GLOBAL MAGNETIC CONFINEMENT FOR THE 1.5D VLASOV-MAXWELL SYSTEM

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Abstract. We establish the global-in-time existence and uniqueness of classical solutions to the "one and one-half" dimensional relativistic Vlasov–Maxwell systems in a bounded interval, subject to an external magnetic field which is infinitely large at the spatial boundary. We prove that the large external magnetic field confines the particles to a compact set away from the boundary. This excludes the known singularities that typically occur due to particles that repeatedly bounce off the boundary. In addition to the confinement, we follow the techniques introduced by Glassey and Schaeffer, who studied the Cauchy problem without boundaries.

1. Introduction

Using external magnetic fields to confine plasmas, has been one of major goals of fusion energy research. It is one of the most promising mechanisms for producing safe new sources of fusion energy. Scientists are particularly interested in designing stable devices to induce confinement (e.g., [Ga, Wh]). In this paper we establish *global-in-time magnetic confinement of a collisionless plasma*, albeit under an assumption of low dimension.

Specifically, we consider the relativistic Vlasov-Maxwell (RVM) system, subject to an external magnetic field B_{ext} in a bounded interval $\Omega = (0, 1)$. We assume a single species of particles with a nonnegative distribution function $f(t, x, v)$, where $t \geq 0$, $x \in \Omega$ and $v \in \mathbb{R}^2$. In this $1\frac{1}{2}$ dimensional model the Vlasov equation is

$$\partial_t f + \hat{v}_1 \partial_x f + (E_1 + \hat{v}_2 \check{B}) \partial_{v_1} f + (E_2 - \hat{v}_1 \check{B}) \partial_{v_2} f = 0, \tag{1.1}$$

where $\check{B} = B(t, x) + B_{\text{ext}}(x)$ with $B_{\text{ext}}(x)$ is a stationary external magnetic field that becomes infinitely large on the boundary. The internal electric and magnetic field with components $E_1(t, x)$, $E_2(t, x)$, $B(t, x)$ satisfies the $1\frac{1}{2}$D Maxwell equations

$$\begin{cases} \partial_t E_1 &= -j_1; \quad \partial_x E_1 = \rho; \\ \partial_t E_2 &= -\partial_x B - j_2, \\ \partial_t B &= -\partial_x E_2. \end{cases} \tag{1.2}$$

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For mathematical simplicity all the physical constants have been normalized. In this relativistic case, the velocity is $\hat{v} = (\hat{v}_1, \hat{v}_2) = v/\sqrt{1+|v|^2}$. The charge density ρ and the current density $j = (j_1, j_2)$ are

$$\rho(t,x) := \int_{\mathbb{R}^2} f(t,x,v)\,dv \quad \text{and} \quad j(t,x) := \int_{\mathbb{R}^2} \hat{v} f(t,x,v)\,dv.$$

We impose the standard initial conditions for the distribution function and the field, namely,

$$f(0,x,v) = f^0(x,v) \geq 0, \quad E_2(0,x) = E_2^0(x), \quad B(0,x) = B^0(x), \tag{1.3}$$

while the initial value for E_1 is already determined by means of the identity $\partial_x E_1 = \rho$ and the specification

$$E_1(0,0) = \lambda \tag{1.4}$$

for a given constant $\lambda \in \mathbb{R}$.

The novelty of this paper lies in the boundary conditions. We assume

$$E_2(t,x)_{|\partial\Omega} = E_2^b(t,x), \qquad B(t,x)_{|\partial\Omega} = B^b(t,x), \tag{1.5}$$

are given functions $E_2^b(t,x)$, $B^b(t,x)$ defined on the boundary. In the sequel we will show that no particle trajectory can reach the boundary $\partial\Omega$ if it begins away from it. Because the particle density $f(t,x,v)$ is constant along each particle trajectory, *no boundary condition is needed for* $f(t,x,v)$, assuming that its initial support does not meet the boundary.

Throughout the paper we take $B_{\text{ext}} = \partial_x \psi_{\text{ext}}(x)$, in which the potential function $\psi_{\text{ext}}(x)$ is assumed to satisfy:

$$\psi_{\text{ext}} \in C^2(\Omega) \quad \text{and} \quad |\psi_{\text{ext}}(x)| \geq \frac{c_0}{\text{dist}(x,\partial\Omega)^\gamma} - \frac{1}{c_0} \qquad \forall\, x \in \Omega \tag{1.6}$$

for some constants $\gamma > 0$ and $c_0 > 0$. In particular, $\psi_{\text{ext}}(x) = \infty$ on the boundary!

We are interested in the well-posedness of the initial-boundary value problem (1.1)–(1.5). In what follows, $C^1(U)$ denotes the standard C^1 function space, and $C^1_0(U)$ consists of functions in $C^1(U)$ that have compact support in U. In particular, $f \in C^1_0([0,T]\times\Omega\times\mathbb{R}^2)$ means that f has compact support in the (x,v)-variable, but has no restriction in the t-variable. We now state our main result.

Theorem 1.1 (Global well-posedness)**.** *Assume that* $f^0 \in C^1_0(\Omega \times \mathbb{R}^2)$ *is nonnegative,* $\lambda \in \mathbb{R}$ *and* E_2^0, B^0, E_2^b, $B^b \in C^1$. *Assume also that the external magnetic field* $B_{ext} = \partial_x \psi_{ext}$ *satisfies* (1.6). *Then the problem* (1.1)–(1.5) *has a unique global-in-time* C^1 *solution* (f, E_1, E_2, B). *Moreover,* f *is nonnegative and* $f \in C^1_0([0,T]\times\Omega\times\mathbb{R}^2)$ *for any* $T > 0$.

Let us mention a few previous results on the global Cauchy problem for the Vlasov–Maxwell system. It is well known that global *weak* solutions exist in the whole three-dimensional space ([DiL]), even in the presence of boundaries ([Gu1, M]). However, it is a famous open problem as to whether such solutions are unique or regular. Concerning classical (smooth) solutions, the authors in [GStr] established the global theory for RVM systems in the whole three-dimensional space under an assumption on the momentum support of the density. Alternative proofs have been given in [BGP, KS]. Subsequently, there was a series of papers [GSc, GSc2, GSc2.5] where the (unconditional) well-posedness and regularity of solutions were established for the $1\frac{1}{2}$, 2, and $2\frac{1}{2}$ dimensional RVM system. The present paper is motivated by [GSc], our novelty being the presence of a boundary.

There have been just a few mathematical studies of the magnetic confinement problem (e.g., [HK, CCM1, CCM2]). Both papers [HK] and [CCM1] are concerned with a plasma with no internal magnetic field but confined by an external magnetic field. In [HK] further assumptions are introduced that reduce the problem to a system for the macroscopic density and electric field. In [CCM1] a Vlasov-Poisson system is considered and an existence-uniqueness theorem is proved. WHAT ABOUT [CCM2], WHICH I DON'T HAVE HANDY?

When confining a plasma modeled by RVM to a spatial domain, singularities are typically created at the boundary and they propagate inside the domain. This is true even for Vlasov-Poisson (VP) systems (i.e., without magnetic fields); see, e.g., [Gu2]. Furthermore, some particles repeatedly bounce off the boundary, making it extremely difficult to analyze their trajectories. To the best of our knowledge, there is no global theory of classical C^1 solutions to the RVM systems in domains with boundaries, even for the simplest RVM model, the $1\frac{1}{2}$ dimensional system (1.1)-(1.2) without an external magnetic field.

However, in our problem with a very intense external magnetic field at the boundary, singularities can be avoided because the particles that come near the boundary are drifted back into the plasma domain. Rigorous details of the confinement are provided in Section 3. The proof of our main theorem then follows along the lines of [GSc].

2. Bounds on the field

The proof of Theorem 1.1 relies on uniform *a priori* estimates. Let us consider a C^1 solution (f, E_1, E_2, B) of the RVM equations (1.1)–(1.5) on a finite time interval $[0, T]$ so that $f(t, x, v) = 0$ at the boundary $x = 0, 1$. We shall derive L^∞ estimates for the fields of such solution. For convenience, we rewrite (1.1) as

$$\partial_t f + \hat{v}_1 \partial_x f + K \cdot \nabla_v f = 0 \tag{2.7}$$

with $K := E + (\hat{v}_2, -\hat{v}_1)\check{B}$. Hereafter we use E to denote the vector (E_1, E_2).

2.1. **Estimate of E_1.** Integrating the Vlasov equation (2.7) in v and using $K \cdot \nabla_v f = \nabla_v \cdot (Kf)$, we obtain

$$\partial_t \rho + \partial_x j_1 = 0. \tag{2.8}$$

Observe that the vanishing condition on $f(t, x, v)$ on the boundary implies $j_1(t, 0) = j_1(t, 1) = 0$ and hence we deduce from (2.8) by integrating in x that

$$\int_\Omega \rho(t, x)\, dx = \int_\Omega \rho(0, x)\, dx = \|f^0\|_{L^1(\Omega\times\mathbb{R}^2)} =: \|f^0\|_1. \tag{2.9}$$

We now exploit (2.8) to estimate the x-component E_1. By $\partial_x E_1 = \rho$ and condition (1.4), we get $E_1(t, x) = \int_0^x \rho(t, y)\, dy + C(t)$ with $C(0) = \lambda = E_1(0, 0)$. Using $\partial_t E_1 = -j_1$ and (2.8), we must have

$$C'(t) = -j_1(t, x) + \int_0^x \partial_x j_1(t, y)\, dy = -j_1(t, 0) = 0.$$

Therefore $C(t) \equiv \lambda$ and hence

$$E_1(t, x) = \int_0^x \rho(t, y)\, dy + \lambda = \int_0^x \int_{\mathbb{R}^2} f(t, y, v)\, dv dy + \lambda. \tag{2.10}$$

We conclude from (2.10) and (2.9) that

$$\|E_1\|_{L^\infty([0,T]\times\Omega)} \le \|f^0\|_1 + \lambda. \tag{2.11}$$

2.2. **Estimate of E_2 and B.** Let $t \in (0, T]$ and $x \in \Omega$ be fixed. Without loss of generality, by symmetry we can assume $x \le 1/2$ in the following calculations. In order to estimate E_2 and B at the point (t, x), our first step is to express these quantities in terms of the initial and boundary data, and the current density j_2. For this purpose, note that the Maxwell equations (1.2) yield

$$\partial_t(E_2 + B) + \partial_x(E_2 + B) = -\partial_x B - j_2 - \partial_x E_2 + \partial_x(E_2 + B) = -j_2 \tag{2.12}$$

and

$$\partial_t(E_2 - B) - \partial_x(E_2 - B) = -\partial_x B - j_2 + \partial_x E_2 - \partial_x(E_2 - B) = -j_2. \tag{2.13}$$

We now consider the following three possibilities, which depend on the relation between x and t.

Case 1: $0 < t \leq x$. Then $0 \leq x - t$ and $x + t \leq 2x \leq 1$. Therefore, it follows from (2.12) and (2.13) that

$$(E_2 + B)(t, x) = (E_2 + B)(0, x - t) - \int_0^t j_2(\tau, x - t + \tau)\, d\tau, \tag{2.14}$$

$$(E_2 - B)(t, x) = (E_2 - B)(0, x + t) - \int_0^t j_2(\tau, x + t - \tau)\, d\tau. \tag{2.15}$$

Adding and subtracting the two quantities respectively yield

$$\begin{aligned} E_2(t, x) = &\frac{1}{2}\Big[E_2^0(x - t) + E_2^0(x + t) + B^0(x - t) - B^0(x + t)\Big] \\ &- \frac{1}{2}\int_0^t \Big[j_2(\tau, x - t + \tau) + j_2(\tau, x + t - \tau)\Big]\, d\tau \end{aligned}$$

and

$$\begin{aligned} B(t, x) = &\frac{1}{2}\Big[E_2^0(x - t) - E_2^0(x + t) + B^0(x - t) + B^0(x + t)\Big] \\ &- \frac{1}{2}\int_0^t \Big[j_2(\tau, x - t + \tau) - j_2(\tau, x + t - \tau)\Big]\, d\tau. \end{aligned}$$

Case 2: $x < t \leq 1 - x$. Then $x - t < 0$ and $x + t \leq 1$. In this case (2.15) is still true, but (2.14) is replaced by

$$(E_2 + B)(t, x) = (E_2 + B)(t - x, 0) - \int_{t-x}^t j_2(\tau, x - t + \tau)\, d\tau. \tag{2.16}$$

Therefore, as in **Case 1** we obtain from (2.16) and (2.15) that

$$\begin{aligned} E_2(t, x) = &\frac{1}{2}\Big[E_2^b(t - x, 0) + E_2^0(x + t) + B^b(t - x, 0) - B^0(x + t)\Big] \\ &- \frac{1}{2}\Big[\int_{t-x}^t j_2(\tau, x - t + \tau)\, d\tau + \int_0^t j_2(\tau, x + t - \tau)\, d\tau\Big] \end{aligned}$$

and

$$\begin{aligned} B(t, x) = &\frac{1}{2}\Big[E_2^b(t - x, 0) - E_2^0(x + t) + B^b(t - x, 0) + B^0(x + t)\Big] \\ &- \frac{1}{2}\Big[\int_{t-x}^t j_2(\tau, x - t + \tau)\, d\tau - \int_0^t j_2(\tau, x + t - \tau)\, d\tau\Big]. \end{aligned}$$

Case 3: $t > 1 - x$. Then $x - t < 0$ and $x + t > 1$. Hence, we have (2.16) and

$$(E_2 - B)(t, x) = (E_2 - B)(t - 1 + x, 1) - \int_{t-1+x}^t j_2(\tau, x + t - \tau)\, d\tau.$$

Consequently,

$$E_2(t,x) = \frac{1}{2}\Big[E_2^b(t-x,0) + E_2^b(t-1+x,1) + B^b(t-x,0) - B^b(t-1+x,1)\Big] - \frac{1}{2}\Big[\int_{t-x}^t j_2(\tau, x-t+\tau)\,d\tau + \int_{t-1+x}^t j_2(\tau, x+t-\tau)\,d\tau\Big]$$

and

$$B(t,x) = \frac{1}{2}\Big[E_2^b(t-x,0) - E_2^b(t-1+x,1) + B^b(t-x,0) + B^b(t-1+x,1)\Big] - \frac{1}{2}\Big[\int_{t-x}^t j_2(\tau, x-t+\tau)\,d\tau - \int_{t-1+x}^t j_2(\tau, x+t-\tau)\,d\tau\Big].$$

We summarize all three cases as follows.

Lemma 2.1. *For any* $t \in (0,T]$ *and* $0 < x \le 1/2$*, we have*

$$E_2(t,x) = \frac{1}{2}[A^+(x-t) + A^-(x+t)] - \frac{1}{2}\Big[\int_{t^+(x)}^t j_2(\tau, x-t+\tau)\,d\tau + \int_{t^-(x)}^t j_2(\tau, x+t-\tau)\,d\tau\Big]$$

and

$$B(t,x) = \frac{1}{2}[A^+(x-t) - A^-(x+t)] - \frac{1}{2}\Big[\int_{t^+(x)}^t j_2(\tau, x-t+\tau)\,d\tau - \int_{t^-(x)}^t j_2(\tau, x+t-\tau)\,d\tau\Big].$$

Here $A^\pm$ *are given explicitly in terms of the initial and boundary data, and*

$$t^+(x) := \begin{cases} 0 & \text{if} \quad t \le x, \\ t-x & \text{if} \quad t > x \end{cases} \qquad \text{and} \qquad t^-(x) := \begin{cases} 0 & \text{if} \quad t \le 1-x, \\ t-1+x & \text{if} \quad t > 1-x. \end{cases}$$

Note that $0 \le t^-(x) \le t^+(x) < t$ because $0 \le x \le \frac{1}{2}$. In Case 1, $t_-(x) = t^+(x) = 0$. In Case 2, $t^-(x) = 0$ but $t^+(x) \neq 0$. In Case 3, neither one is zero. In order to bound E_2 and B, the remaining step is to bound the time integrals of j_2. This is accomplished thanks to the following variation of the cone estimate in [GSc, Lemma 1].

Lemma 2.2 (Key cone estimate)**.** *Let* $t \in (0,T]$ *and* $x \in (0,1/2]$*. Then we have*

$$\int_{t^+(x)}^t \int_{\mathbb{R}^2} |\hat{v}_2| f(\tau, x-t+\tau, v)\,dv d\tau + \int_{t^-(x)}^t \int_{\mathbb{R}^2} |\hat{v}_2| f(\tau, x+t-\tau, v)\,dv d\tau \le \Big[\int_\Omega e(t^-(x), y)\,dy + \int_{t^-(x)}^{t^+(x)} E_2^b(\tau,0) B^b(\tau,0)\,d\tau\Big],$$

where

$$e(\tau, y) := \frac{1}{2}\Big[|E(\tau,y)|^2 + B(\tau,y)^2\Big] + \int_{\mathbb{R}^2} \sqrt{1+|v|^2} f(\tau,y,v)\,dv. \tag{2.17}$$

Proof. Let

$$m(\tau,y) := -\int_{\mathbb{R}^2} v_1 f(\tau,y,v)\,dv - E_2(\tau,y)B(\tau,y).$$

Then by a direct calculation using (1.2) and the definition of j, we obtain

$$\partial_t e - \partial_x m = \Big[\int_{\mathbb{R}^2} \sqrt{1+|v|^2}\,\partial_t f(\tau,y,v)\,dv + \int_{\mathbb{R}^2} v_1\partial_x f(\tau,y,v)\,dv - \int_{\mathbb{R}^2} (\hat{v}_1E_1 + \hat{v}_2E_2) f(\tau,y,v)\,dv\Big].$$

Thus, it follows from the Vlasov equation (2.7) and an integration by part in v that

$$\partial_t e - \partial_x m = \int_{\mathbb{R}^2} \Big[(\nabla_v \sqrt{1+|v|^2}\,)\cdot K - \hat{v}\cdot E\Big] f(\tau,y,v)\,dv.$$

Since $\nabla_v \sqrt{1+|v|^2} = \hat{v}$ and $\hat{v}\cdot K = \hat{v}\cdot E$, we deduce that

$$\partial_t e - \partial_x m = 0 \quad \text{in} \quad [0,T]\times\overline{\Omega}. \tag{2.18}$$

Let us now consider the polygonal region $\Delta := \Delta_1 \cup \Delta_2$, where

$$\Delta_1 := \Big\{(\tau,y) : t^+(x) \le \tau \le t \ \text{ and } \ |y-x| \le t-\tau\Big\}$$

is a triangular region and

$$\Delta_2 := \Big\{(\tau,y) : t^-(x) \le \tau \le t^+(x) \ \text{ and } \ 0 \le y \le x+t-\tau\Big\}$$

is a trapezoidal region. We integrate the energy identity (2.18) over Δ and apply Green's theorem to get

$$0 = \oint_{\partial\Delta} \Big(m\,dt + e\,dx\Big) = \int_{t^-(x)}^{t} (m-e)(\tau, x+t-\tau)\,d\tau + \int_{t}^{t^+(x)} (m+e)(\tau, x-t+\tau)\,d\tau + \int_{t^+(x)}^{t^-(x)} m(\tau,0)\,d\tau + \int_0^1 e(t^-(x),y)\,dy.$$

The first two terms on the right are line integrals on characteristic edges, the third one is an integral on the left edge where $x = 0$, and the last one is an integral on the bottom edge of Δ. Moreover, $m(\tau,0) = -E_2^b(\tau,0)B^b(\tau,0)$ due to the boundary conditions for f and the field. It follows by moving some terms around that

$$\int_{t^+(x)}^{t} (e+m)(\tau, x-t+\tau)\,d\tau + \int_{t^-(x)}^{t} (e-m)(\tau, x+t-\tau)\,d\tau = \int_\Omega e(t^-(x),y)\,dy + \int_{t^-(x)}^{t^+(x)} E_2^b(\tau,0)B^b(\tau,0)\,d\tau. \tag{2.19}$$

Notice that

$$e \pm m = \frac{E_1^2}{2} + \frac{(E_2 \mp B)^2}{2} + \int_{\mathbb{R}^2} (\sqrt{1+|v|^2} \mp v_1) f\, dv \geq \int_{\mathbb{R}^2} \frac{|v_2|}{\sqrt{1+|v|^2}} f\, dv.$$

Therefore, we infer from (2.19) that

$$\int_{t^+(x)}^t \int_{\mathbb{R}^2} |\hat{v}_2| f(\tau, x - t + \tau, v)\, dv d\tau + \int_{t^-(x)}^t \int_{\mathbb{R}^2} |\hat{v}_2| f(\tau, x + t - \tau, v)\, dv d\tau$$
$$\leq \Big[\int_\Omega e(t^-(x), y)\, dy + \int_{t^-(x)}^{t^+(x)} E_2^b(\tau, 0) B^b(\tau, 0)\, d\tau \Big].$$

□

The next lemma states the conservation of energy.

Lemma 2.3. *Let* $e(\tau, y)$ *be given by* (2.17). *Then*

$$\int_\Omega e(t, y)\, dy = \int_\Omega e(0, y)\, dy + \int_0^t \Big[(E_2^b B^b)(\tau, 0) - (E_2^b B^b)(\tau, 1) \Big] d\tau \quad \text{for all} \quad t \in [0, T].$$

Proof. By the identity (2.18) and the boundary condition (1.5), we have

$$\begin{aligned} \partial_t \int_\Omega e(\tau, y)\, dy &= \int_\Omega \partial_t e(\tau, y)\, dy = \int_\Omega \partial_x m(\tau, y)\, dy = m(\tau, 1) - m(\tau, 0) \\ &= \int_{\mathbb{R}^2} v_1 \Big[f(\tau, 0, v) - f(\tau, 1, v) \Big] dv + (E_2 B)(\tau, 0) - (E_2 B)(\tau, 1) \\ &= (E_2^b B^b)(\tau, 0) - (E_2^b B^b)(\tau, 1). \end{aligned}$$

The lemma follows by integration. □

We now combine the preceding results.

Corollary 2.4. *The field is bounded as follows:* $\|E_1\|_{L^\infty([0,T]\times\Omega)} \leq \|f^0\|_1 + \lambda$, *and*

$$\|E_2\|_{L^\infty([0,T]\times\Omega)}, \ \|B\|_{L^\infty([0,T]\times\Omega)} \leq C_1, \tag{2.20}$$

where $C_1 := \|E_2^0\|_{L^\infty(\Omega)} + \|E_2^b\|_{L^\infty([0,T]\times\partial\Omega)} + \|B^0\|_{L^\infty(\Omega)} + \|B^b\|_{L^\infty([0,T]\times\partial\Omega)} + \frac{1}{4}\big[(\|f^0\|_1 + \lambda)^2 + \|E_2^0\|^2_{L^\infty(\Omega)} + \|B^0\|^2_{L^\infty(\Omega)} + 4T\|E_2^b B^b\|_{L^\infty([0,T]\times\partial\Omega)}\big] + \frac{1}{2}\|\sqrt{1+|v|^2} f^0\|_1$.

Proof. The estimate for E_1 is from (2.11) and we only need to prove (2.20). Let $t \in (0, T]$ and $x \in \Omega$. By symmetry we can assume $x \leq 1/2$ as the case $x > 1/2$ is similar. By Lemma 2.1 and the explicit formulas for $A^\pm$ given in the three cases considered above, we have

$$\begin{aligned} |E_2(t, x)|, \ |B_2(t, x)| \leq{} & \|E_2^0\|_{L^\infty(\Omega)} + \|E_2^b\|_{L^\infty([0,T]\times\partial\Omega)} + \|B^0\|_{L^\infty(\Omega)} + \|B^b\|_{L^\infty([0,T]\times\partial\Omega)} \\ & + \frac{1}{2}\Big[\int_{t^+(x)}^t |j_2|(\tau, x - t + \tau)\, d\tau + \int_{t^-(x)}^t |j_2|(\tau, x + t - \tau)\, d\tau \Big]. \end{aligned}$$

But it follows from Lemmas 2.2 and 2.3 that

$$\int_{t^+(x)}^t |j_2|(\tau, x-t+\tau)\,d\tau + \int_{t^-(x)}^t |j_2|(\tau, x+t-\tau)\,d\tau$$
$$\leq \Big[\int_\Omega e(0,y)\,dy + \int_0^{t^+(x)} E_2^b(\tau,0)B^b(\tau,0)\,d\tau - \int_0^{t^-(x)} E_2^b(\tau,1)B^b(\tau,1)\,d\tau \Big]$$
$$\leq \frac{1}{2}\Big[(\|f^0\|_1+\lambda)^2 + \|E_2^0\|^2_{L^\infty(\Omega)} + \|B^0\|^2_{L^\infty(\Omega)}\Big] + \|\sqrt{1+|v|^2}f^0\|_{L^1(\Omega\times\mathbb{R}^2)} + 2T\|E_2^bB^b\|_{L^\infty([0,T]\times\partial\Omega)}.$$

Therefore we obtain the desired estimate (2.20). □

3. Confinement of the particles

Given $(t,x,v) \in (0,T]\times\Omega\times\mathbb{R}^2$. The characteristics of (1.1) corresponding to the point (t,x,v) are the solutions $s \mapsto (X(s),V(s)) = (X(s;t,x,v),V(s;t,x,v))$ to the system

$$\begin{cases} \frac{dX}{ds} = \hat{V}_1(s), \\ \frac{dV_1}{ds} = E_1(s,X) + \hat{V}_2(s)\,\check{B}(s,X), \\ \frac{dV_2}{ds} = E_2(s,X) - \hat{V}_1(s)\,\check{B}(s,X), \\ X(t;t,x,v) = x, \quad V(t;t,x,v) = v. \end{cases} \tag{3.21}$$

Assuming that $E_1, E_2, \check{B} \in C^1([0,T]\times\Omega)$, there exists a unique C^1 solution (X,V) to the system (3.21) in some time interval. It can be uniquely extended to the whole time interval $[0,T]$ as long as the solution $X(s)$ does not reach the boundary $\partial\Omega$. In the next lemma, we show that this is indeed the case thanks to condition (1.6) for the potential of the external magnetic field.

Lemma 3.1 (Confinement property)**.** *Assume that $E,B \in C^1([0,T]\times\Omega)$ satisfy $\partial_t B = -\partial_x E_2$, and that there exist constants C_0, $C_0' > 0$ such that*

$$|E(s,y)| \leq C_0 \quad \textit{and} \quad |B(s,y)| \leq C_0' \quad \textit{for all } (s,y)\in[0,T]\times\Omega. \tag{3.22}$$

Let $(t,x,v)\in(0,T]\times\Omega\times\mathbb{R}^2$ and $(X(s),V(s))$ be a C^1 solution to (3.21) *in the time interval $[t-\alpha,t+\alpha]$ for some $\alpha>0$. Then*

$$dist(X(s),\partial\Omega)^\gamma \geq \frac{c_0}{c_0^{-1}+C_0'+3C_0\alpha+2|v|+|\psi_{ext}(x)|} \quad \forall s\in[t-\alpha,t+\alpha]. \tag{3.23}$$

Proof. By assumption, $X(s)\in\Omega$ for every $s\in(t-\alpha,t+\alpha)$. Since

$$\frac{d}{ds}|V|^2 = 2\Big[V_1\dot{V}_1 + V_2\dot{V}_2\Big] = 2\Big[V_1E_1(s,X) + V_1\hat{V}_2\check{B}(s,X) + V_2E_2(s,X) - V_2\hat{V}_1\check{B}(s,X)\Big]$$
$$= 2V\cdot E(s,X),$$

we deduce from the bound in (3.22) that

$$|V(s)|^2 \le |v|^2 + 2C_0 \int_s^t |V(\tau)|\,|d\tau| \quad \text{for} \quad s \in [t-\alpha, t+\alpha].$$

Hence $u(s) := \sup_{s\le\tau\le t} |V(\tau)|$ satisfies

$$u(s)^2 \le |v|^2 + 2C_0\alpha\, u(s).$$

It follows that $|V(s)| \le u(s) \le |v| + 2C_0\alpha$ and so

$$|V(s)| \le |v| + 2C_0\alpha \quad \forall s \in [t-\alpha, t+\alpha]. \tag{3.24}$$

To estimate $X(s)$, let $\psi(\tau, y) := \int_{\frac{1}{2}}^y B(\tau, z)\,dz$. Then thanks to $\partial_t B = -\partial_x E_2$, we get

$$\partial_t\psi(\tau, y) = -\int_{\frac{1}{2}}^y \partial_x E_2(\tau, z)\,dz = E_2\left(\tau, \frac{1}{2}\right) - E_2(\tau, y). \tag{3.25}$$

Next define

$$p(\tau, y, w) := w_2 + \psi(\tau, y) + \psi_{\text{ext}}(y)$$

where $w = (w_1, w_2) \in \mathbb{R}^2$. Differentiating $p(\tau, y, w)$ along the characteristics and using (3.21) and (3.25), we obtain

$$\begin{aligned}
\frac{d}{ds} p(s, X(s), V(s)) &= \dot V_2 + \partial_t\psi(s, X) + \dot X\,\partial_x\psi(s, X) + \dot X\,\partial_x\psi_{\text{ext}}(X)\\
&= E_2(s, X) - \hat V_1[B(s, X) + B_{\text{ext}}(X)] + \partial_t\psi(s, X) + \hat V_1 B(s, X) + \hat V_1 B_{\text{ext}}(X)\\
&= E_2\left(s, \frac{1}{2}\right).
\end{aligned}$$

Therefore

$$V_2(s) + \psi(s, X(s)) + \psi_{\text{ext}}(X(s)) = v_2 + \psi(t, x) + \psi_{\text{ext}}(x) - \int_s^t E_2\left(\tau, \frac{1}{2}\right) d\tau \tag{3.26}$$

for every $s \in [t-\alpha, t+\alpha]$.

We now show using (1.6) that the path $\tau \in [t-\alpha, t+\alpha] \mapsto X(\tau)$ stays away from $\partial\Omega$ by a specific distance depending on x and v. For this purpose, let $\tau_0 \in (t-\alpha, t+\alpha)$ be arbitrary. Two of the terms in (3.26) are bounded as

$$|\psi(t, x) - \psi(x, X(s))| \le \int_{X(s)}^x |B(s, z)| dz \le C_0'\alpha.$$

By (3.24), $|V(\tau_0)| \le |v| + 2C_0\alpha$. We deduce from (3.26) that

$$\left|\psi_{\text{ext}}(X(\tau_0)) - \psi_{\text{ext}}(x)\right| \le 2|v| + C_0' + 3C_0\alpha.$$

This together with the assumption in (1.6) implies that

$$\text{dist}(X(\tau_0), \partial\Omega)^\gamma \ge \frac{c_0}{c_0^{-1} + C_0' + 3C_0\alpha + 2|v| + |\psi_{\text{ext}}(x)|}.$$

□

Remark 3.2. *The condition* $\partial_t B = -\partial_x E_2$ *is not necessary for the validity of Lemma 3.1. Indeed, an inspection of the above proof reveals that it is enough to assume the quantity* $\partial_t B + \partial_x E_2$ *to be bounded.*

Lemma 3.1 shows that the particles never reach $\partial\Omega$ in a finite time. As a consequence, we obtain the following corollary.

Corollary 3.3. *Let E and B be as in Lemma 3.1. Then for any* $(t, x, v) \in (0, T] \times \Omega \times \mathbb{R}^2$, *the characteristic system* (3.21) *admits a unique* C^1 *solution* $(X(s), V(s))$ *in* $[0, T]$ *with* $X(s) \in \Omega$ *for every* $s \in [0, T]$.

We end this section by giving some direct consequences of Corollary 2.4 and Corollary 3.3 which will be needed in what follows. We still suppose (f, E_1, E_2, B) is a C^1 solution as in Section 2. Then thanks to Corollary 2.4, the conclusion about the characteristics in Corollary 3.3 is true. Since the solution f to (2.7) is constant along such characteristics, we have

$$f(t, x, v) = f^0(X(0; t, x, v), V(0; t, x, v)). \tag{3.27}$$

It follows that

$$\|f\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^2)} = \|f^0\|_{L^\infty(\Omega\times\mathbb{R}^2)} =: \|f^0\|_\infty. \tag{3.28}$$

The next result shows that $f(t, \cdot, \cdot)$ has compact support in both x and v variables.

Lemma 3.4. *Define*

$$P(t) := \sup\Big\{|v| : \; f(t, x, v) \neq 0 \quad \textit{for some} \quad x \in \Omega\Big\},$$
$$\Sigma(t) := \Big\{x \in \Omega : \; f(t, x, v) \neq 0 \quad \textit{for some} \quad v \in \mathbb{R}^2\Big\}.$$

If $spt(f^0) \subset [\epsilon_0, 1 - \epsilon_0] \times [-k_0, k_0]$ *for some* ϵ_0, $k_0 > 0$, *then we have*

$$P(t) \leq k_0 + C_2 t, \tag{3.29}$$

$$dist(\Sigma(t), \partial\Omega)^\gamma \geq \frac{c_0}{c_0^{-1} + 2(k_0 + C_1) + 3C_2 t + \|\psi_{ext}\|_{L^\infty([\epsilon_0, 1-\epsilon_0])}}, \tag{3.30}$$

for $t \in [0, T]$, *where* C_1 *is given by Corollary 2.4 and* $C_2 := 4(\|f^0\|_1 + \lambda + C_1)$.

Proof. Let $(t, x, v) \in (0, T] \times \Omega \times \mathbb{R}^2$ and consider the corresponding characteristic curve $(X(s), V(s))$ given by (3.21). Since $\|E\|_{L^\infty([0,T]\times\Omega)} \leq C_0 := \|f^0\|_1 + \lambda + C_1$ by Corollary 2.4, we have as in the proof of Lemma 3.1 that

$$|V(s)| \leq |v| + 2C_0 t \quad \forall s \in [0, t]. \tag{3.31}$$

Using the fact that $\frac{d}{ds}|V|^2 = 2V \cdot E(s, X)$, we obtain

$$|v|^2 = |V(0)|^2 + 2\int_0^t V(s) \cdot E(s, X)\,ds \le |V(0)|^2 + 2C_0 t\,(|v| + 2C_0 t).$$

It follows that $|v| \le |V(0)| + 4C_0 t$. Together with (3.27) and the support assumption on f^0, this yields

$$P(t) \le k_0 + 4C_0 t.$$

It remains to show (3.30). Fix $t \in (0, T]$ and let $x \in \Sigma(t)$. Then there exists $v \in \mathbb{R}^2$ such that $f(t, x, v) \neq 0$. By (3.29), we have $|v| \le k_0 + C_2 t$. Also, it follows from the formula (3.27) for $f(t, x, v)$ and the assumption on f^0 that the corresponding characteristics $(X(s), V(s))$ must satisfy $X(0) \in [\epsilon_0, 1 - \epsilon_0]$. Moreover, the identity (3.26) is valid for all $s \in [0, t]$, which yields in particular

$$V_2(0) + \psi(0, X(0)) + \psi_{\text{ext}}(X(0)) = v_2 + \psi(t, x) + \psi_{\text{ext}}(x) - \int_0^t E_2(\tau, \frac{1}{2})\,d\tau.$$

Using Corollary 2.4 and (3.31), we deduce that

$$\begin{aligned}|\psi_{\text{ext}}(x)| &\le 2(k_0 + C_1) + (2C_2 + 2C_0 + C_1)t + \|\psi_{\text{ext}}\|_{L^\infty([\epsilon_0, 1-\epsilon_0])}\\ &\le 2(k_0 + C_1) + 3C_2 t + \|\psi_{\text{ext}}\|_{L^\infty([\epsilon_0, 1-\epsilon_0])} =: C.\end{aligned}$$

We infer from this and (1.6) that $\text{dist}(x, \partial\Omega)^\gamma \ge c_0/(c_0^{-1} + C)$ and (3.30) follows. □

By Lemma 3.4 and (3.28), we have

$$\int_{\mathbb{R}^2} f(t, x, v)\,dv = \int_{|v| \le k_0 + C_2 t} f(t, x, v)\,dv \le \|f^0\|_\infty (k_0 + C_2 t)^2.$$

This immediately leads to:

Corollary 3.5. *We have*

$$\|\rho\|_{L^\infty([0,T]\times\Omega)}, \|j\|_{L^\infty([0,T]\times\Omega)} \le \|f^0\|_\infty (k_0 + C_2 T)^2.$$

4. Bounds on derivatives of the fields

In this section we first derive L^∞ estimates for derivatives of the fields and then use them to obtain similar estimates for derivatives of the distribution function f.

Let $k^\pm(t, x) := (E_2 \pm B)(t, x)$. By the arguments leading to Lemma 2.1, we have for every $t \in (0, T]$ and $0 < x \le 1/2$ that

$$\begin{aligned}k^+(t, x) &= \frac{1}{2}A^+(x - t) - \int_{t^+(x)}^t j_2(\tau, x - t + \tau)\,d\tau,\\ k^-(t, x) &= \frac{1}{2}A^-(x - t) - \int_{t^-(x)}^t j_2(\tau, x + t - \tau)\,d\tau,\end{aligned} \tag{4.32}$$

in which $A^{\pm}$ are expressed in terms of the initial and boundary data; see Lemma 2.1. These representation formulas play an important role in the proof of the next result. Before stating it, let θ_0 and θ_1 denote the small constants given by

$$\theta_0^{\gamma} := \frac{c_0}{c_0^{-1} + 2(k_0 + C_1) + 3C_2T + \|\psi_{\text{ext}}\|_{L^\infty([\epsilon_0, 1-\epsilon_0])}}, \tag{4.33}$$

$$\theta_1^{\gamma} := \frac{c_0}{c_0^{-1} + 2(k_0 + C_1) + 3C_2T + \|\psi_{\text{ext}}\|_{L^\infty([\theta_0, 1-\theta_0])}}. \tag{4.34}$$

Notice that the choice of θ_0 ensures that the x-support of $f(t)$ is contained in $[\theta_0, 1-\theta_0]$ for every $t \in [0, T]$ (see Lemma 3.4). On the other hand, Corollary 2.4 and Lemma 3.1 imply that the characteristics $(X(s), V(s))$ corresponding to any point $(t, x, v) \in (0, T] \times [\theta_0, 1-\theta_0] \times \bar{B}_{k_0+C_2T}$ satisfy: $X(s) \in [\theta_1, 1-\theta_1]$ for all $s \in [0, t]$.

Lemma 4.1. *There exists a constant $C_T > 0$ depending only on k_0, T, λ, $\|f^0\|_\infty$, $\|B_{ext}\|_{L^\infty([\theta_0,1-\theta_0])}$, the C^1 norms of E_2^0, B^0 on Ω, and the C^1 norms of $E_2^b(\cdot, x)$, $B^b(\cdot, x)$ on $[0, T]$ ($x = 0,\ 1$) such that*

$$\|\partial_x k^{\pm}\|_{L^\infty([0,T]\times\Omega)} \le C_T.$$

Consequently, we have $\|\partial_x E_2\|_{L^\infty([0,T]\times\Omega)}$, $\|\partial_x B\|_{L^\infty([0,T]\times\Omega)} \le C_T$.

Proof. We employ an argument similar to the proof of [GSc, Lemma 3]. For simplicity, we derive the L^∞ estimates in the region $[0, T] \times (0, 1/2]$ as the case $x > 1/2$ is similar. For (t, x) in such region, it follows from (4.32) by differentiating k^+ in x that

$$\begin{aligned} \partial_x k^+(t, x) &= M(t, x) - \int_{t^+(x)}^{t} \partial_x j_2(\tau, x - t + \tau)\, d\tau \\ &= M(t, x) - \int_{t^+(x)}^{t} \int_{\mathbb{R}^2} \hat{v}_2\, \partial_x f(\tau, x - t + \tau, v)\, dv d\tau \end{aligned} \tag{4.35}$$

with $M(t, x) := \frac{1}{2}(A^+)'(x - t) + j_2\big(t^+(x), x - t + t^+(x)\big)\,(t^+)'(x)$. Notice that by using the explicit formula for A^+, Corollary 3.5 and the fact $|(t^+)'(x)| \le 1$, we obtain

$$\|M\|_{L^\infty([0,T]\times\Omega)} \le C, \tag{4.36}$$

where C depends only on k_0, T, λ, $\|f^0\|_\infty$, the L^∞ norms of the derivatives of E_2^0, B^0 on Ω, and the L^∞ norms of the derivatives of $E_2^b(\cdot, x)$, $B^b(\cdot, x)$ on $[0, T]$ ($x = 0,\ 1$).

We next use the splitting method of Glassey and Strauss in [GStr] and [GSc] to express the operator ∂_x in terms of the two differential operators

$$T_+ = \partial_t + \partial_x \quad \text{and} \quad S = \partial_t + \hat{v}_1 \partial_x.$$

Obviously

$$\partial_x = \frac{T_+ - S}{1 - \hat{v}_1}, \tag{4.37}$$

so that (4.35) can be written as

$$\begin{aligned}\partial_x k^+(t,x) &= M(t,x) - \int_{t^+(x)}^t \int_{\mathbb{R}^2} \frac{\hat{v}_2}{1-\hat{v}_1}\big[(T_+ f)(\tau, x-t+\tau, v) - (Sf)(\tau, x-t+\tau, v)\big]\,dvd\tau \\ &= M(t,x) - \int_{t^+(x)}^t \frac{d}{d\tau}\int_{\mathbb{R}^2} \frac{\hat{v}_2}{1-\hat{v}_1} f(\tau, x-t+\tau, v)\,dv\;d\tau \\ &\quad - \int_{t^+(x)}^t \int_{\mathbb{R}^2} \frac{\hat{v}_2}{1-\hat{v}_1}\nabla_v\cdot(Kf)(\tau, x-t+\tau, v)\,dvd\tau,\end{aligned}$$

where we have used the Vlasov equation $Sf + \nabla_v\cdot(Kf) = 0$. Since f has compact support in v by Lemma 3.4, we easily integrate the last term by parts to arrive at the equation

$$\begin{aligned}\partial_x k^+(t,x) = M(t,x) &- \int_{\mathbb{R}^2} \frac{\hat{v}_2}{1-\hat{v}_1} f(t,x,v)\,dv + \int_{\mathbb{R}^2} \frac{\hat{v}_2}{1-\hat{v}_1} f(t^+(x), x-t+t^+(x), v)\,dv \\ &+ \int_{t^+(x)}^t \int_{\mathbb{R}^2} \nabla_v\Big(\frac{\hat{v}_2}{1-\hat{v}_1}\Big)\cdot(Kf)(\tau, x-t+\tau, v)\,dvd\tau.\end{aligned}$$

We know the support of f in v is contained in the ball $\overline{B_R}$, where $R := k_0 + C_2 T$ with C_2 being given in Lemma 3.4. Using this together with (4.36) and (3.28), we deduce that

$$\begin{aligned}\|\partial_x k^+\|_{L^\infty([0,T]\times(0,\frac{1}{2}])} &\le C + 2\pi R^2 \|f^0\|_\infty \Big\|\frac{\hat{v}_2}{1-\hat{v}_1}\Big\|_{L^\infty(B_R)} \\ &\quad + \Big\|\nabla_v\Big(\frac{\hat{v}_2}{1-\hat{v}_1}\Big)\Big\|_{L^\infty(B_R)} \int_{y-t+t^+(x)}^x \int_{B_R} (|K|f)(y-x+t, y, v)\,dvdy.\end{aligned} \tag{4.38}$$

But it follows from (3.30) and the definition of θ_0 in (4.33) that

$$\int_{y-t+t^+(x)}^x \int_{B_R} (|K|f)(y-x+t, y, v)\,dvdy \le \int_{\theta_0}^{1-\theta_0} \int_{B_R} (|K|f)(y-x+t, y, v)\,dvdy$$

Also, Corollary 2.4 yields $\|K\|_{L^\infty([0,T]\times[\theta_0,1-\theta_0])} \le C' := C_2 + \|B_{\text{ext}}\|_{L^\infty([\theta_0,1-\theta_0])}$. Thus we obtain from (4.38) that

$$\|\partial_x k^+\|_{L^\infty([0,T]\times(0,\frac{1}{2}])} \le C + \pi R^2 \|f^0\|_\infty \Big\{2\Big\|\frac{\hat{v}_2}{1-\hat{v}_1}\Big\|_{L^\infty(B_R)} + C' \Big\|\nabla_v\Big(\frac{\hat{v}_2}{1-\hat{v}_1}\Big)\Big\|_{L^\infty(B_R)}\Big\} \le C_T$$

for some constants C, C_T. By a similar argument for the case $t \in [0,T]$ and $x \in (1/2, 1)$, we infer further that $\|\partial_x k^+\|_{L^\infty([0,T]\times\Omega)} \le C_T$. The bound for $\partial_x k^-$ is obtained in the same manner. The only change is in place of (4.37) we now express $\partial_x = \frac{S-T_-}{1+\hat{v}_1}$ with $T_- = \partial_t - \partial_x$. The differential operator T_- is employed to ensure that

$$\frac{d}{d\tau} f(\tau, x+t-\tau, v) = (T_- f)(\tau, x+t-\tau, v).$$

□

We next exploit the Vlasov and Maxwell equations to derive estimates for all the first derivatives of E, B and f.

Lemma 4.2. *Assume in addition that* $f \in C^2([0,T]\times\Omega\times\mathbb{R}^2)$*. There exists a constant* $C_T > 0$ *depending only on* k_0, T, λ, $\|B_{ext}\|_{C^1([\theta_1,1-\theta_1])}$*, the* C^1 *norms of* f^0, E_2^0, B^0*, and the* C^1 *norms of* $E_2^b(\cdot,x)$, $B^b(\cdot,x)$ *on* $[0,T]$ *(*$x = 0,\ 1$*) such that*

$$\|f\|_{C^1([0,T]\times\overline{\Omega}\times\mathbb{R}^2)} + \|E\|_{C^1([0,T]\times\overline{\Omega})} + \|B\|_{C^1([0,T]\times\overline{\Omega})} \le C_T.$$

Proof. We begin with the fields E and B. Since $\partial_t E_1 = -j_1$ and $\partial_x E_1 = \rho$, we get from Corollary 3.5 that $\|\nabla E_1\|_{L^\infty([0,T]\times\overline{\Omega})} \le 2\|f^0\|_\infty (k_0 + C_2 T)^2$. Using $\partial_t E_2 = -\partial_x B - j_2$, Lemma 4.1 and Corollary 3.5, we also get an L^∞ bound for ∇E_2. These together with Corollary 2.4 give $\|E\|_{C^1([0,T]\times\overline{\Omega})} \le C_T$. On the other hand, the C^1 estimate for B is a consequence of the fact $\partial_t B = -\partial_x E_2$, Lemma 4.1 and Corollary 3.5.

Next we estimate the derivatives of f. By differentiating the Vlasov equation (2.7) with respect to x and v respectively, one has

$$(\partial_t + \hat{v}_1\partial_x + K\cdot\nabla_v)(\partial_x f) = -\partial_x K\cdot\nabla_v f,$$
$$(\partial_t + \hat{v}_1\partial_x + K\cdot\nabla_v)(\nabla_v f) = -(\nabla_v\hat{v}_1)\partial_x f - (\nabla_v\cdot K)\nabla_v f.$$

Let $R := k_0 + C_2 T$. Integrating the two equations along the characteristics and using the remark just before Lemma 4.1, we obtain

$$\|\partial_x f(t)\|_{L^\infty([\theta_0,1-\theta_0]\times\bar{B}_R)} \le \|\partial_x f^0\|_\infty + \int_0^t \|\partial_x K\|_{L^\infty([0,T]\times[\theta_1,1-\theta_1]\times\mathbb{R}^2)}\|\nabla_v f(s)\|_\infty\, ds$$

and

$$\|\nabla_v f(t)\|_{L^\infty([\theta_0,1-\theta_0]\times\bar{B}_R)}$$
$$\le \|\nabla_v f^0\|_\infty + \int_0^t \Big[\|\nabla_v\hat{v}_1\|_\infty\|\partial_x f(s)\|_\infty + \|\nabla_v\cdot K\|_{L^\infty([0,T]\times[\theta_1,1-\theta_1]\times\mathbb{R}^2)}\|\nabla_v f(s)\|_\infty\Big]\, ds.$$

Observe that $\|\nabla_v\hat{v}_1\|_\infty \le 2$. Moreover, the C^1 bounds for E, B and the assumption for $B_{\rm ext}$ imply that $\|\partial_x K\|_{L^\infty([0,T]\times[\theta_1,1-\theta_1]\times\mathbb{R}^2)} \le C_T$ and $\|\nabla_v\cdot K\|_{L^\infty([0,T]\times[\theta_1,1-\theta_1]\times\mathbb{R}^2)} \le C_T$, where C_T now depends also on $\|B_{\rm ext}\|_{C^1([\theta_1,1-\theta_1])}$. Therefore, it follows from the above two inequalities and the fact $f(t)$ is supported in $[\theta_0, 1-\theta_0]\times\bar{B}_R$ that

$$\|\partial_x f(t)\|_\infty \le C\left(1 + \int_0^t \|\nabla_v f(s)\|_\infty\, ds\right), \tag{4.39}$$

$$\|\nabla_v f(t)\|_\infty \le C\left(1 + \int_0^t \left[\|\partial_x f(s)\|_\infty + \|\nabla_v f(s)\|_\infty\right] ds\right). \tag{4.40}$$

Letting $u(s) := \|\partial_x f(s)\|_\infty + \|\nabla_v f(s)\|_\infty$, we get $u(t) \le 2C\left(1 + \int_0^t u(s)\, ds\right)$, so that $u(t) \le 2Ce^{2Ct} \le 2Ce^{2CT}$ for $t\in[0,T]$, giving the bounds for $\|\partial_x f\|_\infty$ and $\|\nabla_v f\|_\infty$. The identity

$$\partial_t f = -\hat{v}_1\partial_x f - K\cdot\nabla_v f,$$

also yields the bound on $\|\partial_t f\|_\infty$. □

5. Proof of the main result

Proof of Theorem 1.1, the uniqueness part. Suppose that $(\tilde{f}, \tilde{E}, \tilde{B})$ and (f^*, E^*, B^*) are two global C^1 solutions to the problem (1.1)–(1.5). Define

$$f := \tilde{f} - f^*, \quad E := \tilde{E} - E^* \quad \text{and} \quad B := \tilde{B} - B^*.$$

Then we have

$$E_1(t,x) = \int_0^x \int_{\mathbb{R}^2} f(t,y,v)\, dv dy \tag{5.41}$$

and

$$\partial_t f + \hat{v}_1 \partial_x f + [E^* + (\hat{v}_2, -\hat{v}_1)(B^* + B_{\text{ext}})] \cdot \nabla_v f = -[E + (\hat{v}_2, -\hat{v}_1)B] \cdot \nabla_v \tilde{f}. \tag{5.42}$$

Let $T > 0$ be arbitrary. Lemma 3.1 implies that the characteristics for equation (5.42) never reach $\partial\Omega$. So, integrating (5.42) along characteristics and using $f(0,\cdot,\cdot) \equiv 0$, we obtain for every $t \in [0,T]$ that

$$\|f(t)\|_\infty \le \|\nabla_v \tilde{f}\|_\infty \int_0^t \Big(\|E(s)\|_\infty + \|B(s)\|_\infty \Big)\, ds. \tag{5.43}$$

The relation (5.41) and Lemma 3.4 yield

$$\|E_1(t)\|_\infty \le \int_0^1 \int_{B_{k_0+C_2T}} f(t,y,v)\, dv dy \le C_T \|f(t)\|_\infty. \tag{5.44}$$

On the other hand, we infer from the representation formulas for $\tilde{E}_2$, $\tilde{B}$ and E_2^*, B^* given by Lemma 2.1 that

$$\|E_2(t)\|_\infty, \|B(t)\|_\infty \le \int_0^t \int_{B_{k_0+C_2T}} |\hat{v}_2|\, \|f(\tau)\|_\infty\, dv d\tau \le C_T \int_0^t \|f(s)\|_\infty\, ds. \tag{5.45}$$

Letting $h(s) := \sup_{\tau\in[0,s]} \|f(\tau)\|_\infty$, it follows from (5.43)–(5.45) that there exists a constant $C > 0$ depending on C_T and $\|\nabla_v \tilde{f}\|_\infty = \|\nabla_v \tilde{f}\|_{L^\infty\left([0,T]\times\overline{\Omega}\times\bar{B}_{k_0+C_2T}\right)} < \infty$ such that

$$\|f(t)\|_\infty \le C \int_0^t h(s)\, ds, \ \forall t \in [0,T].$$

Thus $h(t) \le C \int_0^t h(s)\, ds$, $\forall t \in [0,T]$, so that $h \equiv 0$, and hence $f(t) = 0$ for every $t \in [0,T]$. This together with (5.44) and (5.45) gives also $\|E(t)\|_\infty = \|B(t)\|_\infty = 0$. Thus we conclude that $\tilde{f}(t) \equiv f^*(t)$, $\tilde{E}(t) \equiv E^*(t)$ and $\tilde{B}(t) \equiv B^*(t)$ for all $t \in [0,T]$. The global uniqueness follows since $T > 0$ is arbitrary. □

Proof of Theorem 1.1, the existence part. Given our results obtained in Sections 2–4, the proof of the existence of a global C^1 solution follows via a the standard iteration scheme. This procedure is presented in [GSc] and [G, Chapter 5], and we shall only indicate the main points. By a standard density argument, one can assume in addition that $\psi_{\text{ext}} \in$

$C^3(\Omega)$, $f^0 \in C_0^2(\Omega \times \mathbb{R}^2)$, E_2^0, $B^0 \in C^2(\overline{\Omega})$ and $E_2^b(\cdot, x), B^b(\cdot, x) \in C^2([0,\infty))$ at each $x = 0, 1$.

Let $T > 0$ be arbitrary. We recursively define a sequence of solutions $\{(f^n, E^n, B^n)\}$ to the corresponding linear equations and show that it converges to a solution of the nonlinear problem (1.1)–(1.5). For the initial step ($n = 0$), we take $f^0(t, x, v) := f^0(x, v)$, and

$$E^0(t,x) := \int_0^x \int_{\mathbb{R}^2} f^0(y,v)\,dvdy + \lambda, \quad E_2^0(t,x) := E_2^0(x), \quad B^0(t,x) := B^0(x).$$

For $n \in \mathbb{N}$, assume that E_1^{n-1}, E_2^{n-1}, $B^{n-1} \in C^2([0,T] \times \overline{\Omega})$ are already given. Let $K^{n-1} := E^{n-1} + (\hat{v}_2, -\hat{v}_1)(B^{n-1} + B_{\text{ext}})$ and denote $(X^n(s), V^n(s))$ the solution of the characteristics system associated to a point $(t, x, v) \in [0,T] \times \Omega \times \mathbb{R}^2$. That is,

$$\begin{cases} \frac{dX^n}{ds} = \hat{V}_1^n(s), \\ \frac{dV^n}{ds} = K^{n-1}(s, X^n, V^n), \\ X^n(t;t,x,v) = x, \quad V^n(t;t,x,v) = v. \end{cases} \tag{5.46}$$

Notice that Lemma 3.1 and Remark 3.2 ensure that the characteristic $X(s)$ never reaches $\partial\Omega$. Since $K^{n-1} \in C^2([0,T] \times \Omega \times \mathbb{R}^2)$, we know that $(X^n, V^n) \in C^2([0,T]; \mathbb{R}^3)$. We define the n-th iterate of the distribution function by

$$f^n(t,x,v) := f^0(X^n(0), V^n(0)).$$

Then $f^n \in C^2([0,T] \times \Omega \times \mathbb{R}^2)$ and it satisfies the initial value problem

$$\begin{cases} \partial_t f^n + \hat{v}_1 \partial_x f^n + K^{n-1} \cdot \nabla_v f^n = 0, \\ f^n(0,x,v) = f^0(x,v). \end{cases} \tag{5.47}$$

Moreover, Lemma 3.4 shows that f^n has compact support in the x and v variables, i.e. $f^n \in C_0^2([0,T] \times \Omega \times \mathbb{R}^2)$. Therefore, the functions

$$\rho^n(t,x) := \int_{\mathbb{R}^2} f^n(t,x,v)\,dv \quad \text{and} \quad j^n(t,x) := \int_{\mathbb{R}^2} \hat{v} f^n(t,x,v)\,dv$$

are in $C_0^2([0,T] \times \Omega)$. Next, we define

$$E_1^n(t,x) = \int_0^x \rho^n(t,y)\,dy + \lambda \tag{5.48}$$

and E_2^n, B^n as the solution of

$$\begin{cases} \partial_t E_2^n = -\partial_x B^n - j_2^n, & \partial_t B^n = -\partial_x E_2^n, \\ E_2^n(0,x) = E_2^0(x), & B^n(0,x) = B^0(x), \\ E_2^n(t,x)_{|\partial\Omega} = E_2^b(t,x), & B^n(t,x)_{|\partial\Omega} = B^b(t,x). \end{cases} \tag{5.49}$$

As in Section 2, we know that E_2^n and B^n must be given by the formulas in Lemma 2.1 with j_2 being replaced by j_2^n. We deduce that $E_1^n, E_2^n, B^n \in C^2([0,T] \times \overline{\Omega})$.

Then it follows from the same reasoning leading to Lemma 4.2 that there exists a constant $C_T > 0$ depending only on $k_0,\ T,\ \lambda,\ \|B_{\mathrm{ext}}\|_{C^1([\theta_1,1-\theta_1])}$, the C^1 norms of $f^0,\ E_2^0,\ B^0$, and the C^1 norms of $E_2^b(\cdot,x),\ B^b(\cdot,x)$ on $[0,T]$ $(x = 0,\ 1)$ such that

$$\|f^n\|_{C^1([0,T]\times\overline{\Omega}\times\mathbb{R}^2)} + \|E^n\|_{C^1([0,T]\times\overline{\Omega})} + \|B^n\|_{C^1([0,T]\times\overline{\Omega})} \le C_T.$$

Moreover, by following the arguments in [G, Section 5.8] we see that $\{(f^n, E_1^n, E_2^n, B^n)\}$ is a Cauchy sequence in the C^1 norm. Consequently, there exist $f \in C^1([0,T]\times\overline{\Omega}\times\mathbb{R}^2)$ and $E_1, E_2, B \in C^1([0,T]\times\overline{\Omega})$ such that $f^n \to f$, $E_1^n \to E_1$, $E_2^n \to E_2$, $B^n \to B$ uniformly for $t \in [0,T]$, $x \in \overline{\Omega}$, $v \in \mathbb{R}^2$, together with all their first derivatives. In particular, the function f, E_2 and B satisfy the initial and boundary conditions (1.3)–(1.5). Note also that $f \in C_0^1([0,T]\times\Omega\times\mathbb{R}^2)$ since the (x,v)- support of f^n is bounded uniformly in n by Lemma 3.4.

Passage to the limit in (5.47) yields the Vlasov equation. On the other hand, passage to the limit in (5.48) and (5.49) yields

$$\begin{cases} E_1(t,x) = \int_0^x \rho(t,y)\,dy + \lambda, \\ \partial_t E_2 = -\partial_x B - j_2, \quad \partial_t B = -\partial_x E_2. \end{cases}$$

Thus (f, E_1, E_2, B) is a C^1 solution to the problem (1.1)–(1.5) in the time interval $[0,T]$. Due to the arbitrariness of T and the uniqueness of C^1 solutions presented earlier, we infer that the problem (1.1)–(1.5) admits a global classical solution (f, E, B) with $f \in C^1([0,\infty)\times\overline{\Omega}\times\mathbb{R}^2)$ and $E, B \in C^1([0,\infty)\times\overline{\Omega})$. Moreover, $f \in C_0^1([0,T]\times\Omega\times\mathbb{R}^2)$ for every $T > 0$.

We finally note that the non-negativity of the solution f is inherited from that of f^0 as f is constant along the characteristics. □

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