k-1} W\rangle &= \langle K\partial_x^k W, \partial_x^{k-1} W_t\rangle \\ &\quad + \langle (K_x + (\alpha_x/\alpha)K)\partial_x^{k-1} W, \partial_x^{k-1} W_t\rangle - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0).\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{dt}\langle K\partial_x^k W, \partial_x^{k-1} W\rangle &= \langle K_t\partial_x^k W, \partial_x^{k-1} W\rangle + \langle 2K\partial_x^k W, \partial_x^{k-1} W_t\rangle \\
&\quad + \langle (K_x + (\alpha_x/\alpha)K)\partial_x^{k-1} W, \partial_x^{k-1} W_t\rangle - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0) \\
&= \langle 2K\partial_x^k W, \partial_x^{k-1}[(-A^0)^{-1}(AW_x + (BW_x)_x + M_1\bar W_x + (M_2\bar W_x)_x)]\rangle \\
&\quad + \langle (K_x + (\alpha_x/\alpha)K)\partial_x^{k-1} W, \\
&\qquad \partial_x^{k-1}[(-A^0)^{-1}(AW_x + (BW_x)_x + M_1\bar W_x + (M_2\bar W_x)_x)]\rangle \\
&\quad - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0) \\
&\le -2\langle K(A^0)^{-1}A\partial_x^k W, \partial_x^k W\rangle + \epsilon\|w^I\|_k^2 + Cc_*^2\|w^{II}\|_{k+1}^2 \\
&\quad + C\zeta\|w^I\|_0^2 + C\sum_{l=1}^k \langle \omega(x)\partial_x^l w^I, \partial_x^l w^I\rangle - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0)
\end{aligned}$$

for ϵ small.

Using (4.8), we obtain from the above

$$\begin{aligned}
\frac{d}{dt}\langle K\partial_x^k W, \partial_x^{k-1} W\rangle \le& -\frac{\theta_2}{3}\|\partial_x^k w^I\|_0^2 + Cc_*^2\|w^{II}\|_{k+1}^2 + \epsilon\|w^I\|_{k-1} \\
&+ C\zeta\|w^I\|_0^2 + C\sum_{l=1}^k \langle \omega(x)\partial_x^l w^I, \partial_x^l w^I\rangle \qquad (4.46)\\
&- K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0). \qquad (4.47)
\end{aligned}$$

4.1.6. *Final estimates.* We are ready to conclude our result. First combining the estimate (4.26) with (4.17), we easily obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle\Big) \\
&\le -\Big(\frac{\theta}{8}\|w_{xx}^{II}\|_0^2 + \frac{1}{4}\langle \omega(x) w_x^I, w_x^I\rangle\Big) \\
&\quad + C\Big(\zeta\|w^I\|_1^2 + \langle |\bar W_x| w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big) + I_b^1 \\
&\quad - \frac{M}{4}\Big(\langle \omega(x) w^I, w^I\rangle + \theta\|w_x^{II}\|_0^2\Big) + CM\zeta\|w^I\|_0^2 \\
&\quad + MC(c_*)\|w^{II}\|_0^2 + MI_b^0
\end{aligned}$$

By choosing M sufficiently large such that $M\theta \gg C(c_*)$, and noting that $c_*\theta_1|\bar W_x| \le \omega(x)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle\Big) \\
&\le -\Big(\theta\|w^{II}\|_2^2 + \langle \omega(x) w^I, w^I\rangle + \langle \omega(x) w_x^I, w_x^I\rangle\Big) \qquad (4.48)\\
&\quad + C\Big(\zeta\|w^I\|_1^2 + C(c_*)\|w^{II}\|_0^2\Big) + I_b^1 + MI_b^0.
\end{aligned}$$

We shall treat the boundary terms later. Now we employ the estimate (4.45) to absorb the term $\|w^I\|_1$ into the left hand side. Indeed, fixing c_* large as above, adding (4.48) with (4.45) times ϵ, and choosing ϵ, ζ sufficiently small such that $\epsilon C(c_*) \ll \theta, \epsilon \ll 1$ and $\zeta \ll \epsilon\theta_2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle KW_x, W\rangle\Big)\\
&\quad\le -\Big(\theta\|w^{II}\|_2^2 + \langle\omega(x)w^I, w^I\rangle + \langle\omega(x)w^I_x, w^I_x\rangle\Big)\\
&\qquad + C\Big(\zeta\|w^I\|_1^2 + C(c_*)\|w^{II}\|_0^2\Big) - \frac{\theta_2\epsilon}{2}\|w^I_x\|_0^2\\
&\qquad + C\epsilon\Big(C(c_*)\|w^{II}\|_2^2 + \zeta\|w^I\|_0^2 + \langle\omega(x)w^I, w^I\rangle + \langle\omega(x)w^I_x, w^I_x\rangle\Big)\\
&\qquad + I_b^1 + MI_b^0 - \epsilon K_0 W_0\cdot(W_0)_t\\
&\quad\le -\frac{1}{2}\Big(\theta\|w^{II}\|_2^2 + \theta_2\epsilon\|w^I_x\|_0^2\Big) + C(c_*)\Big(\zeta\|w^I\|_0^2 + \|w^{II}\|_0^2\Big) + I_b
\end{aligned}$$

where $I_b := I_b^1 + MI_b^0 - \epsilon K_0 W_0\cdot(W_0)_t$.

By a view of boundary terms I_b^0, I_b^1, we treat the term I_b in each inflow/outflow case separately. Recalling the inequality (4.25), $|w^{II}_x(0)| \le C\|w^{II}\|_2$. Thus, using this, for the inflow case we have

$$\begin{aligned}
I_b &\le M|W(0)|^2 + C|W_t(0)|^2 + M|w^{II}_x(0)||w^{II}(0)|\\
&\le \frac{\theta}{2}\|w^{II}\|_2^2 + M^2|W(0)|^2 + C|W_t(0)|^2.
\end{aligned}$$

Meanwhile, for the outflow case, with $M\theta_1 \gg 1$ and $K_0W_0\cdot(W_0)_t \sim w_0^{II}w_{0t}^I + w_0^I w_{0t}^{II}$, we have I_b is bounded by

$$\begin{aligned}
&-\frac{\theta_1}{2}(|w^I_x(0)|^2 + |w^I(0)|^2)\\
&\qquad + C(|w^{II}_t(0)|^2 + |w^{II}(0)|^2) + \epsilon(|w^{II}_x(0)|^2 + |w^I_t(0)|^2)
\end{aligned}$$

which, together with ϵ being sufficiently small and the facts that

$$|w^I_t(0)| \le C(|w^I_x(0)| + |w^{II}_x(0)| + |W(0)|)$$

obtained from solving the hyperbolic equation and the embedding inequality

$$|w^{II}_x(0)| \le C\|w^{II}\|_2,$$

yields

$$I_b \le -\frac{\theta_1}{2}(|w^I_x(0)|^2 + |w^I(0)|^2) + \frac{\theta}{2}\|w^{II}\|_2^2 + C(|w^{II}(0)|^2 + |w^{II}_t(0)|^2)$$

for the outflow case. Now by Cauchy-Schwarz's inequality and by positivity definite of A^0, it is easy to see that

$$\mathcal{E} := \langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle KW_x, W\rangle \sim \|W\|^2_{H^1_\alpha} \sim \|W\|^2_{H^1}. \tag{4.49}$$

The last equivalence is due to the fact that α is bounded above and below away from zero. Thus the above derives

$$\frac{d}{dt}\mathcal{E}(W)(t) \le -\theta_3 \mathcal{E}(W)(t) + C(c_*)\Big(\|W(t)\|_{L^2}^2 + \mathcal{B}_1(t)^2\Big),$$

for some positive constant θ_3, which by the Gronwall inequality yields

$$\|W(t)\|_{H^1}^2 \le Ce^{-\theta t}\|W_0\|_{H^1}^2 + C(c_*)\int_0^t e^{-\theta(t-\tau)}\Big(\|W(\tau)\|_{L^2}^2 + \mathcal{B}_1(\tau)^2\Big)d\tau, \tag{4.50}$$

where $W(x,0) = W_0(x)$ and

$$\mathcal{B}_1(\tau)^2 := \mathcal{O}(|W(0,\tau)|^2 + |W_t(0,\tau)|^2) = \mathcal{O}(|(h_1,h_2)|^2 + |(h_1,h_2)_t|^2) \tag{4.51}$$

for the inflow case, and

$$\mathcal{B}_1(\tau)^2 := \mathcal{O}(|w^{II}(0,\tau)|^2 + |w_t^{II}(0,\tau)|^2) = \mathcal{O}(|h|^2 + |h_t|^2) \tag{4.52}$$

for the outflow case.

Similarly, by induction, we shall derive the same estimates for W in H^s. To do that, let us define

$$\mathcal{E}_1(W) := \langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle KW_x, W\rangle$$
$$\mathcal{E}_k(W) := \langle A^0 \partial_x^k W, \partial_x^k W\rangle + M\mathcal{E}_{k-1}(W) + \epsilon\langle K\partial_x^k W, \partial_x^{k-1} W\rangle.$$

Then by Cauchy-Schwarz inequality, it is easy to see that $\mathcal{E}_k(W) \sim \|W\|_{H^k}^2$, and by induction, we obtain

$$\frac{d}{dt}\mathcal{E}_s(W)(t) \le -\theta_3 \mathcal{E}_s(W)(t) + C(c_*)(\|W(t)\|_{L^2}^2 + \mathcal{B}_h(t)^2),$$

for some positive constant θ_3, which by the Gronwall inequality yields

$$\|W(t)\|_{H^s}^2 \le Ce^{-\theta t}\|W_0\|_{H^s}^2 + C(c_*)\int_0^t e^{-\theta(t-\tau)}(\|W(\tau)\|_{L^2}^2 + \mathcal{B}_h(\tau)^2)d\tau, \tag{4.53}$$

where $W(x,0) = W_0(x)$ and $\mathcal{B}_h$ is defined as in (1.22) and (1.23).

4.1.7. *The general case.* Following [MaZ4], the general case that hypotheses (A1)-(A3) hold can easily be covered via following simple observations. First, we may express matrix A in (4.4) as

$$A(W+\bar W) = \hat A + (\zeta + |\bar W_x|)\begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1)\end{pmatrix} \tag{4.54}$$

where $\hat A$ is a symmetric matrix obeying the same derivative bounds as described for A, identical to A in the 11 block and obtained in other blocks jk by

$$\begin{aligned} A^{jk}(W+\bar W) &= A^{jk}(\bar W) + A^{jk}(W+\bar W) - A^{jk}(\bar W) \\ &= A^{jk}(W_+) + \mathcal{O}(|W_x| + |\bar W_x|) = A^{jk}(W_+) + \mathcal{O}(\zeta + |\bar W_x|). \end{aligned} \tag{4.55}$$

Replacing A by $\hat{A}$ in the k^{th} order Friedrichs-type bounds above, we find that the resulting error terms may be expressed as

$$\langle \partial_x^k \mathcal{O}(\zeta + |\bar{W}_x|)|W|, |\partial_x^{k+1} w^{II}|\rangle,$$

plus lower order terms, easily absorbed using Young's inequality, and boundary terms

$$\mathcal{O}(\sum_{i=0}^{k} |\partial_x^i w^{II}(0)||\partial_x^k w^I(0)|)$$

resulting from the use of integration by parts as we deal with the 12−block. However these boundary terms were already treated somewhere as before (see (4.35)). Hence we can recover the same Friedrichs-type estimates obtained above. Thus we may relax $(A1')$ to $(A1)$.

The second observation is that, because of the favorable terms

$$c_* \theta_1 \langle |\bar{W}_x| \partial_x^k w^I, \partial_x^k w^I \rangle$$

occurring in the lefthand sides of the Friedrichs-type estimates (4.42), we need the Kawashima-type bound only to control the contribution to $|\partial_x^k w^I|^2$ coming from x near $+\infty$; more precisely, we require from this estimate only a favorable term

$$-\theta_2 \langle (1 - \mathcal{O}(\zeta + |\bar{W}_x|)) \partial_x^k w^I, \partial_x^k w^I \rangle$$

rather than $\theta_2 \|\partial_x^k w^I\|_0^2$ as in (4.46). But, this may easily be obtained by substituting for K a skew-symmetric matrix-valued function $\hat{K} := K(W_+)$, and using the fact that

$$\Re e(K(A^0)^{-1}A + B)(W_+) \geq \theta_2 > 0,$$

and same as (4.55), $K = \hat{K} + \mathcal{O}(\zeta + |\bar{W}_x|)$, we have

$$\Re e(K(A^0)^{-1}A + B)(W) \geq \theta_2(1 - \mathcal{O}(\zeta + |\bar{W}_x|)) > 0.$$

Thus we may relax $(A2')$ to $(A2)$.

Finally, notice that the term $g(\tilde{W}_x) - g(\bar{W}_x)$ in the perturbation equation may be Taylor expanded as

$$\begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \bar{W}_x) + g_1(\bar{W}_x, \tilde{W}_x) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix}$$

The first term, since it vanishes in the first component and since $|\bar{W}_x|$ decays at plus spatial infinity, yields by Young's inequality the estimate

$$\Big\langle \begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \bar{W}_x) + g_1(\bar{W}_x, \tilde{W}_x) \end{pmatrix}, \begin{pmatrix} w_x^I \\ w_x^{II} \end{pmatrix} \Big\rangle \leq C\Big(\langle (\zeta + |\bar{W}_x|) w_x^I, w_x^I \rangle + \|w_x^{II}\|_0^2\Big)$$

which can be treated in the Friedrichs-type estimates. The $(0, O(|W_x|^2)^T$ nonlinear term may be treated as other source terms in the energy estimates. Specifically, the worst-case term

$$\langle \partial_x^k W, \partial_x^k \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix} \rangle = -\langle \partial_x^{k+1} w^{II}, \partial_x^{k-1} \mathcal{O}(|W_x|^2) \rangle - \partial_x^k w^{II}(0) \partial_x^{k-1} \mathcal{O}(|W_x|^2)(0)$$

may be bounded by

$$\|\partial_x^{k+1} w^{II}\|_{L^2} \|W\|_{W^{2,\infty}} \|W\|_{H^k} - \partial_x^k w^{II}(0)\partial_x^{k-1}\mathcal{O}(|W_x|^2)(0).$$

The boundary term will contribute to energy estimates in the form (4.33) of I_b, and thus we may use the parabolic equations to get rid of this term as we did in (4.34). Thus, we may relax $(A3')$ to $(A3)$, completing the proof of the general case $(A1)-(A3)$ and the proposition. □

4.2. **Energy estimate II.** We require also the following estimate:

Lemma 4.3 ([HR])**.** *Under the hypotheses of Theorem 1.7, let* $E_0 := \|(1+|x|^2)^{3/4}U_0\|_{H^4}$*, and suppose that, for* $0 \le t \le T$*, the* $W^{2,\infty}$ *norm of the solution* U *of* (5.2) *remains bounded by some constant* $C > 0$*. Then, for all* $0 \le t \le T$*,*

$$\|(1+|x|^2)^{3/4}U(x,t)\|_{H^4}^2 \le ME_0e^{Mt}. \tag{4.56}$$

Proof. This follows by standard Friedrichs symmetrizer estimates carried out in the weighted H^4 norm. □

Remark 4.4. *An immediate consequence of Lemma 4.3, by Sobolev embedding:* $W^{3,\infty} \subset H^4$ *and equation* (5.2)*, is that, if* E_0 *and* $\|U\|_{H^4}$ *are uniformly bounded on* $[0,T]$*, then*

$$(1+|x|)^{3/2}\Big[|U| + |U_t| + |U_x| + |U_{xt}|\Big](x,t) \tag{4.57}$$

is uniformly bounded on $[0,T]$ *as well.*

5. Stability analysis

In this section, we shall prove Theorems 1.6 and 1.7. Following [HZ, MaZ3], define the nonlinear perturbation $U=(u,v)$ by

$$U(x,t) := \tilde U(x,t) - \bar U(x), \tag{5.1}$$

we obtain

$$U_t - LU = Q(U,U_x)_x, \tag{5.2}$$

where linearized operator

$$LU := -(AU)_x + (BU_x)_x \tag{5.3}$$

where

$$AU := dF(\bar U)U - (dB(\bar U)U)\bar U_x, \quad B = B(\bar U)$$

and the second-order Taylor remainder:

$$Q(U,U_x) = F(\bar U + U) - F(\bar U) + A(\bar U)U + (B(\bar U+U) - B(\bar U))U_x$$

satisfying

$$
\begin{aligned}
|Q(U,U_x)| &\le C(|U||U_x| + |U|^2)\\
|\Pi_1 Q(U,U_x)_x| &\le C(|U||U_x| + |U|^2)\\
|Q(U,U_x)_x| &\le C(|U||U_{xx}| + |U_x|^2 + |U||U_x|)\\
|Q(U,U_x)_{xx}| &\le C(|U||U_{xx}| + |U||U_{xxx}| + |U_x||U_{xx}| + |U_x|^2)
\end{aligned}
\tag{5.4}
$$

so long as $|U|$ remains bounded.

For boundary conditions written in $U-$coordinates, (B) gives

$$
\begin{aligned}
h(t) = \tilde h(t) - \bar h &= (\tilde W(U+\bar U) - \tilde W(\bar U))(0,t)\\
&= (\partial \tilde W/\partial \tilde U)(\bar U_0)U(0,t) + \mathcal{O}(|U(0,t)|^2).
\end{aligned}
\tag{5.5}
$$

in inflow case and

$$
\begin{aligned}
h(t) = \tilde h(t) - \bar h &= (\tilde w^{II}(U+\bar U) - \tilde w^{II}(\bar U))(0,t)\\
&= (\partial \tilde w^{II}/\partial \tilde U)(\bar U_0)U(0,t) + \mathcal{O}(|U(0,t)|^2)\\
&= m\begin{pmatrix}\bar b_1 & \bar b_2\end{pmatrix}(\bar U_0)U(0,t) + \mathcal{O}(|U(0,t)|^2)\\
&= mB(\bar U_0)U(0,t) + \mathcal{O}(|U(0,t)|^2).
\end{aligned}
\tag{5.6}
$$

5.1. **Integral formulation.** We obtain the following:

Lemma 5.1 (Integral formulation)**.** *We have*

$$
\begin{aligned}
U(x,t) = &\int_0^\infty G(x,t;y)U_0(y)\,dy\\
&+ \int_0^t \Big(\tilde G_y(x,t-s;0)BU(0,s) + G(x,t-s;0)AU(0,s)\Big)\,ds\\
&+ \int_0^t\int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds\\
&- \int_0^t\int_0^\infty \tilde G_y(x,t-s;y)\Pi_2 Q(U,U_y)(y,s)\,dy\,ds
\end{aligned}
\tag{5.7}
$$

where $U(y,0) = U_0(y)$.

Proof. From the duality (see [ZH, Lemma 4.3]), we find that $G(x,t-s;y)$ considered as a function of y,s satisfies the adjoint equation

$$
(\partial_s - L_y)^* G^*(x,t-s;y) = 0,
\tag{5.8}
$$

or

$$
-G_s - (GA)_y + GA_y = (G_yB)_y.
\tag{5.9}
$$

in the distributional sense, for all $x,y,t>s>0$, where the adjoint operator of L_y is defined by

$$
L_y^* V := V_y^* A + (V_y^* B)_y,
\tag{5.10}
$$

with $V^* = V^{tr}$.

Likewise, for boundary conditions, we have, by duality

(iii') for all $x, t > 0$, $G(x,t;0) \equiv 0$ in the outflow case $\bar{A}_* < 0$; and $G(x,t;0)B = 0$ in the inflow case $\bar{A}_* > 0$, noting that no boundary condition need be applied on the hyperbolic part for the adjoint equations in the inflow case.

Thus, integrating G against (5.2), we obtain for any classical solution that

$$\begin{aligned}\int_0^t \int_0^\infty & G(x,t-s;y)Q(U,U_y)_y(y,s)\,dy\,ds = \\ & \int_0^t \int_0^\infty G(x,t-s;y)(\partial_s - L_y)U(y,s)\,dy\,ds \\ & =: I_1 + I_2.\end{aligned} \tag{5.11}$$

Integrating by parts and using the boundary conditions (iii') on the boundary $y = 0$, we get

$$\begin{aligned} I_1 &= \int_0^t \int_0^\infty G(x,t-s;y)\partial_s U(y,s)\,dy\,ds \\ &= \int_0^t \int_0^\infty \partial_s G(x,t-s;y)U(y,s)\,dy\,ds \\ &\quad + \int_0^\infty G(x,0;y)U(y,t)\,dy - \int_0^\infty G(x,t;y)U(y,0)\,dy \end{aligned}$$

where note that

$$U(x,t) = \int_0^\infty G(x,0;y)U(y,t)\,dy$$

and also

$$\begin{aligned} I_2 &= \int_0^t \int_0^\infty G(x,t-s;y)(-L_y)U(y,s)\,dy\,ds \\ &= \int_0^t \int_0^\infty G(x,t-s;y)((AU)_y - (BU_y)_y)(y,s)\,dy\,ds \\ &= \int_0^t \int_0^\infty (-G_y A - (G_y B)_y)U(y,s)\,dy\,ds \\ &\quad - \int_0^t G_y(x,t-s;0)BU(0,s)ds - \int_0^t G(x,t-s;0)AU(0,s)ds \end{aligned}$$

Combining these estimates, and noting that $G_y B = \tilde{G}_y B$ since $HB \equiv 0$, we obtain (5.7) by rearranging and integrating by parts the last term of

$$\begin{aligned}\int_0^t \int_0^\infty & G(x,t-s;y)Q(U,U_y)_y(y,s)\,dy\,ds \\ & = \int_0^t \int_0^\infty (H + \tilde{G})(x,t-s;y)Q(U,U_y)_y(y,s)\,dy\,ds\end{aligned} \tag{5.12}$$

□

As an expression for U_x, we obtain the following.

Lemma 5.2 (Integral formulation for U_x). *We have*

(5.13)
$$\begin{aligned}
U_x(x,t) = &\int_0^\infty G_x(x,t;y)U_0(y)\,dy - \int_0^t H(x,t-s;0)\Pi_1 Q(U,U_y)_y(0,s)\,ds\\
&+ \int_0^t \Big[\tilde G_{xy}(x,t-s;0)BU(0,s) + G_x(x,t-s;0)AU(0,s)\Big]\,ds\\
&+ \int_0^t\int_0^\infty (H_x - H_y)(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds\\
&- \int_0^t\int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_{yy}(y,s)\,dy\,ds\\
&- \int_0^{t-1}\int_0^\infty \tilde G_{xy}(x,t-s;y)\Pi_2 Q(U,U_y)(y,s)\,dy\,ds\\
&+ \int_{t-1}^t\int_0^\infty \tilde G_x(x,t-s;y)\Pi_2 Q(U,U_y)_y(y,s)\,dy\,ds
\end{aligned}$$

where $U(y,0) = U_0(y)$.

Proof. Differentiating the formulation (5.7) for $U(x,t)$ with respect to x and noting that

$$\begin{aligned}
\int_0^t\int_0^\infty H_x\phi\,dy\,ds = &\int_0^t\int_0^\infty (H_x - H_y)\phi\,dy\,ds\\
&- \int_0^t\int_0^\infty H(x,t-s;y)\phi_y(y,s)\,dy\,ds - \int_0^t H(x,t-s;0)\phi(0,s)ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t\int_0^\infty \tilde G_{xy}\psi\,dy\,ds = &\int_0^{t-1}\int_0^\infty \tilde G_{xy}\psi\,dy\,ds\\
&- \int_{t-1}^t\int_0^\infty \tilde G_x\psi_y\,dy\,ds - \int_{t-1}^t \tilde G_x(x,t-s;0)\psi(0,s)ds
\end{aligned}$$

are valid for any smooth functions ϕ, ψ, we obtain the lemma. □

5.2. **Convolution estimates.** To establish stability, we use the following lemmas proved in [HZ, HR, RZ].

Lemma 5.3 (Linear estimates I). *Under the assumptions of Theorem 1.7,*

$$\begin{aligned}
&\int_0^{+\infty} |\tilde G(x,t;y)|(1+|y|)^{-3/2}\,dy \le C(\theta+\psi_1+\psi_2)(x,t),\\
&\int_0^{+\infty} |\tilde G_x(x,t;y)|(1+|y|)^{-3/2}\,dy \le C(\theta+\psi_1+\psi_2)(x,t),
\end{aligned} \tag{5.14}$$

and so the latter is dominated by $\psi_1+\psi_2$, *for* $0\le t\le+\infty$, *some* $C>0$.

Lemma 5.4 (Linear estimates II). *Under the assumptions of Theorem 1.7, if* $|U_0(x)| + |\partial_x U_0(x)| \le E_0(1+|x|)^{-3/2}, E_0 > 0$, *then, for some* $\theta > 0$,

$$\begin{aligned} \int_0^{+\infty} H(x,t;y)U_0(y)\,dy &\le CE_0 e^{-\theta t}(1+|x|)^{-3/2}, \\ \int_0^{+\infty} H_x(x,t;y)U_0(y)\,dy &\le CE_0 e^{-\theta t}(1+|x|)^{-3/2}, \end{aligned} \tag{5.15}$$

and so both are dominated by $CE_0(\psi_1+\psi_2)$, *for* $0 \le t \le +\infty$, *some* $C > 0$.

Lemma 5.5 (Nonlinear estimates I). *Under the assumptions of Theorem 1.7,*

$$\begin{aligned} \int_0^t \int_0^{+\infty} |\tilde G_y(x,t-s;y)|\Psi(y,s)\,dyds &\le C(\theta+\psi_1+\psi_2)(x,t), \\ \int_0^{t-1} \int_0^{+\infty} |\tilde G_{xy}(x,t-s;y)|\Psi(y,s)\,dyds &\le C(\theta+\psi_1+\psi_2)(x,t), \end{aligned} \tag{5.16}$$

for $0 \le t \le +\infty$, *some* $C > 0$, *where*

$$\Psi(y,s) := (\theta+\psi_1+\psi_2)^2(y,s). \tag{5.17}$$

Lemma 5.6 (Nonlinear estimates II). *Under the assumptions of Theorem 1.7,*

$$\begin{aligned} \int_0^t \int_0^{+\infty} H(x,t-s;y)\Upsilon(y,s)\,dyds &\le C(\psi_1+\psi_2)(x,t) \\ \int_0^t \int_0^{+\infty} (H_x - H_y)(x,t-s;y)\Upsilon(y,s)\,dyds &\le C(\psi_1+\psi_2)(x,t) \\ \int_{t-1}^t \int_0^{+\infty} |\tilde G_x(x,t-s;y)|\Upsilon(y,s)\,dyds &\le C(\psi_1+\psi_2)(x,t) \end{aligned} \tag{5.18}$$

for all $0 < t < +\infty$, *some* $C > 0$, *where*

$$\Upsilon(y,s) := s^{-1/4}(\theta+\psi_1+\psi_2)(y,s) \tag{5.19}$$

We require also the following estimate accounting boundary effects.

Lemma 5.7 (Boundary estimates I). *Under the assumptions of Theorem 1.7, if* $|h(t)| + |h'(t)| \le E_0(1+t)^{-1}$,

$$\begin{aligned} \int_0^t H(x,t-s;0)h(s)\,ds &\le CE_0(\psi_1+\psi_2)(x,t) \\ \int_0^t H_x(x,t-s;0)h(s)\,ds &\le CE_0(\psi_1+\psi_2)(x,t), \end{aligned} \tag{5.20}$$

for $0 \le t \le +\infty$, *some* $C > 0$.

Proof. Note that $H(x,t;0)\equiv 0$ for the outflow case $A_*<0$. Consider the inflow case $A_*>0$ (and thus $\bar a_*>0$). We have

$$\begin{aligned}
&\Big|\int_0^t H(x,t-s;0)h(s)\,ds\Big|\\
&\qquad = e^{-\eta_0 x/\bar a_*}|h(-\frac{1}{\bar a_*}(x-\bar a_* t))|\\
&\qquad \le e^{-\eta_0|x|}(1+|x-\bar a_* t|)^{-1}\le CE_0(\psi_1+\psi_2)(x,t),\\
&\Big|\int_0^t H_x(x,t-s;0)h(s)\,ds\Big|\\
&\qquad \le e^{-\eta_0 x/\bar a_*}\Big(|h|+|h'|\Big)(-\frac{1}{\bar a_*}(x-\bar a_* t))|\\
&\qquad \le e^{-\eta_0|x|}(1+|x-\bar a_* t|)^{-1}\le CE_0(\psi_1+\psi_2)(x,t),
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 5.8 (Boundary estimates II)**.** *Under the assumptions of Theorem 1.7, if* $|h(t)|\le E_0(1+t)^{-1-\epsilon}$ *and* $|h'(t)|\le E_0(1+t)^{-1}$,

$$\begin{aligned}
&\Big|\int_0^t \Big(\tilde G_y(x,t-s;0)Bh(s)+G(x,t-s;0)Ah(s)\Big)\,ds\Big|\\
&\qquad\qquad \le CE_0(\theta+\psi_1+\psi_2)(x,t)\\
&\Big|\int_0^t \Big(\tilde G_{xy}(x,t-s;0)Bh(s)+G_x(x,t-s;0)Ah(s)\Big)\,ds\Big|\\
&\qquad\qquad \le CE_0(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}\tag{5.21}$$

for $0\le t\le +\infty$*, some* $C>0$.

Proof. We first give the estimate on $\int_0^{t-1}$, where $G_y(x,t-s;0)$ and $\tilde G_{xy}(x,t-s;0)$ are nonsingular. We have

$$\Big|\int_0^{t-1}\tilde G_y(x,t-s;0)Bh(s)\,ds\Big|\le C\int_1^t|\tilde G_y(x,\tau;0)|(1+t-\tau)^{-1-\epsilon}\,d\tau.\tag{5.22}$$

We shall estimate the integral for each term $(1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}$, appearing in $\tilde G_y(x,\tau;0)$, and omit the $\mathcal{O}(e^{-\eta(x+t)})$ term, which is negligible. First, for $a_k<0$, using $e^{-|x-a_k\tau|^2/M\tau}\le e^{-x^2/Mt}e^{-\eta\tau}$ for some $\eta>0$, we have

$$\begin{aligned}
&\int_1^t(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-|x-a_k\tau|^2/M\tau}\,d\tau\\
&\qquad\le e^{-x^2/Mt}\Big(\int_1^{t/2}+\int_{t/2}^t\Big)(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-\eta\tau}\,d\tau\\
&\qquad\le e^{-x^2/Mt}\Big((1+t)^{-1}+(1+t)^{-1/2}e^{-\eta t}\Big),
\end{aligned}\tag{5.23}$$

which is clearly bounded by $C(\theta+\psi_1)(x,t)$. For $a_k>0$, we consider three distinct regions depending on x and t. First for $x\ge a_k t$, we further divide the estimates into two cases: $(1,t/2)$ and $(t/2,t)$. For $\tau\in(1,t/2)$, we have $e^{-|x-a_k\tau|^2/M\tau}\le e^{-x^2/Mt}e^{-\eta\tau}$ for some $\eta>0$ and thus as above the integral is bounded by $C(\theta+\psi_1)(x,t)$. For $\tau\in(t/2,t)$, we write $x-a_k\tau=x-a_kt+a_k(t-\tau)$ and thus

$$\begin{aligned}\int_{t/2}^t&(1+\tau)^{-1/2}(1+t-\tau)^{-1-\epsilon}e^{-|x-a_k\tau|^2/M\tau}\,d\tau\\&\le e^{-(x-a_kt)^2/Mt}\int_{t/2}^t(1+\tau)^{-1/2}(1+t-\tau)^{-1-\epsilon}e^{-a_k(t-\tau)^2/M\tau}\,d\tau\\&\le C(1+t)^{-1/2}e^{-(x-a_kt)^2/Mt}\int_{t/2}^t(1+t-\tau)^{-1-\epsilon}\,d\tau\le C\theta(x,t).\end{aligned}$$

Next, consider the case: $x\le a_kt/2$. Divide the analysis into cases: $(1,3t/4)$ and $(3t/4,t)$. For $\tau\in(1,3t/4)$, use the change of variable $s:=(x-a_k\tau)/\sqrt{\tau}$ to get

$$\begin{aligned}\int_1^{3t/4}&(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-|x-a_k\tau|^2/M\tau}\,d\tau\\&\le(1+t)^{-1}\int_1^{3t/4}(1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}\,d\tau\\&\le(1+t)^{-1}\int_{-\infty}^{+\infty}e^{-s^2/M}\,ds\le(1+t)^{-1},\end{aligned}\tag{5.24}$$

which is bounded by $C\psi_1(x,t)$. For $\tau\in(3t/4,t)$, we have $e^{-|x-a_k\tau|^2/M\tau}\le e^{-\eta\tau}$ for some $\eta>0$ and thus

$$\begin{aligned}\int_{3t/4}^t&(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-|x-a_k\tau|^2/M\tau}\,d\tau\\&\le(1+t)^{-1/2}\int_{3t/4}^te^{-\eta\tau}\,d\tau\le C(1+t)^{-1/2}e^{-\eta t}\le C\theta(x,t).\end{aligned}\tag{5.25}$$

Finally, consider the case $x\in(a_kt/2,a_kt)$. We write $x=aa_kt$ with $a:=\frac{x}{a_kt}$. We again divide the estimate into three regions: $(1,at),(at,\frac{1+a}{2}t)$, and $(\frac{1+a}{2}t,t)$. For $\tau\in(1,at)$, we have $(1+t-\tau)^{-1}\le C(1+t)^{-1}\le C\psi_1(x,t)$ and

$$\int_1^{at}(1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}\,d\tau\le\int_0^{+\infty}e^{-s^2/M}\,ds\le C.\tag{5.26}$$

For $\tau \in (at, \frac{1+a}{2}t)$, we have $(1+t-\tau)^{-1} \le C(1+|x-a_kt|)^{-1}$ and by change of variable $s := (x-a_k\tau)/\tau$,

$$\begin{aligned}\int_{at}^{\frac{1+a}{2}t} &(1+\tau)^{-1/2} e^{-|x-a_k\tau|^2/M\tau}\, d\tau \\ &\le \int_0^{\frac{1-a}{1+a}} e^{-\tau^2/M}\, d\tau \le C(1-a) \le Ct^{-1}|x-a_kt|.\end{aligned} \tag{5.27}$$

Thus the integral is bounded by $Ct^{-1} \le C\psi_1(x,t)$. For $\tau \in (\frac{1+a}{2}t, t)$, we have $|x-a_k\tau| \ge |x-a_k\frac{1+a}{2}t| = \frac{a_k}{2}|1-a|t = \frac{|x-a_kt|}{2}$, and thus

$$\begin{aligned}\int_{\frac{1+a}{2}t}^{t} &(1+\tau)^{-1/2}(1+t-\tau)^{-1-\epsilon} e^{-|x-a_k\tau|^2/M\tau}\, d\tau \\ &\le (1+t)^{-1/2} e^{-|x-a_kt|^2/2Mt} \int_{\frac{1+a}{2}t}^{t} (1+t-\tau)^{-1-\epsilon}\, d\tau \\ &\le C(1+t)^{-1/2} e^{-|x-a_kt|^2/2Mt} \le C\theta(x,t).\end{aligned} \tag{5.28}$$

Therefore, combining all these estimates, we obtain

$$\Big| \int_0^{t-1} \tilde G_y(x,t-s;0) B h(s)\, ds \Big| \le C(\theta+\psi_1)(x,t). \tag{5.29}$$

We also have similar estimates for G_{xy} on the nonsingular part $\int_0^{t-1}$.

Next, to bound the singular part $\int_{t-1}^t$, we integrate (5.9) in y from 0 to $+\infty$ to obtain

$$\tilde G_y B + GA = -\int_0^{+\infty} G(x,t-s;y) A_y\, dy + \int_0^{+\infty} G_s(x,t-s;y)\, dy. \tag{5.30}$$

Substituting in the lefthand side of (5.21), and integrating by parts in s, we obtain

$$\begin{aligned}\int_{t-1}^{t} (\tilde G_y B + GA) h(s)\, ds = &\int_0^1 \Big(\int_0^{+\infty} A_y(y) G(x,\tau;y)\, dy \Big) h(t-\tau)\, d\tau \\ &- \int_0^1 \Big(\int_0^{+\infty} G(x,\tau;y)\, dy \Big) h'(t-\tau)\, d\tau \\ &+ \Big(\int_0^{+\infty} G(x,1;y)\, dy \Big) h(t-1),\end{aligned} \tag{5.31}$$

which by $\int |G| dy \le C$ is bounded by $\max_{0\le\tau\le1}(|h|+|h'|)(t-\tau)$.

Combining this with the following more straightforward estimate (for large x, $|x| > a_n^+ t$)

(5.32)

$$\begin{aligned}\Big|\int_{t-1}^t \tilde G_y(x,t-s;0)Bh(s)\,ds\Big| &\le \int_0^1 |\tilde G_y(x,\tau;0)|Bh(t-\tau)\,d\tau\\ &\le C\max_{0\le\tau\le 1}|h(t-\tau)|\int_0^1 \tau^{-1/2}e^{-|x|^2/C\tau}\,d\tau\\ &\le C\max_{0\le\tau\le 1}|h(t-\tau)|\int_0^1 \tau^{-1}e^{-|x|^2/C\tau}\,d\tau\\ &= C|x|^{-2}\max_{0\le\tau\le 1}|h(t-\tau)|\\ &\quad\times\int_0^1 (|x|^2/\tau)e^{-|x|^2/C\tau}\,d\tau\\ &\le C\max_{0\le\tau\le 1}|h(t-\tau)||x|^{-2},\end{aligned}$$

(5.33)

$$\begin{aligned}\Big|\int_{t-1}^t \tilde G(x,t-s;0)Ah(s)\,ds\Big| &\le \int_0^1 |\tilde G(x,\tau;0)|Ah(t-\tau)\,d\tau\\ &\le C\max_{0\le\tau\le 1}|h(t-\tau)|\int_0^1 \tau^{-1/2}e^{-|x|^2/C\tau}\,d\tau\\ &\le C\max_{0\le\tau\le 1}|h(t-\tau)||x|^{-2},\end{aligned}$$

and the estimate (5.20) for H term (thus together with (5.33) for $G = \tilde G + H$), we find that the contribution from $\int_{t-1}^t$ has norm bounded by

$$\max_{0\le\tau\le 1}(|h|+|h'|)(t-\tau)(1+|x|)^{-2} \le CE_0(\psi_1+\psi_2)(x,t).$$

Combining this estimate with the one for $\int_0^{t-1}$, we obtain the first inequality in (5.21). For second inequality, we first differentiate (5.31) with respect to x to get

(5.34)

$$\begin{aligned}\int_{t-1}^t (\tilde G_{xy}B + G_xA)h(s)\,ds &= \int_0^1\Big(\int_0^{+\infty} A_y(y)G_x(x,\tau;y)\,dy\Big)h(t-\tau)\,d\tau\\ &\quad-\int_0^1\Big(\int_0^{+\infty} G_x(x,\tau;y)\,dy\Big)h'(t-\tau)\,d\tau\\ &\quad+\Big(\int_0^{+\infty} G_x(x,1;y)\,dy\Big)h(t-1),\end{aligned}$$

which, by $\int_0^1\int |G_x|dyd\tau \le C\int_0^1 \tau^{-1/2}d\tau \le C$, is bounded by $\max_{0\le\tau\le 1}(|h|+|h'|)(t-\tau)$, similarly as above.

For the large x, clearly we still have similar estimates as (5.32) and (5.33) for $\tilde{G}_{xy}$ and $\tilde{G}_x$. These, estimate (5.20) for H_x, and (5.34) yield the contribution from $\int_{t-1}^t$ as above, which together with the estimate for $\int_0^{t-1}$ completes the proof of (5.21). □

5.3. **Linearized stability.** In this subsection, we shall give the proof of Theorem 1.6. We first need the following estimates:

Lemma 5.9 ([MaZ4]). *Under the assumptions of Theorem 1.6,*

$$\begin{aligned} \Big|\int_0^{+\infty} \tilde{G}(\cdot,t;y)f(y)\,dy\Big|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-1/r)}|f|_{L^q},\\ \Big|\int_0^{+\infty} H(\cdot,t;y)f(y)\,dy\Big|_{L^p} &\le Ce^{-\eta t}|f|_{L^p}, \end{aligned} \tag{5.35}$$

for all $t\ge 0$, some C, $\eta>0$, for any $1\le q\le p$ and $f\in L^q\cap L^p$, where $1/r+1/q=1+1/p$.

Lemma 5.10. *Under the assumptions of Theorem 1.6, if $|h(t)|\le E_0(1+t)^{-1-\epsilon}$,*

$$\begin{aligned} \Big|\int_0^t \Big(\tilde{G}_y(x,t-s;0)Bh(s)+G(x,t-s;0)Ah(s)\Big)\,ds\Big|_{L^p}&\\ \le CE_0(1+t)^{-\frac{1}{2}(1-1/p)}& \end{aligned} \tag{5.36}$$

for $0\le t\le +\infty$, some $C>0$.

Proof. This follows at once by the boundary estimate (5.21) and the fact that $|(\theta+\psi_1+\psi_2)(\cdot,t)|_{L^p}\le C(1+t)^{-\frac{1}{2}(1-1/p)}$. □

Proof of Theorem 1.6. Sufficiency of (D) for linearized stability (the main point here) follows easily by applying the above lemmas to the following representation for solution $U(x,t)$ of the linearized equations (1.13)

$$\begin{aligned} U(x,t) = &\int_0^\infty G(x,t;y)U_0(y)\,dy\\ &+\int_0^t \Big(\tilde{G}_y(x,t-s;0)BU(0,s)+G(x,t-s;0)AU(0,s)\Big)\,ds \end{aligned}$$

where $U(y,0)=U_0(y)$ and $|U(0,s)|\le C|h(s)|\le C(1+s)^{-1-\epsilon}$ by (1.14) in the inflow case, and $|BU(0,s)|\le C|h(s)|\le C(1+s)^{-1-\epsilon}$ by (1.15) in the outflow case, noting that $G(x,t;0)\equiv 0$ in this case. Necessity follows by a much simpler argument, restricting x, y to a bounded set and letting $t\to\infty$, noting that G is given by the ODE evolution of the spectral projection onto the finite set of zeros of D in $\Re\lambda\ge 0$, necessarily nondecaying, plus an $O(e^{-\eta t})$ error, $\eta>0$, from which we find that asymptotic decay implies nonexistence of any such zeros; see Proposition 7.7 and Corollary 7.8, [MaZ3] for details. □

5.4. **Nonlinear argument.** In this subsection, we shall give the proof of Theorem 1.7. In fact, with the above preparations, the proof of nonlinear stability is also straightforward.

Lemma 5.11 (H^4 local theory)**.** *Under the hypotheses of Theorem 1.7, then, for T sufficiently small depending on the H^4-norm of U_0, there exists a unique solution $U(x,t) \in L^\infty(0,T;H^4(x))$ of* (5.2) *satisfying*

$$|U(t)|_{H^4} \le C|U_0|_{H^4} \tag{5.37}$$

for all $0 \le t \le T$.

Proof. Short-time existence, uniqueness, and stability are described in [Z2, Z4], using a standard (bounded high norm, contractive low norm) contraction mapping argument. We omit the details. □

Lemma 5.12. *Under the hypotheses of Theorem 1.7, let $U \in L^\infty(0,T;H^4(x))$ satisfy* (5.2) *on $[0,T]$, and define*

$$\zeta(t) := \sup_{x, 0\le s\le t} \Big[(|U|+|U_x|)(\theta+\psi_1+\psi_2)^{-1}(x,t)\Big]. \tag{5.38}$$

If $\zeta(T)$ and $|U_0|_{H^4}$ are bounded by ζ_0 sufficiently small, then, for some $\epsilon > 0$, (i) the solution U, and thus ζ extends to $[0,T+\epsilon]$, and (ii) ζ is bounded and continuous on $[0,T+\epsilon]$.

Proof. Boundedness and smallness of $|U(t)|_{H^4}$ on $[0,T]$ follow by Proposition 4.1, provided smallness of $\zeta(T)$ and $|U_0|_{H^4}$. By Lemma 5.11, this implies the existence, boundedness of $|U(t)|_{H^4}$ on $[0,T+\epsilon]$, for some $\epsilon > 0$, and thus, by Lemma 4.3, boundedness and continuity of ζ on $[0,T+\epsilon]$. □

Proof of Theorem 1.7. We shall establish:

Claim. For all $t \ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \le C_2(E_0 + \zeta(t)^2). \tag{5.39}$$

From this result, provided $E_0 < 1/4C_2^2$, we have that by continuous induction

$$\zeta(t) < 2C_2E_0 \tag{5.40}$$

for all $t \ge 0$. From (5.40) and the definition of ζ in (5.38) we then obtain the bounds of (1.24). Thus, it remains only to establish the claim above.

Proof of Claim. We must show that $(|U|+|U_x|)(\theta+\psi_1+\psi_2)^{-1}$ is bounded by $C(E_0+\zeta(t)^2)$, for some $C > 0$, all $0 \le s \le t$, so long as ζ remains sufficiently small. First we need an estimate for $U(0,s)$ and $U_s(0,s)$. For the inflow case, by boundary condition estimate (5.5) and by the hypotheses on $h(s)$, we have

$$|U(0,s)| \le C(h(s) + |U(0,s)|^2) \le C(E_0(1+s)^{-1-\epsilon} + |U(0,s)|^2) \tag{5.41}$$

from which by continuity of $|U(0,t)|$ (Remark 4.4) and smallness of E_0, we obtain a similar estimate to (5.40):

$$|U(0,s)| \le CE_0(1+s)^{-1-\epsilon}. \tag{5.42}$$

Similarly for an estimate of $U_t(0,t)$, by taking the derivative of (5.5), we get

$$\begin{aligned} |U_s(0,s)| &\le C(h'(s) + |U||U_s|(0,s)) \\ &\le C(E_0(1+s)^{-1} + |U(0,s)||U_s(0,s)|) \\ &\le C(E_0(1+s)^{-1} + |U_s(0,s)|^2) \end{aligned} \tag{5.43}$$

which by the same argument as above yields

$$|U_s(0,s)| \le CE_0(1+s)^{-1}. \tag{5.44}$$

Next, for the outflow case with boundary condition (5.6), we have

$$\begin{aligned} |BU(0,s)| &\le CE_0(1+s)^{-1-\epsilon} + \mathcal{O}(|U(0,s)|^2) \\ |(BU)_s(0,s)| &\le CE_0(1+s)^{-1} + \mathcal{O}(|U||U_s|(0,s)). \end{aligned} \tag{5.45}$$

Now by (5.38), we have for all $t \ge 0$ and some $C > 0$ that

$$|U(x,t)| + |U_x(x,t)| \le \zeta(t)(\theta + \psi_1 + \psi_2)(x,t), \tag{5.46}$$

and therefore

$$\begin{aligned} |Q(U,U_y)(y,s)| &\le C\zeta(t)^2\Psi(y,s) \\ |\Pi_1 Q(U,U_y)_y(y,s)| &\le C\zeta(t)^2\Psi(y,s) \end{aligned} \tag{5.47}$$

with $\Psi = (\theta + \psi_1 + \psi_2)^2$ as defined in (5.17), for $0 \le s \le t$.

As an estimate for $U(x,t)$, we use the representation (5.7) of $U(x,t)$:

$$\begin{aligned} |U(x,t)| &= \Big| \int_0^\infty G(x,t;y)U_0(y)\,dy \Big| \\ &+ \Big| \int_0^t (\tilde{G}_y(x,t-s;0)BU(0,s) + G(x,t-s;0)AU(0,s))\,ds \Big| \\ &+ \Big| \int_0^t \int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds \Big| \\ &+ \Big| \int_0^t \int_0^\infty \tilde{G}_y(x,t-s;y)\Pi_2 Q(U,U_y)(y,s)\,dy\,ds \Big|, \end{aligned}$$

where by applying Lemmas 5.3-5.6 together with (5.47), we have

$$\begin{aligned} &\Big| \int_0^\infty G(x,t;y)g(y)\,dy \Big| \\ &\qquad \le E_0 \int_0^\infty (|\tilde{G}(x,t;y)| + |H(x,t;y)|)(1+|y|)^{-3/2}\,dy \\ &\qquad \le CE_0(\theta + \psi_1 + \psi_2)(x,t) \end{aligned} \tag{5.48}$$

$$
\begin{aligned}
&\Big|\int_0^t\int_0^\infty \tilde G_y(x,t-s;y)Q(U,U_y)(y,s)\,dy\,ds\Big|\\
&\qquad\le C\zeta(t)^2\int_0^t\int_0^\infty |\tilde G_y(x,t-s;y)|\Psi(y,s)\,dy\,ds\\
&\qquad\le C\zeta(t)^2(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{5.49}
$$

$$
\begin{aligned}
&\Big|\int_0^t\int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds\Big|\\
&\qquad\le C\zeta(t)^2\int_0^t\int_0^\infty H(x,t-s;y)(\theta+\psi_1+\psi_2)^2\,dy\,ds\\
&\qquad\le C\zeta(t)^2\int_0^t\int_0^\infty H(x,t-s;y)\Upsilon(y,s)\,dy\,ds\\
&\qquad\le C\zeta(t)^2(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{5.50}
$$

and, for the boundary term, we apply the estimate (5.42) and Lemma 5.8, yielding

$$
\begin{aligned}
&\Big|\int_0^t(\tilde G_y(x,t-s;0)BU(0,s)+G(x,t-s;0)AU(0,s))\,ds\Big|\\
&\qquad\le C(E_0+\zeta(t)^2)(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{5.51}
$$

for the inflow. Whereas, for the outflow case, noting that $G(x,t-s;0)\equiv 0$ in the outflow case, we apply the estimate (5.45), (5.46) and Lemma 5.8 to give the same estimate as above, yielding

$$
\Big|\int_0^t\tilde G_y(x,t-s;0)BU(0,s)\,ds\Big|\le C(E_0+\zeta(t)^2)(\theta+\psi_1+\psi_2)(x,t)
$$

where we used (5.46) for $|U(0,s)|\le\zeta(t)(1+s)^{-1}$ and thus by (5.45), $|BU(0,s)|\le C(E_0+\zeta(t)^2)(1+s)^{-1-\epsilon}$.

Therefore, combining the above estimates, we obtain

$$
|U(x,t)|(\theta+\psi_1+\psi_2)^{-1}(x,t)\le C(E_0+\zeta(t)^2).
\tag{5.52}
$$

To derive the same estimate for $|U_x(x,t)|$, we first obtain by using Proposition 4.1,

$$
\begin{aligned}
|U(t)|^2_{H^4}&\le Ce^{-\theta t}|U_0|^2_{H^4}+C\int_0^t e^{-\theta(t-\tau)}\Big[|U(\tau)|^2_{L^2}+\mathcal{B}_h(\tau)^2\Big]d\tau\\
&\le C(E_0+\zeta(t)^2)t^{-1/2},
\end{aligned}
$$

where $\mathcal{B}_h$ is the boundary function defined in Proposition 4.1, and thus by the one dimensional Sobolev embedding: $|U(t)|_{W^{3,\infty}}\le C|U(t)|_{H^4}$,

$$
\begin{aligned}
|Q(U,U_x)_x|&\le C(\zeta^2(t)+4C^2E_0^2)\Upsilon\\
|Q(U,U_x)_{xx}|&\le C(\zeta^2(t)+4C^2E_0^2)\Upsilon
\end{aligned}
\tag{5.53}
$$

where $\Upsilon=t^{-1/4}(\theta+\psi_1+\psi_2)$.

Now again applying Lemmas 5.3-5.8 together with (5.53), (5.44), and (5.45), we have obtained the desired estimate, that is, bounded by $(\zeta^2(t) + CE_0)(\theta+\psi_1+\psi_2)(x,t)$, for most terms in the formulation (5.13) of $U_x(x,t)$, except one boundary term:

$$\int_0^t H(x,t-s;0)|\Pi_1 Q(U,U_y)_y(0,s)|\,ds,$$

which is bounded by $CE_0(\psi_1+\psi_2)(x,t)$ by using (5.4), (5.46), and Lemma 5.7, and noting that

$$|\Pi_1 Q(U,U_y)_y(0,s)| \le \zeta(t)|h(s)|(\theta+\psi_1+\psi_2)(0,s) \le C\zeta(t)|h(s)|.$$

Therefore, together with (5.52), we have obtained

$$(|U(x,t)| + |U_x(x,t)|)(\theta+\psi_1+\psi_2)^{-1}(x,t) \le C(E_0+\zeta(t)^2) \tag{5.54}$$

as claimed, which completes the proof of Theorem 1.7. □

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