TOPOGRAPHY INFLUENCE ON THE LAKE EQUATIONS IN BOUNDED DOMAINS

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Abstract. We investigate the influence of the topography on the lake equations which describe the two-dimensional horizontal velocity of a three-dimensional incompressible flow. We show that the lake equations are structurally stable under Hausdorff approximations of the fluid domain and L^p perturbations of the depth. As a byproduct, we obtain the existence of a weak solution to the lake equations in the case of singular domains and rough bottoms. Our result thus extends earlier works by Bresch and Métivier treating the lake equations with a fixed topography and by Gérard-Varet and Lacave treating the Euler equations in singular domains.

Contents

1. Introduction

The lake equations are introduced in the physical literature as a two-dimensional geophysical model to describe the evolution of the vertically averaged horizontal component of the three-dimensional velocity of an incompressible Euler flow; see for example [5, 2, 10, 1] and the references therein for physical discussions and derivation of the model. Precisely, the lake equations with prescribed initial

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and boundary conditions are

$$\begin{cases} \partial_t(bv) + \mathrm{div}\,(bv \otimes v) + b\nabla p = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega, \\ \mathrm{div}\,(bv) = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega, \\ (bv)\cdot \nu = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ v(0,x) = v^0(x) & \text{for } x \in \Omega. \end{cases} \tag{1.1}$$

Here $v = v(t,x)$ denotes the two-dimensional horizontal component of the fluid velocity, $p = p(t,x)$ the pressure, $b = b(x)$ the vertical depth which is assumed to be varying in x, $\Omega \subset \mathbb{R}^2$ is the spatial bounded domain of the fluid surface, and ν denotes the inward-pointing unit normal vector on $\partial\Omega$.

In case that b is a constant, (1.1) simply becomes the well-known two-dimensional Euler equations, and the well-posedness is widely known since the work of Wolibner [12] or Yudovich [13]. When the depth b varies but is bounded away from zero, the well-posedness is established in Levermore, Oliver and Titi [10]. Most recently, Bresch and Métivier [1] extended the work in [10] by allowing the varying depth to vanish on the boundary of the spatial domain. In this latter situation, the corresponding equations for the stream function are degenerate near the boundary and the elliptic techniques for degenerate equations are needed to obtain the well-posedness.

In this paper, we are interested in stability and asymptotic behavior of the solutions to the above lake equations under perturbations of the fluid domain or rather perturbations of the geometry of the lake which is described by the pair (Ω, b). Our main result roughly asserts that the lake equations are persistent under these topography perturbations. That is, if we let (Ω_n, b_n) be any sequence of lakes which converges to (Ω, b) (in the sense of Definition 1.4), then the weak solutions to the lake equations on (Ω_n, b_n) converge to the weak solution on the limiting lake (Ω, b). In particular, we obtain strong convergence of velocity in L^2 and we allow the limiting domain Ω to be very singular as long as it can be approximated by smooth domains Ω_n in the Hausdorff sense. The depth b is merely assumed to be bounded. As a byproduct, *we establish the existence of global weak solutions of the equations* (1.1) *for very rough lakes* (Ω, b).

Let us make our assumptions on the lake more precise. We assume that the (limiting) lake (Ω, b) has a finite number of islands, namely:

(H1) $\Omega := \widetilde{\Omega} \setminus \Big(\bigcup_{k=1}^{N} \mathcal{C}^k\Big)$, where $\widetilde{\Omega}$, $\mathcal{C}^k$ are bounded simply connected subsets of $\mathbb{R}^2$, $\widetilde{\Omega}$ is open, and $\mathcal{C}^k$ are disjoints and compact subsets of $\widetilde{\Omega}$.

We assume that the boundary is the only place where the depth can vanish, namely:

(H2) There is a positive constant M such that

$$0 < b(x) \leq M \qquad \text{in } \Omega$$

and in addition, for any compact set $K \subset \Omega$ there exists positive number θ_K such that $b(x) \geq \theta_K$ on K.

In the case of smooth lakes, we add another hypothesis. Near each piece of boundary, we allow the shore to be either of non-vanishing or vanishing topography with constant slopes in the following sense:

(H3) There are small neighborhoods $\mathcal{O}^0$ and $\mathcal{O}^k$ of $\partial\widetilde{\Omega}$ and $\partial\mathcal{C}^k$ respectively, such that, for $0 \leq k \leq N$,

$$b(x) = c(x)\,[d(x)]^{a_k} \qquad \text{in } \mathcal{O}^k \cap \Omega, \tag{1.2}$$

where $c(x), d(x)$ are bounded C^3 functions in the neighborhood of the boundary, $c(x) \geq \theta > 0$, $a_k \geq 0$. Here the geometric function $d(x)$ satisfies $\Omega = \{d > 0\}$ and $\nabla d \neq 0$ on $\partial\Omega$.

In particular, around each obstacle $\mathcal{C}^k$, we have either *Non-vanishing topography* when $a_k = 0$, in which case $b(x) \geq \theta$ or *Vanishing topography* if $a_k > 0$ in which case $b(x) \to 0$ as $x \to \partial\mathcal{C}^k$. As (H3) will be only considered for smooth lakes $\partial\Omega \in C^3$, we note that up to a change of c, θ, we may take $d(x) = \mathrm{dist}(x, \partial\Omega)$.

1.1. **Weak formulations.** As in the case of the 2D Euler equations, it is crucial to use the notion of generalized vorticity, which is defined by

$$\omega := \frac{1}{b}\mathrm{curl}\, v = \frac{1}{b}(\partial_1 v_2 - \partial_2 v_1).$$

Indeed, taking the curl of the momentum equation, it follows that the vorticity formally verifies the following transport equation

$$\partial_t(b\omega) + \mathrm{div}\,(bv\omega) = 0. \tag{1.3}$$

Thanks to the condition $\mathrm{div}\,(bv) = 0$, we will show in Lemma 3.1 that the L^p norm of $b^{\frac{1}{p}}\omega$ is a conserved quantity for any $p \in [1, \infty]$, which provides an important estimate on the solution.

When Ω is not regular, the divergence free condition and the boundary condition $bv^0 \cdot \nu|_{\partial\Omega} = 0$ have to be understood in a weak sense:

$$\int_\Omega b(x)v^0(x) \cdot h(x)\, dx = 0, \tag{1.4}$$

for any test function h in the function space $G(\Omega)$ defined by

$$G(\Omega) := \Big\{ w \in L^2(\Omega) \ : \ w = \nabla p, \quad \text{for some } p \in H^1_{\mathrm{loc}}(\Omega) \Big\}.$$

For $bv^0 \in L^2(\Omega)$, such a condition in (1.4) is equivalent to

$$bv^0 \in \mathcal{H}(\Omega), \tag{1.5}$$

where

$$\mathcal{H}(\Omega) = \text{ the closure in } L^2 \text{ of } \{\varphi \in C^\infty_c(\Omega) \mid \mathrm{div}\,\varphi = 0\}. \tag{1.6}$$

This equivalence can be found, for instance, in [3, Lemma III.2.1]. Moreover, in [3] the author points out that if Ω is a regular bounded domain and if bv^0 is a sufficiently smooth function, then bv^0 verifies (1.4) if and only if $\mathrm{div}\, bv^0 = 0$ and $bv^0 \cdot \nu|_{\partial\Omega} = 0$.

Similarly to (1.4), the weak form of the divergence free and tangency conditions on bv also reads:

$$\forall h \in C^\infty_c\left([0, +\infty); G(\Omega)\right), \qquad \int_{\mathbb{R}_+}\int_\Omega b(x)v(t,x) \cdot h(t,x)\, dxdt = 0. \tag{1.7}$$

Next, we introduce several notions of global weak solutions to the lake equations. The first is in terms of the velocity.

Definition 1.1. *Let v^0 be a vector field such that*

$$\mathrm{div}\,(bv^0) = 0 \ \textit{in } \Omega, \quad bv^0 \cdot \nu = 0 \ \textit{on } \partial\Omega, \ \textit{weakly (in the sense of (1.4))}$$

and

$$\frac{\mathrm{curl}\, v^0}{b} \in L^\infty(\Omega).$$

We say that v is a global weak solution of the velocity formulation of the lake equations (1.1) with initial velocity v^0 if

i) $\frac{\mathrm{curl}\, v}{b} \in L^\infty(\mathbb{R}_+ \times \Omega)$ *and* $\sqrt{b}v \in L^\infty(\mathbb{R}_+; L^2(\Omega))$;

ii) $\mathrm{div}\,(bv) = 0$ *in* Ω *and* $bv \cdot \nu = 0$ *on* $\partial\Omega$ *in the sense of* (1.7);

iii) the momentum equation in (1.1) *is verified in the distributional sense. That is, for all divergence-free vector test functions* $\Phi \in C^\infty_c([0,\infty) \times \overline{\Omega})$ *tangent to the boundary, there holds that*

$$\int_0^\infty \int_\Omega \Phi_t \cdot v\, dxdt + \int_0^\infty \int_\Omega (bv \otimes v) : \nabla\Big(\frac{\Phi}{b}\Big)\, dxdt + \int_\Omega \Phi(0,x) \cdot v^0(x)\, dx = 0. \tag{1.8}$$

We emphasize that the test functions Φ are allowed to be in $C_c^\infty([0,\infty)\times\overline{\Omega})$ rather than in $C_c^\infty([0,\infty)\times\Omega)$. Namely, for any test functions Φ belonging to $C_c^\infty([0,\infty)\times\overline{\Omega})$, there exists $T>0$ such that $\Phi\equiv 0$ for any $t>T$, and such that $\Phi(t,\cdot)\in C^\infty(\overline{\Omega})$ for any t, in the sense that $D^k\Phi(t,\cdot)$ is bounded and uniformly continuous on Ω for any $k\geq 0$ (see e.g. [3]).

In the above definition, it does not appear immediately clear how to make sense of (1.8) for test functions supported up to the boundary due to the term Φ/b which would then blow up at the boundary. For this reason, let us introduce a weak *interior* solution v of the velocity formulation to be the weak solution v as in Definition 1.1 with the test functions Φ in (1.8) being supported *inside* the domain, i.e. $\Phi\in C_c^\infty([0,\infty)\times\Omega)$. For this weaker solution, (1.8) then makes sense under the regularity (i) when $b\in W^{1,\infty}_{\rm loc}(\Omega)$ (because (H2) gives an estimate of b^{-1} locally in space). Later on in Appendix A, we show that (1.8) indeed makes sense with the test functions supported up to the boundary when the lake is smooth, even in the case of vanishing topography.

The second formulation of weak solutions is in terms of the vorticity and reads as follows.

Definition 1.2. *Let (v^0,ω^0) be a pair such that*

$$\operatorname{div}(bv^0)=0 \text{ in } \Omega,\quad bv^0\cdot\nu=0 \text{ on } \partial\Omega \qquad \text{weakly (in the sense of (1.4)))} \tag{1.9}$$

and

$$\omega^0\in L^\infty(\Omega),\quad \operatorname{curl} v^0=b\omega^0 \qquad \text{(in the distributional sense).} \tag{1.10}$$

We say that (v,ω) is a global weak solution of the vorticity formulation of the lake equations on (Ω,b) with initial condition (v^0,ω^0) if

i) $\omega\in L^\infty(\mathbb{R}_+\times\Omega)$ and $\sqrt{b}v\in L^\infty(\mathbb{R}_+;L^2(\Omega))$;

ii) $\operatorname{div}(bv)=0$ in Ω and $bv\cdot\nu=0$ on $\partial\Omega$ in the sense of (1.7);

iii) $\operatorname{curl} v=b\omega$ in the distributional sense;

iv) the transport equation (1.3) is verified in the sense of distribution. That is, for all test functions $\varphi\in C_c^\infty([0,\infty)\times\overline{\Omega})$ such that $\partial_\tau\varphi|_{\partial\Omega}\equiv 0$ (i.e. constant on each piece of boundary), there holds

$$\int_0^\infty\int_\Omega \varphi_t b\omega\,dxdt+\int_0^\infty\int_\Omega \nabla\varphi\cdot vb\omega\,dxdt+\int_\Omega\varphi(0,x)b\omega^0(x)\,dx=0. \tag{1.11}$$

We also introduce a weaker intermediate notion: weak *interior* solution of the vorticity formulation to be the weak solution (v,ω) as in Definition 1.2 with the test functions being supported *inside* the domain: i.e. $\varphi\in C_c^\infty([0,\infty)\times\Omega)$.

We will establish the relations between these definitions in Appendix A. For example, when the lake is smooth, all velocity and vorticity formulations are equivalent.

Following the proof of Yudovich [13], Levermore, Oliver and Titi [10] established existence and uniqueness of a global weak solution (with the vorticity formulation) in the case of non-vanishing topography, assuming the lake is smooth and simply connected. Recently, Bresch and Métivier [1] extended the well-posedness to the case of vanishing topography. In both of these works, Ω is assumed to be simply connected, $\partial\Omega\in C^3$, and $b\in C^3(\overline{\Omega})$. The essential tool in establishing the well-posedness is a Calderón-Zygmund type inequality. This inequality is highly non trivial to obtain if the depth vanishes, and the proof requires to work with degenerate elliptic equations.

In Section 2, we shall sketch the proof of the well-posedness of the lake equations under our current setting (H1)-(H3):

Theorem 1.3. *Let (Ω,b) be a lake verifying Assumptions (H1)-(H3) and $(\partial\Omega,b)\in C^3\times C^3(\overline{\Omega})$. Then for any pair (v^0,ω^0) such that $b^{-1}\operatorname{curl} v^0=\omega^0\in L^\infty(\Omega)$, there exists a unique global weak solution (v,ω) to the lake equations that verifies both the velocity and vorticity formulations. Furthermore, we have that*

$$\omega\in C(\mathbb{R}_+,L^r(\Omega)),\quad v\in C(\mathbb{R}_+,W^{1,r}(\Omega)),\quad v\cdot\nu=0 \text{ on } \partial\Omega,$$

for arbitrary r in $[1,\infty)$ and the circulations of v around $\mathcal{C}^k$ are conserved for any $k=1\dots N$.

When the domain is not simply connected, the vorticity alone is not sufficient to determine the velocity uniquely from (1.9)-(1.10). We will then introduce in Section 2.1 the generalized circulation for lake equations, derive the Biot-Savart law (the law which yields the velocity in term of the vorticity and circulations), and prove the Kelvin's theorem concerning conservation of the circulation.

1.2. **Assumptions.** For each $n \geq 1$, let (Ω_n, b_n) be a lake of either vanishing or non-vanishing or mixed-type topography as described above in (H1)-(H3) with constants $\theta_n, M_n, a_{0,n}, \dots, a_{N,n}$ and function $d_n(x)$.

In what follows, we write $(\Omega_0, b_0) = (\Omega, b)$, which will play the role of the limiting lake. We assume that these lakes have the same finite number of islands N, namely for any $n \geq 0$

$$\Omega_n := \widetilde{\Omega}_n \setminus \Big(\bigcup_{k=1}^{N} \mathcal{C}_n^k \Big),$$

where $\widetilde{\Omega}_n, \mathcal{C}_n^k$ are simply connected subsets of $\mathbb{R}^2$, $\widetilde{\Omega}_n$ is open, and $\mathcal{C}_n^k \subset \widetilde{\Omega}_n$ are disjoint and compact. In addition, let D be a big enough subset so that $\Omega_n \subset D$, $n \geq 0$.

Definition 1.4. *Assume that $(\partial\Omega_n, b_n) \in C^3 \times C^3(\overline{\Omega_n})$ for all $n \geq 1$. We say that the sequence of lakes (Ω_n, b_n) converges to the lake (Ω, b) as $n \to \infty$ if there hold*

- $\widetilde{\Omega}_n \to \widetilde{\Omega}$ *in the Hausdorff sense;*
- $\mathcal{C}_n^k \to \mathcal{C}^k$ *in the Hausdorff sense;*
- $\|b_n\|_{L^\infty(\Omega_n)}$ *is uniformly bounded and for any compact set $K \subset \Omega$ there exist positive θ_K and sufficiently large $n_0(K)$ such that $b_n(x) \geq \theta_K$ for all $x \in K$ and $n \geq n_0(K)$;*
- $b_n \to b$ *in*[1] $L^1_{\text{loc}}(\Omega)$.

Here Ω_n converges to Ω in the Hausdorff sense if and only if the Hausdorff distance between Ω_n and Ω converges to zero. See for example [4, Appendix B] for more details about the Hausdorff topology, in particular the Hausdorff convergence implies the following proposition: for any compact set $K \subset \Omega$, there exists $n_K > 0$ such that $K \subset \Omega_n$ for all $n \geq n_K$, which gives sense to the fourth item of the above definition.

Definition 1.4, allows in particular the limit $a_n \to a_0 = 0$, with a_n introduced as in (H3). This means that the passage from a lake of the vanishing type in which the slope gets steeper and steeper to a lake of non-vanishing type is allowed. This appears to be complicated to deduce from the analysis in [1], where the condition $a_0 > 0$ is crucial. Remarkably, it turns out that uniform estimates of the velocity in $W^{1,p}$ are not needed in order to pass to the limit. As will be shown, L^2 estimates are sufficient.

1.3. **Main results.** As mentioned, a velocity field is uniquely determined by its vorticity and its circulation around each obstacle. We recall that when the velocity field v is continuous, the circulation around each obstacle $\mathcal{C}^k$ is classically defined by

$$\gamma_{\text{cl}}^k := \oint_{\partial\mathcal{C}^k} v \cdot d\mathbf{s}.$$

However, with a low regularity velocity field as in our definitions of weak solutions, such a path integral might not be well defined a priori. We are led to introduce the generalized circulation

$$\gamma^k(v) := \int_\Omega \operatorname{div}(\chi^k v^\perp)\, dx$$

where χ^k is some smooth cut-off function that is equal to one in a neighborhood of $\mathcal{C}^k$ and zero away from $\mathcal{C}^k$. Observing that $\operatorname{div}(\chi^k v^\perp) = -\nabla^\perp \chi^k \cdot v - \chi^k \operatorname{curl} v$, the generalized circulation is well defined for the weak solution v by condition (i) in Definitions 1.1 and 1.2 (indeed, (H2) implies that v belongs to $L^2(\operatorname{supp} \nabla^\perp \chi^k)$). The generalized circulation does not depend on the choice of the cutoff function (indeed, if χ_k and $\tilde{\chi}_k$ are equal to one in a neighborhood of $\mathcal{C}^k$ and zero in a neighborhood of $\partial\widetilde{\Omega} \cup_{j\neq k} \mathcal{C}^j$, then we can integrate by part to verify that $\int_\Omega \operatorname{div}((\chi^k - \tilde{\chi}^k)v^\perp)dx = 0$). Actually, we will show in the proof of Proposition A.4 that the notions of generalized circulation and classical circulation are equivalent for smooth lakes. Most importantly, the velocity field is uniquely determined

[1]Note that since b_n is uniformly bounded in L^∞, we directly see that the convergence holds in L^p, $p < \infty$.

by the vorticity and the circulations; see Section 2. We refer to [7] for an alternative definition of weak circulation.

Our assumptions on the convergence of the initial data are in terms of the vorticity and circulations. Precisely, we assume that the initial vorticity ω_n^0 is uniformly bounded:

$$\|\omega_n^0\|_{L^\infty(\Omega_n)} \leq M_0, \tag{1.12}$$

for some positive M_0, and there holds the convergence

$$\omega_n^0 \rightharpoonup \omega^0 \text{ weakly in } L^1(D), \tag{1.13}$$

as $n \to \infty$. Here ω_n^0 is extended to be zero in $D \setminus \Omega_n$. Concerning the circulations, we assume that the sequence $\gamma_n = \{\gamma_n^k\}_{1\leq k\leq N} \in \mathbb{R}^N$ converges to a given vector $\gamma = \{\gamma^k\}_{1\leq k\leq N}$ in the sense that

$$\sum_{k=1}^{N} |\gamma_n^k - \gamma^k| \to 0, \tag{1.14}$$

as $n \to \infty$. Then, for each $n \geq 1$, we define the initial velocity field v_n^0 to be the unique solution of the following elliptic problem in Ω_n:

$$\operatorname{div}(b_n v_n^0) = 0, \quad (b_n v_n^0) \cdot \nu|_{\partial\Omega_n} = 0, \quad \operatorname{curl} v_n^0 = b_n \omega_n^0, \quad \gamma_n^k(v_n^0) = \gamma_n^k \ \forall 1 \leq k \leq N. \tag{1.15}$$

The existence and uniqueness of v_n^0 are established in Section 2.

Our first main theorem is concerned with the stability of the lake equations:

Theorem 1.5. *Let (Ω, b) be a lake satisfying Assumptions (H1)-(H3) with $(\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega})$. Assume that there is a sequence of lakes (Ω_n, b_n) which converges to (Ω, b) in the sense of Definition 1.4. Assume also that $(\omega_n^0, \gamma_n, v_n^0)$ are as in* (1.12)–(1.15)*. Let (v_n, ω_n) be the unique weak solution of the lake equations* (1.1) *on the lake (Ω_n, b_n) with initial velocity v_n^0, $n \geq 1$. Then, there exists a pair (v, ω) so that*

$$v_n \to v \text{ strongly in } L^2_{\rm loc}(\mathbb{R}_+; L^2(D)), \qquad \omega_n \rightharpoonup \omega \text{ weak-}* \text{ in } L^\infty(\mathbb{R}_+ \times D).$$

Furthermore, (v, ω) is the unique weak solution of the lake equations on the lake (Ω, b) with initial vorticity ω^0 and initial circulation $\gamma \in \mathbb{R}^N$.

This theorem, whose proof will be given in Section 3, links together various results on the lake equations, namely the flat bottom case (Euler equations [13]), non-vanishing topography [10] and vanishing topography [1]. Indeed, we allow the limit $a_n \to 0$ (passing from vanishing topography to non vanishing topography), or the limit $\theta_n \to 0$ if $b_n = b + \theta_n$ where b verifies (H2)-(H3) (passing from non vanishing topography to vanishing topography). The convergence of the solutions of the Euler equations when the domains converge in the Hausdorff topology is a recent result established by Gérard-Varet and Lacave [4], based on the γ-convergence on open sets (a brief overview of this notion is given in Appendix B). The present paper can be regarded as a natural extension of [4] to the lake equations i.e. to the case of non-flat bottoms b_n when we consider a weak notion of convergence of b_n.

The γ-convergence is an H^1_0 theory on the stream function (or an L^2 theory on the velocity). Bresch and Métivier have obtained estimates in $W^{2,p}$ for any $2 \leq p < +\infty$ (namely, the Calderón-Zygmund inequality) for the stream function, which is necessary for the uniqueness problem or to give a sense to the velocity formulation. For our interest in the sequential stability of the lake solutions, it turns out that we can treat our problem without having to derive uniform estimates in $W^{2,p}$, which appear hard to obtain. In fact, we will first prove the convergence of a subsequence of v_n to v and show that the limiting function v is indeed a solution of the limiting lake equations. Since the Calderón-Zygmund inequality is verified for the solution of the limiting lake equations, the uniqueness yields that the whole sequence indeed converges to the unique solution in (Ω, b).

In addition, since the Calderón-Zygmund inequality is not used in the compactness argument, it follows that the existence of a weak solution to the lake equations with non-smooth domains or non-smooth topography can be obtained as a limit of solutions to the lake equations with smooth domains. Our second main theorem is concerned with non-smooth lakes which do not necessarily verify (H3).

Theorem 1.6. *Let (Ω, b) be a lake satisfying (H1)-(H2). We assume that for every $1 \leq k \leq N$, $\mathcal{C}^k$ has a positive Sobolev H^1 capacity. For any $\omega^0 \in L^\infty(\Omega)$ and $\gamma \in \mathbb{R}^N$, there exists a global weak solution (v, ω) of the lake equations in the vorticity formulation on the lake (Ω, b) with initial vorticity ω^0 and initial circulation $\gamma \in \mathbb{R}^N$. This solution enjoys a Biot-Savart decomposition and its circulations are conserved in time. If we assume in addition that $b \in W^{1,\infty}_{\text{loc}}(\Omega)$ then (v, ω) is also a global weak* interior *solution in the velocity formulation.*

Let us mention that we do not assume any regularity of $\partial\Omega$; for instance, $\partial\Omega$ can be the Koch snowflake. To obtain solutions for the vorticity formulation, we do not need any regularity on b either; it might not even be continuous. But even in the case where we assume the bottom to be locally lipschitz, choosing $b_n := b + \frac{1}{n}$ we can consider a zero slope: $b(x) = e^{-1/d(x)}$ or non constant: $b(x) = d(x)^{a(x)}$; our theorem states that (v, ω) is a solution of the vorticity formulation and an *interior* solution of the velocity formulation. Such a result might appear surprising, because the known existence result requires that the lake domain is smooth, namely $(\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega})$ and (H3).

The Sobolev H^1 capacity of a compact set $E \subset \mathbb{R}^2$ is defined by

$$\operatorname{cap}(E) := \inf\{\|v\|^2_{H^1(\mathbb{R}^2)},\ v \geq 1 \text{ a.e. in a neighborhood of } E\},$$

with the convention that $\operatorname{cap}(E) = +\infty$ when the set in the r.h.s. is empty. We refer to [6] for an extensive study of this notion (the basic properties are listed in [4, Appendix A], in particular we recall that a material point has a zero capacity whereas the capacity of a Jordan arc is positive).

Apparently, in such non-smooth lake domains, the Calderón-Zygmund inequality is no longer valid, and hence the well-posedness is delicate. For existence, our construction of the solution follows by approximating the non-smooth lake by an increasing sequence of smooth domains in which the solutions are given from Theorem 1.3.

Finally, we leave out the question of uniqueness in the case of non-smooth lakes. We refer to [8] for a uniqueness result for the 2D Euler equations in simply-connected domains with corners. In [8] the velocity is shown in general not to belong to $W^{1,p}$ for all p (precisely, if there is a corner of angle $\alpha > \pi$, then the velocity is no longer bounded in $L^p \cap W^{1,q}$, $p > p_\alpha$, $q > q_\alpha$ with $p_\alpha \to 4$ and $q_\alpha \to 4/3$ as $\alpha \to 2\pi$).

2. Well-posedness of the lake equations for smooth lake

In this section, we sketch the proof of existence of the lake equations in a non-simply connected domain (Theorem 1.3). The proof can be outlined as follows:

- we first prove existence of a global weak interior solution in the vorticity formulation. The proof follows by adding an artificial viscosity (as was done in [9]) and obtaining compactness for the vanishing viscosity problem (Section 2.2);
- as the lake is smooth, we then use the Calderón-Zygmund inequality established in [1], which in turn implies that for arbitrary $r \geq 1$, $\omega \in C(\mathbb{R}_+, L^r(\Omega))$, $v \in C(\mathbb{R}_+, W^{1,r}(\Omega))$, and $v \cdot \nu = 0$ on $\partial\Omega$;
- thanks to the regularity close to the boundary, we can show by a continuity argument that the vorticity equation (1.11) is indeed verified for test functions supported all the way to the boundary (Proposition A.5). The existence of a global weak solution in the vorticity formulation (with conserved circulations) is then established. The solution also verifies the velocity formulation due to the equivalence of the two formulations (Proposition A.4).
- finally, uniqueness of a global weak solution is shown in Section 2.3 by following the celebrated method of Yudovich.

Essentially, this outline of the proof was introduced by Yudovich in his study of two-dimensional Euler equations [13], and it was used in [10, 1] in the case of the lake equations. We shall provide the proof with more details as it will be crucial in our convergence proof later on.

Throughout this section, we fix a smooth lake (Ω, b) namely:

$$(\Omega, b) \text{ satisfying Assumptions (H1)-(H3) (see Section 1) and } (\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega}). \tag{2.1}$$

We allow the lake to have either vanishing or non-vanishing topography. We shall begin the section by deriving the Biot-Savart law. We then obtain the well-posedness of the lake equations (1.1) in the sense of Definitions 1.1 and 1.2.

2.1. **Auxiliary elliptic problems.** Let us introduce the function space

$$X := \Big\{ f \in H^1_0(\Omega) \; : \; b^{-1/2}\nabla f \in L^2(\Omega) \Big\}.$$

We will sometimes write the function space as X_b instead of X to emphasize the dependence on b. Clearly, $(X, \|\cdot\|_X)$ is a Hilbert space with inner product $\langle f, g\rangle_X := \langle b^{-1/2}\nabla f, b^{-1/2}\nabla g\rangle_{L^2}$ and norm $\|f\|_X := \langle f, f\rangle_X^{1/2}$. Our first remark is concerned with the density of $C^\infty_c(\Omega)$ in X.

Lemma 2.1. *Let (Ω, b) be a smooth lake in the sense of (2.1). Then $C^\infty_c(\Omega)$ is dense in X with respect to the norm $\|\cdot\|_X$.*

The proof relies on a variant of the Hardy's inequality. As it was noted in the introduction, we can consider that d is the distance to the boundary in (1.2). With the notation:

$$\partial\Omega_R := \{x \in \Omega : 0 \le d(x) \le R\}, \tag{2.2}$$

we establish the following Hardy type inequality:

Lemma 2.2. *Let (Ω, b) be a smooth lake in the sense of (2.1). Then the following inequality holds uniformly for every $f \in H^1_0(\Omega)$ and any positive R:*

$$\|b^{-1/2}(f/d)\|_{L^2(\partial\Omega_R)} \lesssim \|b^{-1/2}\nabla f\|_{L^2(\partial\Omega_R)}. \tag{2.3}$$

Here in Lemma 2.2 and throughout the paper, the notation $g \lesssim h$ is used to mean a uniform bound $g \le Ch$, for some universal constant C that is independent of the underlying parameter (in (2.3), small $R > 0$ and f).

Proof of Lemma 2.2. We start with the following claim: for any $f \in H^1_0(\Omega)$ and any positive R, there holds that

$$\int_{R\le d(x)\le 2R} |f(x)|^2\,dx \lesssim R^2 \int_{d(x)\le 2R} |\nabla f(x)|^2\,dx. \tag{2.4}$$

The claim follows directly from the fundamental theorem of Calculus and the standard Hölder's inequality at least for smooth compactly supported functions. By density, it extends to $H^1_0(\Omega)$.

Next, by (2.4) and the classical Hardy's inequality, the lemma follows easily for functions $f \in H^1_0(\Omega)$ whose support is away from the set $\bigcup_{k:a_k>0}\mathcal{O}_k$. It suffices to consider functions f that are supported in the set $\mathcal{O}_j$ for $a_j > 0$. Again by (2.4), we can write

$$\begin{aligned}
\Big\|b^{-1/2}(f/d)\Big\|^2_{L^2(\partial\Omega_R)} &= \sum_{k\in\mathbb{N}^*}\int_{R\le 2^k d(x)\le 2R}\left(\frac{f(x)}{d(x)}\right)^2\frac{dx}{b(x)}\\
&\lesssim \sum_{k\in\mathbb{N}^*}(R2^{-k})^{-(a_j+2)}\int_{R\le 2^k d(x)\le 2R}|f(x)|^2\,dx\\
&\lesssim \sum_{k\in\mathbb{N}^*}(R2^{-k})^{-a_j}\int_{2^k d(x)\le 2R}|\nabla f(x)|^2\,dx\\
&\lesssim \int_{\partial\Omega_R}\left(\sum_{k\in\mathbb{N}^*:\ 2^k d(x)\le 2R}(R2^{-k})^{-a_j}\right)|\nabla f(x)|^2\,dx.
\end{aligned}$$

Since the summation in the parentheses in the last line above is bounded by b^{-1}, the integral on the righthand side is bounded by $\|b^{-1/2}\nabla f\|^2_{L^2(\partial\Omega_R)}$. The lemma is thus proved. ☐

Proof of Lemma 2.1. Fix $\varepsilon > 0$ and $f \in X$. It suffices to construct a cut-off function $\chi \in C_c^1(\Omega)$ such that

$$\|(1-\chi)f\|_X \leq \varepsilon. \tag{2.5}$$

The lemma would then follow simply by approximating the compactly supported function χf with its C_c^∞ mollifier functions.

Now since $f \in X$, there exists a positive R_ϵ such that

$$\int_{\partial\Omega_{R_\epsilon}} |\nabla f(x)|^2 \frac{dx}{b(x)} \leq \varepsilon^2. \tag{2.6}$$

Let us introduce a cut-off function $\eta \in C^\infty(\mathbb{R}_+)$ such that $0 \leq \eta \leq 1$, $\eta(z) \equiv 1$ if $z \geq 1$ and $\eta(z) \equiv 0$ if $z \leq 1/2$ and define

$$\chi(x) = \eta(d(x)/(R_\epsilon)).$$

Clearly, $\chi \in C_c^1(\Omega)$. In addition, we note that $\nabla[(1-\chi)f] = (1-\chi)\nabla f - f\nabla\chi$. It then follows by (2.6) that

$$\int_\Omega (1-\chi(x))^2 |\nabla f(x)|^2 \frac{dx}{b(x)} \leq \int_{\partial\Omega_{R_\epsilon}} |\nabla f(x)|^2 \frac{dx}{b(x)} \leq \varepsilon^2.$$

Meanwhile using the fact that

$$|f\nabla\chi| = |R_\epsilon^{-1} f \eta'(d(x)/R_\epsilon)\nabla d(x)| \leq |(f/d)(x)| \|\eta'\|_{L^\infty}$$

and Lemma 2.2, we obtain

$$\int_\Omega |f(x)\nabla\chi(x)|^2 \frac{dx}{b(x)} \leq \|\eta'\|_{L^\infty} \int_{\partial\Omega_{R_\epsilon}} \frac{|f(x)|^2}{d(x)^2} \frac{dx}{b(x)} \lesssim \int_{\partial\Omega_{R_\epsilon}} |\nabla f(x)|^2 \frac{dx}{b(x)} \lesssim \varepsilon^2.$$

This yields (2.5) which completes the proof of the lemma. □

Next, we consider the following auxiliary elliptic problem

$$\operatorname{div}\Big[\frac{1}{b}\nabla\psi\Big] = f \text{ in } \Omega, \quad \text{with} \quad \psi_{|\partial\Omega} = 0. \tag{2.7}$$

Proposition 2.3. *Let (Ω, b) be a smooth lake in the sense of (2.1). Given $f \in L^2(\Omega)$, there exists a unique (distributional) solution $\psi \in X$ of the problem (2.7).*

Proof. Let us introduce the functional

$$E(\psi) := \int_\Omega \Big(\frac{1}{2b}|\nabla\psi|^2 + f\psi\Big)\, dx.$$

Since $f \in L^2$, the functional $E(\cdot)$ is well-defined on X. Let $\psi_k \in X$ be a minimizing sequence. Thanks to the Poincaré inequality and the fact that b is bounded, ψ_k is uniformly bounded in X. Up to a subsequence, we assume that $\psi_k \rightharpoonup \psi$ weakly in X. By the lower semi-continuity of the norm, it follows that

$$E(\psi) \leq \liminf_{k\to\infty} E(\psi_k).$$

Hence, $\psi \in X$ is indeed a minimizer. In addition, by minimization, the first variation of $E(\psi)$ reads

$$\int_\Omega \Big(\frac{1}{b}\nabla\varphi\cdot\nabla\psi + \varphi f\Big)\, dx = 0, \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{2.8}$$

which shows that ψ is a solution of (2.7). We recall that the Dirichlet boundary condition is encoded in the function space X. For the uniqueness, let us assume that $\psi \in X$ is a solution with $f \equiv 0$. Then, (2.8) simply reads $\langle\varphi, \psi\rangle_X = 0$, for arbitrary $\varphi \in C_c^\infty(\Omega)$. It follows by density (see Lemma 2.1) that $\|\psi\|_X = 0$ and so $\psi = 0$. This proves the uniqueness as claimed. □

Definition 2.4. *We say that Φ is a b-harmonic function if*

$$\Phi \in H^1_0(\widetilde{\Omega}), \quad b^{-1/2}\nabla\Phi \in L^2(\Omega),$$

where $\widetilde{\Omega}$ is as introduced in (H1), so that Φ solves the problem

$$\operatorname{div}\Big[\frac{1}{b}\nabla\Phi\Big] = 0 \text{ in } \Omega, \qquad \text{and } \partial_\tau\Phi = 0 \text{ on } \partial\Omega.$$

We denote by $\mathcal{H}_b$ the space of b-harmonic functions.

We remark that since a b-harmonic function Φ belongs to $H^1(\Omega)$, we can define its trace at the boundary, and so $\partial_\tau\Phi = 0$ should be understood as its trace being constant on each connected component of $\partial\Omega$.

Proposition 2.5. *Let (Ω, b) be a smooth lake in the sense of (2.1). For $1 \le k \le N$, there exists a unique b-harmonic function φ^k such that*

$$\varphi^k = 0 \text{ on } \partial\widetilde{\Omega}, \quad \varphi^k = \delta_{ik} \text{ on } \partial\mathcal{C}^i, \ \forall i = 1\dots N.$$

Moreover, the family $\{\varphi^k\}_{k=1..N}$ forms a basis for the set of b-harmonic functions.

Proof. Let $\delta = \frac{1}{10}\min_{i\neq j}\{\operatorname{dist}(\mathcal{C}^i,\mathcal{C}^j), \operatorname{dist}(\mathcal{C}^i,\partial\widetilde{\Omega})\}$. For each k, we introduce a cut-off function $\chi^k \in C^\infty_c(\widetilde{\Omega})$ which is supported in a δ-neighborhood of $\mathcal{C}^k$ and satisfies

$$\chi^k(x) = 0 \text{ if } d(x,\mathcal{C}^k) > \delta, \quad \chi^k(x) = 1 \text{ if } d(x,\mathcal{C}^k) < \delta/2. \tag{2.9}$$

In particular,

$$\chi^k = \delta_{ik} \text{ in a neighborhood of } \mathcal{C}^i, \ \forall i = 1\dots N.$$

By Proposition 2.3, there exists a unique solution $\tilde\varphi^k \in X$ to the problem

$$\operatorname{div}\Big[\frac{1}{b}\nabla\tilde\varphi^k\Big] = -\operatorname{div}\Big[\frac{1}{b}\nabla\chi^k\Big] \text{ in } \Omega, \quad \tilde\varphi^k = 0 \text{ on } \partial\Omega.$$

Indeed, since $\nabla\chi^k$ is smooth and vanishes near the boundaries, the right-hand side of the above problem clearly belongs to $L^2(\Omega)$. Now if we define

$$\varphi^k := \tilde\varphi^k + \chi^k, \tag{2.10}$$

the existence of a b-harmonic function φ^k follows at once as claimed.

The uniqueness follows from the uniqueness result in Proposition 2.3: indeed, let φ^1 and φ^2 be two b-harmonic functions which have the same trace on each component of $\partial\Omega$. Then, $\Phi := \varphi^1 - \varphi^2$ satisfies (2.7) with $f = 0$ and belongs to $H^1_0(\Omega)$ hence to $\Phi \in X$, which is the function space where the uniqueness of solutions to (2.7) was proved.

Finally, since any b-harmonic function by definition is constant on each connected component of $\partial\Omega$, it follows clearly that the family $\{\varphi^k\}_{1\le k\le N}$ forms a basis of $\mathcal{H}_b$. □

To recognize the divergence free condition (1.4), we need the following simple lemma:

Lemma 2.6. *Let (Ω, b) be a lake satisfying assumptions (H1)-(H2) (not necessarily smooth). Let $\psi \in X$, $c_k \in \mathbb{R}$ and $\chi^k \in C^\infty(\Omega)$ as introduced in (2.9). Then the vector function*

$$v := \frac{\nabla^\perp(\psi + \sum_{k=1}^N c_k\chi^k)}{b}$$

satisfies

$$\operatorname{div}(bv) = 0 \text{ in } \Omega, \quad bv\cdot\nu = 0 \text{ on } \partial\Omega \quad \text{weakly (in the sense of (1.4))}. \tag{2.11}$$

Conversely, let v be a vector field so that $bv \in L^2(\Omega)$ and (2.11) holds. Then there exists $\psi \in H^1_0(\widetilde{\Omega})$ such that

$$bv = \nabla^\perp\psi \text{ in } \overline{\Omega} \text{ and } \partial_\tau\psi = 0 \text{ on } \partial\Omega.$$

Proof. As $\psi \in X \subset H^1_0(\Omega)$, we can easily check that $\nabla^\perp \psi$ belongs to $\mathcal{H}(\Omega)$ (see (1.6)) and then satisfies (1.4) (by the equivalence proved in [3, Lemma III.2.1]). Moreover, since $\nabla^\perp \chi^k$ is smooth and vanishes in a neighborhood of the boundary, an integration by parts implies that $\nabla^\perp \chi^k$ verifies the boundary condition (1.4). This gives (2.11).

The second one is a classical statement which does not depend on the regularity of $\partial\Omega$. Indeed, as (1.4) is equivalent to (1.5) when $bv \in L^2(\Omega)$, we can find a sequence of divergence-free vector fields $v_n \in C^\infty_c(\Omega)$, such that $v_n \to bv$ in $L^2(\Omega)$. Then v_n is supported in a smooth set, and we can use the classical Hodge-De Rham theorem: $v_n = \nabla^\perp \psi_n$ where ψ_n is constant near the boundaries. Choosing ψ_n such that $\psi_n(x) \equiv 0$ in a neighborhood of $\partial\widetilde{\Omega}$, we then infer by Poincaré inequality that ψ_n is a Cauchy sequence in $H^1_0(\widetilde{\Omega})$ (where we have extending ψ_n in $\mathcal{C}^k$ by constant values). Passing to the limit, we infer that $bv = \nabla^\perp \psi$ for some $\psi \in H^1_0(\widetilde{\Omega})$ such that $\partial_\tau \psi \equiv 0$ on $\partial\Omega$. □

Remark 2.7. Let (Ω, b) be a smooth lake in the sense of (2.1). If $\psi \in X$ and Φ is a simili harmonic function, then $bv := \nabla^\perp(\psi + \Phi)$ verifies

$$\operatorname{div}(bv) = 0 \text{ in } \Omega, \quad bv \cdot \nu = 0 \text{ on } \partial\Omega, \quad \text{weakly (in the sense of (1.4))}.$$

Indeed, Proposition 2.5 states that there exists c_k such that $\Phi \equiv \sum_{k=1}^N c_k \varphi^k$, so using (2.10), we can decompose bv as $bv = \nabla^\perp(\tilde\psi + \sum_{k=1}^N c_k \chi^k)$ with $\tilde\psi \in X$. Then Lemma 2.6 can be applied.

With the regularity considered in Definition 2.4, it is not clear that $\int_{\partial\mathcal{C}^k} \frac{\nabla^\perp \Phi}{b} \cdot \hat\tau \, ds$ is well defined. Using χ^k defined as in (2.9), we introduce the generalized circulation of a vector field v around $\mathcal{C}^k$ by

$$\gamma^k(v) := \int_\Omega \operatorname{div}\left[\chi^k v^\perp\right] dx = -\int_\Omega \operatorname{curl}\left[\chi^k v\right] dx = -\int_\Omega \left(\nabla^\perp \chi^k \cdot v + \chi^k \operatorname{curl} v\right) dx. \tag{2.12}$$

Lemma 2.8. *Let (Ω, b) be a smooth lake in the sense of (2.1). If ψ is a b-harmonic function such that the generalized circulation of the vector field $\frac{\nabla^\perp\psi}{b}$ around each $\mathcal{C}^k$ is equal to zero for all k, then ψ must be identically zero.*

Proof. Set

$$v := \frac{1}{b}\nabla^\perp \psi.$$

We begin the proof with the following claim: there exists a function f such that

$$v = \nabla f. \tag{2.13}$$

We observe that

$$\operatorname{curl}(v) = \operatorname{div}\left(\frac{1}{b}\nabla\psi\right) = 0. \tag{2.14}$$

From Remark 2.7, we get that

$$\operatorname{div}(bv) = 0 \text{ in } \Omega, \quad bv \cdot \nu = 0 \text{ on } \partial\Omega, \quad \text{weakly (in the sense of (1.4))}.$$

As b is regular, by local elliptic regularity we deduce from (2.14) that v is a continuous function in Ω. Thus we can define the classical circulation $\oint v \cdot \tau ds$ along any closed path and we infer by the curl free property that this circulation does not depend on the homotopy class of the path. Next, choose c_k a closed curve supported in the region where $\chi^k = 0$ so that c_k is homotopic to $\partial\mathcal{C}^k$ (see (2.9) for the definition of χ^k). Let c'_k be another homotopic path supported in $\{x \in \Omega,\ \chi^k(x) = 1\}$. We let A^k be the region bounded by c_k and c'_k. Using (2.12), we then compute that

$$\int_{c_k} v \cdot \tau ds = \int_{c'_k} v \cdot \tau ds = \left(\int_{c'_k} - \int_{c_k}\right)(\chi^k v)\cdot \tau ds = -\int_{A^k} \operatorname{curl}(\chi^k v)\, dx = -\int_\Omega \operatorname{curl}(\chi^k v)\, dx = 0,$$

where we have used the fact that

$$\operatorname{curl}(\chi^k v) = \chi^k \operatorname{curl}(v) - v^\perp \cdot \nabla \chi^k$$

vanishes outside of A^k.

Since any smooth loop can be decomposed into a finite number of loops which are either homotopic to the trivial loop or homotopic to one of the c_k's, we have that the circulation of v along any closed curve vanishes. Therefore, fixing P an arbitrary point in Ω and letting

$$f(Q) = \int_{\gamma_{PQ}} v \cdot \tau ds$$

where γ_{PQ} is any (smooth) path from P to Q, we obtain (2.13).

As ψ is a b-harmonic function, we have that $\frac{\nabla^\perp \psi}{\sqrt{b}} = \sqrt{b} v = \sqrt{b} \nabla f$ belongs to $L^2(\Omega)$, hence,

$$\int_\Omega b|v|^2 \, dx = \int_\Omega \frac{\nabla^\perp \psi}{\sqrt{b}} \cdot \sqrt{b} \nabla f \, dx. \tag{2.15}$$

Moreover, by Proposition 2.5 and (2.10) we can decompose ψ as

$$\psi = \sum_{k=1}^N c_k \varphi^k = \psi^0 + \sum_{k=1}^N c_k \chi^k$$

where $\psi^0 \in X$. By density of $C_c^\infty(\Omega)$ in X (see Lemma 2.1) we have

$$\int_\Omega \frac{\nabla^\perp \psi^0}{\sqrt{b}} \cdot \sqrt{b} \nabla f \, dx = \lim_{n\to\infty} \int_\Omega \frac{\nabla^\perp \psi_n}{\sqrt{b}} \cdot \sqrt{b} \nabla f \, dx = \lim_{n\to\infty} \int_\Omega \nabla^\perp \psi_n \cdot \nabla f \, dx = 0$$

where we have integrated by parts and used that $\psi_n \in C_c^\infty(\Omega)$. Moreover, as χ^k is smooth and $\nabla \chi^k$ vanishes close to the boundary, we also have by an integration by parts:

$$\int_\Omega \frac{\nabla^\perp \chi^k}{\sqrt{b}} \cdot \sqrt{b} \nabla f \, dx = \int_\Omega \nabla^\perp \chi^k \cdot \nabla f \, dx = 0$$

for all k.

Putting together these two relations, (2.15) implies that v is equal to zero, from which we conclude that ψ is constant in Ω. Since $\psi \in H_0^1(\widetilde{\Omega})$, ψ vanishes in Ω as claimed. ☐

Proposition 2.9. *Let (Ω, b) be a smooth lake in the sense of (2.1). There exists a basis $\{\psi^k\}_{k=1}^N$ of $\mathcal{H}_b$ which satisfies*

$$\gamma^i\Big(\frac{\nabla^\perp \psi^k}{b}\Big) = \delta_{ik}, \qquad \forall\ i, k.$$

Proof. Consider the linear mapping

$$\Psi : \mathcal{H}_b \to \mathbb{R}^N, \quad \Psi(g) = (\gamma_1, \dots, \gamma_N), \quad \gamma_i := \gamma^i\Big(\frac{\nabla^\perp g}{b}\Big).$$

Lemma 2.8 states that Ψ is one-to-one and Proposition 2.5 implies that $\dim \mathcal{H}_b = N$. Consequently, Ψ is bijective and we can define $\psi^i = \Psi^{-1}(e_i)$ where e_i is the i-th vector in the canonical basis of $\mathbb{R}^N$. ☐

Proposition 2.10 (Decomposition)**.** *Let (Ω, b) be a smooth lake in the sense of (2.1). Let $\omega \in L^\infty(\Omega)$ and $\gamma_0 = (\gamma_0^1, \cdots, \gamma_0^N) \in \mathbb{R}^N$. Then there exists a unique vector field v such that $\sqrt{b} v \in L^2(\Omega)$,*

$$\operatorname{div}(bv) = 0 \text{ in } \Omega, \quad bv \cdot \nu = 0 \text{ on } \partial\Omega, \quad \text{weakly (in the sense of (1.4))} \tag{2.16}$$

and

$$\operatorname{curl}(v) = b\omega \quad \text{in} \quad \mathcal{D}'(\Omega), \qquad \gamma^i(v) = \gamma_0^i. \tag{2.17}$$

Moreover, we have the following Biot-Savart formula:

$$v = b^{-1} \nabla^\perp \psi^0 + \sum_{i=1}^N \alpha^i b^{-1} \nabla^\perp \psi^i, \tag{2.18}$$

where $\psi^i \in \mathcal{H}_b$ *is the function defined as in Proposition 2.9 above,* $\alpha^i = \gamma_0^i + \int_\Omega b\omega\varphi^i\,dx$, φ^i *defined as in Proposition 2.5, and* $\psi^0 \in X$ *the unique solution (see Proposition 2.3) of the problem*

$$\operatorname{div}\,(\frac{1}{b}\nabla\psi^0) = b\omega \quad \text{in } \mathcal{D}'(\Omega), \quad \psi^0 \in X.$$

Proof. We begin by showing that the vector field defined as

$$u := b^{-1}\nabla^\perp\psi^0 + \sum_{i=1}^N \Big(\gamma_0^i + \int_\Omega b\omega\varphi^i\,dx\Big) b^{-1}\nabla^\perp\psi^i$$

verifies (2.16)-(2.17). By definition, $\sqrt{b}u \in L^2(\Omega)$. The curl condition in (2.17) is obvious from the definitions of ψ^0 and the b-harmonic functions. Condition (2.16) comes from Remark 2.7. The hardest part is to compute the circulation of $b^{-1}\nabla^\perp\psi^0$. By the definition (2.10) of φ^i, we use that $\chi^i = \varphi^i - \tilde{\varphi}^i$ with $\tilde{\varphi}^i \in X$, to get:

$$\gamma^i(b^{-1}\nabla^\perp\psi^0) = -\int_\Omega \Big(\nabla^\perp\varphi^i\cdot b^{-1}\nabla^\perp\psi^0 + \varphi^i b\omega\Big)\,dx + \int_\Omega \operatorname{div}\Big[\tilde{\varphi}^i b^{-1}\nabla\psi^0\Big]\,dx. \tag{2.19}$$

Now, for any $\Phi \in C_c^\infty(\Omega)$, we note that

$$\int_\Omega \operatorname{div}\Big[\Phi b^{-1}\nabla\psi^0\Big]\,dx = 0,$$

and as $\tilde{\varphi}^i$ belongs to X and C_c^∞ is dense in X, we deduce from the fact that both $b^{-1/2}\nabla\psi^0$ and $b\omega$ belong to $L^2(\Omega)$ that

$$\int_\Omega \operatorname{div}\Big[\tilde{\varphi}^i b^{-1}\nabla\psi^0\Big]\,dx = \int_\Omega \Big[b^{-1/2}(\nabla\tilde{\varphi}^i - \nabla\Phi)\cdot b^{-1/2}\nabla\psi^0 - (\tilde{\varphi}^i - \Phi)b\omega\Big]\,dx = 0. \tag{2.20}$$

Concerning the first term on the right hand side of (2.19), as $\psi^0 \in X$ we can again use the density of C_c^∞ in X to state that the fact that φ^i is a b-harmonic function reads as

$$\begin{aligned}\int_\Omega \nabla^\perp\varphi^i\cdot b^{-1}\nabla^\perp\psi^0\,dx &= \int_\Omega b^{-1}\nabla\varphi^i\cdot\nabla\psi^0\,dx\\ &= \int_\Omega b^{-1/2}\nabla\varphi^i\cdot b^{-1/2}(\nabla\psi^0 - \nabla\Phi)\,dx + \int_\Omega b^{-1}\nabla\varphi^i\cdot\nabla\Phi\,dx = 0.\end{aligned}$$

Therefore, putting these last two equalities together with (2.19) gives that $\gamma^i(u) = -\int_\Omega b\omega\varphi^i\,dx + \Big(\gamma_0^i + \int_\Omega b\omega\varphi^i\,dx\Big) = \gamma_0^i$, which shows that u verifies (2.16)-(2.17).

To prove the uniqueness of u, let v be another vector field such that $\sqrt{b}v \in L^2(\Omega)$ and v satisfies (2.16)-(2.17).

By Lemma 2.6, there exists $\psi \in H_0^1(\widetilde{\Omega})$ such that

$$bv = \nabla^\perp\psi \text{ in } \mathcal{D}'(\overline{\Omega}), \quad \partial_\tau\psi = 0 \text{ on } \partial\Omega.$$

So

$$\Psi := \psi - \Big(\psi^0 + \sum_{i=1}^N \Big(\gamma_0^i + \int_\Omega b\omega\varphi^i\,dx\Big)\psi^i\Big)$$

is a b-harmonic function such that the circulation of $\frac{\nabla^\perp\Psi}{b}$ around each $\mathcal{C}^k$ is equal to zero. Lemma 2.8 gives that $v = u$, which ends the proof. □

2.2. **Existence of a global weak solution.** In this subsection we prove the existence of a global weak interior solution for the vorticity formulation:

Lemma 2.11. *Let* (Ω, b) *be a smooth lake in the sense of* (2.1). *Let* $\omega^0 \in L^\infty(\Omega)$ *and* $\{\gamma_0^i\}_{1\le i\le N}$ *fixed numbers and define* v^0 *by* (2.18). *Then there exists a global weak interior solution to the lake equations on* (Ω, b) *in the vorticity formulation (see Definition 1.2). Moreover, the circulations of this solution are conserved.*

The original idea comes from Yudovich: we introduce an artificial viscosity $\varepsilon \mathrm{div}\,(b\nabla\omega_\varepsilon)$ in the vorticity equations, assuming the Dirichlet condition for the vorticity at the boundary. This viscosity is artificial, because of the boundary condition: in the (physically relevant) Navier-Stokes equations, the Dirichlet condition is given on v, which does not imply the Dirichlet condition on the vorticity. Although the inviscid problem in Navier-Stokes equations is a hard issue, the limit problem $\varepsilon \to 0$ with the artificial viscosity is possible to achieve. Actually, to use directly a result from [9], we consider $b_\varepsilon := b+\varepsilon \geq \varepsilon > 0$, and we approximate ω^0 by $\omega^0_\varepsilon \in C^\infty_c$ such that $\|\omega^0_\varepsilon\|_{L^\infty} \leq 2\|\omega^0\|_{L^\infty}$. As b_ε is strictly positive, standard arguments for Navier-Stokes equations [9] gives the existence and uniqueness of a global solution

$$\omega_\varepsilon \in C([0,\infty); H^1_0(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty); H^2(\Omega))$$

of the problem (in the sense of distribution)

$$\begin{cases} \partial_t(b_\varepsilon\omega_\varepsilon) + b_\varepsilon v_\varepsilon \cdot \nabla\omega_\varepsilon - \varepsilon \mathrm{div}\,(b_\varepsilon\nabla\omega_\varepsilon) = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega, \\ v_\varepsilon = \dfrac{1}{b_\varepsilon}\nabla^\perp\psi^0_\varepsilon[\omega_\varepsilon] + \displaystyle\sum_{i=1}^N \frac{\gamma^i_0 + \int_\Omega b_\varepsilon\omega_\varepsilon\varphi^i_\varepsilon\,dx}{b_\varepsilon}\nabla^\perp\psi^i_\varepsilon, & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega, \\ \omega_\varepsilon = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ \omega_\varepsilon(0,x) = \omega^0_\varepsilon(x) & \text{for } x \in \Omega, \end{cases} \tag{2.21}$$

where $\varepsilon > 0$ is arbitrary and γ^i_0 are given independently of ε and t. The above system is exactly the problem studied in [9]. Indeed the authors work in non-simply connected domains, and Lemma 5 therein is similar to our decomposition (Proposition 2.10). In their case, the tangential part $v \cdot \tau$ is clearly defined (as $b_\varepsilon > 0$) so their definition of the circulation as an integral along $\partial\mathcal{C}^k$ is the same as our generalized circulation. In this work, the test functions are compactly supported in Ω: $\varphi \in C^\infty_c([0,\infty)\times\Omega)$. Indeed, for Navier-Stokes equations, the general framework is of H^{-1} to $H^1_0(\Omega)$ and test functions in $C^\infty_c([0,\infty)\times\Omega)$ are sufficient because the Dirichlet boundary condition is already encoded by the fact that the velocity (here the vorticity) belongs to H^1_0. Moreover, we have the "energy relation":

$$\|\sqrt{b_\varepsilon}\omega_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)} + \varepsilon\int_0^t \|\sqrt{b_\varepsilon}\nabla\omega_\varepsilon(s,\cdot)\|^2_{L^2(\Omega)}\,ds \leq \|\sqrt{b_\varepsilon}\omega^0_\varepsilon\|^2_{L^2(\Omega)}, \quad \forall t \geq 0. \tag{2.22}$$

Next, the idea is to pass to the limit $\varepsilon \to 0$. Let us perform this limit as follows:

- by integration by parts and Poincaré inequality on $\widetilde{\Omega}$ we have that (thanks to the tangency condition of v_ε):

$$\begin{aligned} \|\sqrt{b_\varepsilon}v_\varepsilon\|^2_{L^2(\Omega)} &\leq \|b_\varepsilon\omega_\varepsilon\|_{L^2(\Omega)}\|\psi_\varepsilon\|_{L^2(\Omega)} \leq C_2\sqrt{M+\varepsilon}\|\sqrt{b_\varepsilon}\omega^0_\varepsilon\|_{L^2(\Omega)}\|\nabla\psi_\varepsilon\|_{L^2(\Omega)} \\ &\leq 2C_2|\Omega|^{\frac12}(M+1)^{3/2}\|\omega^0\|_{L^\infty(\Omega)}\|\sqrt{b_\varepsilon}v_\varepsilon\|_{L^2(\Omega)}, \end{aligned}$$

  where ψ_ε is the stream function vanishing on $\partial\widetilde{\Omega}$ associated to $b_\varepsilon v_\varepsilon$:

$$\psi_\varepsilon := \psi^0_\varepsilon[\omega_\varepsilon] + \sum_{i=1}^N (\gamma^i_0 + \int_\Omega b_\varepsilon\omega_\varepsilon\varphi^i_\varepsilon\,dx)\psi^i_\varepsilon.$$

  Hence $\sqrt{b_\varepsilon}v_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\Omega))$, uniformly in ε:

$$\|\sqrt{b_\varepsilon}v_\varepsilon(t)\|_{L^2} \lesssim 1. \tag{2.23}$$

- for ε fixed, since $b_\varepsilon \geq \epsilon > 0$, we observe that $\partial_t\omega_\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-1}(\Omega))$ and also that $\omega_\varepsilon \in C(\mathbb{R}_+; L^2(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\Omega))$. Hence one can multiply the vorticity equation by some power of ω_ε to get for all time, and for all $p \geq 1$,

$$\|(b_\varepsilon)^{\frac1p}\omega_\varepsilon(t,\cdot)\|_{L^p} \leq \|(b_\varepsilon)^{\frac1p}\omega^0_\varepsilon\|_{L^p} \leq (M+1)^{\frac1p}\|\omega^0_\varepsilon\|_{L^p} \leq 2[(M+1)(|\Omega|+1)]^{\frac1p}\|\omega^0\|_{L^\infty}.$$

  As the constant at the right hand side is uniform in p, we infer that

$$\|\omega_\varepsilon(t,\cdot)\|_{L^\infty} \leq 2\|\omega^0\|_{L^\infty}. \tag{2.24}$$

Therefore, Banach-Alaoglu theorem implies that, up to a subsequence, ω_ε converges weak-$*$ to ω in $L^\infty(\mathbb{R}_+ \times \Omega)$, and $\sqrt{b_\varepsilon}v_\varepsilon$ converges weak-$*$ to $\sqrt{b}v$ in $L^\infty(\mathbb{R}_+; L^2(\Omega))$, as $\varepsilon \to 0$. This weak convergence is sufficient to get (i), (ii) and (iii) in Definition 1.2. Moreover, by construction (see (2.21)), $\gamma^i(v_\varepsilon(t)) = \gamma^i_0$ for all $t \in \mathbb{R}_+$, $i = 1 \dots N$. Hence, the weak limit is also sufficient to pass to the limit in the circulation definition (2.12) which implies that the circulations of v is conserved.

To get (iv), we will pass to the limit in equation (2.21), but we need a strong convergence of the velocity.

Without loss of generality, we may restrict ourselves to $\mathcal{O}$ a smooth simply connected open subset of Ω such that $\overline{\mathcal{O}} \subset \Omega$. We introduce the Leray projector $\mathbb{P}_\mathcal{O}$ from $L^2(\mathcal{O})$ to $\mathcal{H}(\mathcal{O})$ (see (1.6) for the definition), i.e. $\mathbb{P}_\mathcal{O}$ is the unique operator such that

$$v_\varepsilon = \mathbb{P}_\mathcal{O} v_\varepsilon + \nabla q_\varepsilon, \quad \operatorname{div}(\mathbb{P}_\mathcal{O} v_\varepsilon) = 0, \quad (\mathbb{P}_\mathcal{O} v_\varepsilon)\cdot \nu|_{\partial\mathcal{O}} = 0. \tag{2.25}$$

All the details about the Leray projector can be found e.g. in [3]. In particular, it is known that such a projector is orthogonal in L^2, and so

$$\|\mathbb{P}_\mathcal{O} v_\varepsilon\|^2_{L^2(\mathcal{O})} + \|\nabla q_\varepsilon\|^2_{L^2(\mathcal{O})} \le \|v_\varepsilon\|^2_{L^2(\mathcal{O})} \le \theta_\mathcal{O}^{-1}\|\sqrt{b_\varepsilon}v_\varepsilon\|^2_{L^2(\Omega)}$$

for some positive constant $\theta_\mathcal{O}$ (see (H2) for the last inequality, because $\overline{\mathcal{O}} \subset \Omega$ is a compact subset of Ω). This together with (2.23) yields that, up to a subsequence, ∇q_ε and $\mathbb{P}_\mathcal{O} v_\varepsilon$ converge weak-$*$ in $L^\infty([0,T]; L^2(\mathcal{O}))$ to ∇q and $\mathbb{P}_\mathcal{O} v$, with $v = \mathbb{P}_\mathcal{O} v + \nabla q$, as $\varepsilon \to 0$. Besides, since $\operatorname{curl}(\mathbb{P}_\mathcal{O} v_\varepsilon) = \operatorname{curl}(v_\varepsilon) = b_\varepsilon \omega_\varepsilon$ is uniformly bounded in L^∞ and using (2.25), we see that $\{\mathbb{P}_\mathcal{O} v_\varepsilon(t)\}$ always remains inside a compact set of $L^2(\mathcal{O})$ (indeed, by the standard Calderón-Zygmund theorem on $\mathcal{O}$, $\sum_j \|\partial_j \mathbb{P}_\mathcal{O} v_\varepsilon(t)\|_{L^2} \lesssim \|\operatorname{curl}(v_\varepsilon(t))\|_{L^2}$).

As (2.21) is verified for test functions $\varphi \in C^\infty_c(\mathcal{O})$ with $\mathcal{O}$ a simply connected smooth domain, like in Proposition A.2 in Appendix A we infer that we have a velocity type equation:

$$-\int_0^\infty \int_\Omega \Phi \cdot \partial_t v_\varepsilon \, dxdt + \int_0^\infty \int_\Omega \omega_\varepsilon b_\varepsilon v_\varepsilon \cdot \Phi^\perp \, dxdt - \varepsilon \int_0^\infty \int_\Omega b_\varepsilon \nabla \omega_\varepsilon \cdot \Phi^\perp \, dxdt = 0$$

for all divergence-free $\Phi \in C^\infty_c(\mathcal{O})$. For such a test function, using (2.22), (2.23) and (2.24), we obtain that

$$\begin{aligned}
|\langle \mathbb{P}_\mathcal{O} v_\varepsilon(t_2), \Phi\rangle - \langle \mathbb{P}_\mathcal{O} v_\varepsilon(t_1), \Phi\rangle| &= |\langle v_\varepsilon(t_2), \Phi\rangle - \langle v_\varepsilon(t_1), \Phi\rangle| \\
&= \left| \int_{t_1}^{t_2} \int_\Omega b_\varepsilon \omega_\varepsilon v_\varepsilon \cdot \Phi^\perp \, dxdt - \varepsilon \int_{t_1}^{t_2} \int_\Omega b_\varepsilon \Phi^\perp \cdot \nabla \omega_\varepsilon \, dxdt \right| \\
&\lesssim \|\Phi\|_{L^2} \big[\sqrt{M+1} \|\sqrt{b_\varepsilon} v_\varepsilon\|_{L^\infty_t L^2} \|\omega_\varepsilon\|_{L^\infty_{t,x}} |t_1 - t_2| \\
&\quad + \sqrt{\varepsilon(M+1)} \|\sqrt{\varepsilon b_\varepsilon} \nabla \omega_\varepsilon\|_{L^2_{t,x}} |t_1 - t_2|^{\frac12} \big] \\
&\lesssim \|\Phi\|_{L^2} C(\omega^0) \left[|t_1 - t_2| + \sqrt{\varepsilon} |t_1 - t_2|^{\frac12} \right].
\end{aligned}$$

By density, we note that the above estimates remain valid for $\Phi \in \mathcal{H}(\mathcal{O})$. However, $\mathbb{P}_\mathcal{O}$ is an orthogonal projector, hence for any $\Phi \in L^2(\mathcal{O})$, we can write that

$$\langle \mathbb{P}_\mathcal{O} v_\varepsilon(t), \Phi\rangle = \langle \mathbb{P}_\mathcal{O} v_\varepsilon(t), \mathbb{P}_\mathcal{O}\Phi\rangle.$$

Thus, for any $\Phi \in L^2(\mathcal{O})$, since $\mathbb{P}_\mathcal{O}\Phi \in \mathcal{H}(\mathcal{O})$, there holds

$$\begin{aligned}
|\langle \mathbb{P}_\mathcal{O} v_\varepsilon(t_2), \Phi\rangle - \langle \mathbb{P}_\mathcal{O} v_\varepsilon(t_1), \Phi\rangle| &\lesssim \|\mathbb{P}_\mathcal{O}\Phi\|_{L^2} C(\omega^0) \left[|t_1 - t_2| + \sqrt{\varepsilon} |t_1 - t_2|^{\frac12} \right] \\
&\lesssim \|\Phi\|_{L^2} C(\omega^0) \left[|t_1 - t_2| + \sqrt{\varepsilon} |t_1 - t_2|^{\frac12} \right]
\end{aligned}$$

which implies that the family $\{\mathbb{P}_\mathcal{O} v_\varepsilon\}$ is equicontinuous in $L^2(\mathcal{O})$. Since we have seen that it takes values in a compact set, the Arzela-Ascoli theorem gives us the precompactness of $\{\mathbb{P}_\mathcal{O} v_\varepsilon\}$ in $C([0,T]; L^2(\mathcal{O}))$.

Finally, we can now pass to the limit in (2.21). We recall that for any $\varphi \in C^\infty_c([0,T) \times \mathcal{O})$, the first equation in (2.21) reads

$$0 = \int_0^\infty \int_\Omega \left[\varphi_t b_\varepsilon \omega_\varepsilon + b_\varepsilon \omega_\varepsilon v_\varepsilon \cdot \nabla\varphi - \varepsilon b_\varepsilon \nabla \omega_\varepsilon \cdot \nabla \varphi \right] dxdt + \int_\Omega \varphi(0,x) b_\varepsilon \omega_\varepsilon(x) \, dx. \tag{2.26}$$

Clearly, thanks to (2.22), we can pass to the limit as $\varepsilon \to 0$ in all the (linear) terms except the nonlinear term: $b_\varepsilon \omega_\varepsilon v_\varepsilon \cdot \nabla \varphi$. For the remaining term, using the relation (A.1), we get

$$\begin{aligned}\int_0^\infty \int_\Omega b_\varepsilon \omega_\varepsilon v_\varepsilon \cdot \nabla \varphi \, dxdt &= \int_0^\infty \int_\Omega (\operatorname{curl} v_\varepsilon) v_\varepsilon^\perp \cdot \nabla^\perp \varphi \, dxdt \\ &= \int_0^\infty \int_\Omega \Big[\operatorname{div}(b_\varepsilon v_\varepsilon \otimes v_\varepsilon) \cdot \frac{\nabla^\perp \varphi}{b_\varepsilon} - \frac{1}{2} \nabla |v_\varepsilon|^2 \cdot \nabla^\perp \varphi \Big] \, dxdt \\ &= \int_0^\infty \int_\Omega \operatorname{div}(b_\varepsilon v_\varepsilon \otimes v_\varepsilon) \cdot \frac{\nabla^\perp \varphi}{b_\varepsilon} \, dxdt.\end{aligned}$$

In addition, we can write

$$b_\varepsilon v_\varepsilon \otimes v_\varepsilon = b_\varepsilon \mathbb{P}_{\mathcal{O}} v_\varepsilon \otimes v_\varepsilon + b_\varepsilon \nabla q_\varepsilon \otimes \mathbb{P}_{\mathcal{O}} v_\varepsilon + b_\varepsilon \nabla q_\varepsilon \otimes \nabla q_\varepsilon,$$

in which $\mathbb{P}_{\mathcal{O}}$ is the Leray projector defined as above. The integration involving the first two terms on the right hand side converges to its limit by taking integration by parts and using a weak-strong convergence argument. For the last term, we further compute:

$$\operatorname{div}[b_\varepsilon \nabla q_\varepsilon \otimes \nabla q_\varepsilon] = \frac{b_\varepsilon}{2} \nabla(|\nabla q_\varepsilon|^2) + (\nabla b_\varepsilon \cdot \nabla q_\varepsilon + b_\varepsilon \Delta q_\varepsilon) \nabla q_\varepsilon.$$

Here we note from (2.25) that $\operatorname{div} v_\varepsilon = \Delta q_\varepsilon$, and as $\operatorname{div} b_\varepsilon v_\varepsilon = 0$, we get that $\Delta q_\varepsilon = -\frac{\nabla b_\varepsilon}{b_\varepsilon} \cdot v_\varepsilon$. Hence, we have

$$\operatorname{div}[b_\varepsilon \nabla q_\varepsilon \otimes \nabla q_\varepsilon] = \frac{b_\varepsilon}{2} \nabla(|\nabla q_\varepsilon|^2) - (\nabla b_\varepsilon \cdot \mathbb{P}_{\mathcal{O}} v_\varepsilon) \nabla q_\varepsilon.$$

This yields

$$\int_0^\infty \int_\Omega \operatorname{div}(b_\varepsilon v_\varepsilon \otimes v_\varepsilon) \cdot \frac{\nabla^\perp \varphi}{b_\varepsilon} \, dxdt = \int_0^\infty \int_\Omega \Big[\frac{1}{2} \nabla(|\nabla q_\varepsilon|^2) \cdot \nabla^\perp \varphi + (\nabla b_\varepsilon \cdot \mathbb{P}_{\mathcal{O}} v_\varepsilon) \nabla q_\varepsilon \cdot \frac{\nabla^\perp \varphi}{b_\varepsilon} \Big] \, dxdt$$

in which the first integral vanishes, whereas the second integral passes to the limit again by a weak-strong convergence argument.

By putting these into (2.26) and using the same algebra as just performed, it follows in the limit that

$$\int_0^\infty \int_\Omega \varphi_t b\omega \, dxdt + \int_0^\infty \int_\Omega \nabla \varphi \cdot v b\omega \, dxdt + \int_\Omega \varphi(0,x) b\omega^0(x) \, dx = 0 \tag{2.27}$$

for all $\varphi \in C_c^\infty([0,T) \times \mathcal{O})$. Recall that $\mathcal{O}$ was an arbitrary smooth simply connected domain in Ω. This proves that the identity (2.27) holds for all $\varphi \in C_c^\infty([0,\infty) \times \Omega)$.

To conclude, we have shown that (v, ω) is an interior weak solution of the lake equations in the vorticity formulation, which completes the proof of Lemma 2.11.

2.3. **Well-posedness of a global weak solution.** In this subsection, we use the Calderón-Zygmund inequality (2.28), below (see also Bresch and Métivier [1]), to upgrade our solution (v, ω) to a weak solution in the vorticity formulation, which is then equivalent to a weak solution in the velocity formulation. Using again the Calderón-Zygmund inequality, we prove that weak solutions in the velocity formulation are unique, which ends the proof of Theorem 1.3.

Gain of regularity for smooth lakes. First, we recall the main result of Bresch and Métivier in [1]: if the lake is smooth with constant slopes, then we have a Calderón-Zygmund type inequality. Namely:

Proposition 2.12 ([1, Theorem 2.3])**.** *Let (Ω, b) be a smooth lake in the sense of (2.1). Let $f \in L^p(\Omega)$ for $p > 4$ and $bv \in L^2(\Omega)$. If the triplet (b, v, f) verifies the following elliptic problem*

$$\operatorname{div}(bv) = 0 \text{ in } \Omega, \quad bv \cdot \nu = 0 \text{ on } \partial\Omega, \quad \text{weakly (in the sense of (1.4))}$$

and

$$\operatorname{curl} v = f \text{ in } \mathcal{D}'(\Omega),$$

then $v \in C^{1-\frac{2}{p}}(\overline{\Omega})$ *and* $\nabla v \in L^p(\Omega)$. *Moreover, there exists a constant* C *which depends only on* Ω *and* b *so that for any* $p > 4$

$$\|\nabla v\|_{L^p(\Omega)} \leq Cp\Big(\|f\|_{L^p(\Omega)} + \|bv\|_{L^2(\Omega)}\Big). \tag{2.28}$$

In addition,

$$v \cdot \nu = 0 \text{ on } \partial\Omega.$$

This inequality is well known in the case of non-degenerate topography ($b \geq \theta_0 > 0$) and it was extended by Bresch and Métivier in the case of a depth which vanishes at the shore like $d(x)^a$ for $a > 0$. The authors decompose the domain in two pieces: one which is far from the boundary where they use classical elliptic estimates, and one near the boundary. As for the latter piece, they flatten the boundary and reduce the problem to study a degenerate elliptic equation with coefficients vanishing at the boundary of a half-plane.

This decomposition in several subdomains explains why we have the terms $\|bv\|_{L^2}$ in the right hand side part of the Calderón-Zygmund inequality (2.28), coming from the support of the gradient of some cut-off functions. We remark also that we can easily have some islands with vanishing (where a_k can be different from a_0) or non vanishing topography, which gives a lake where the Calderón-Zygmund inequality holds true.

By Lemma 2.11 there exists (v, ω) verifying the elliptic problem ii)-iii) in Definition 1.2. Then, Proposition 2.12 states that ∇v belongs to L^p for any $p > 4$. This estimate is crucial to prove that (v, ω) is actually a global weak solution to the vorticity formulation (Proposition A.5), which is also a global weak solution to the velocity formulation (Proposition A.4), because the circulations are conserved. The Calderón-Zygmund inequality will be also the key for the uniqueness. Let us also note that we deduce in the proof of Proposition A.4 that the regularity of v implies the equivalence between the notions of generalized circulation and classical circulation.

By using the renormalized solutions in the sense of DiPerna-Lions, it follows that $\omega \in C([0,\infty); L^p(\Omega))$ and $v \in C([0,\infty); W^{1,p}(\Omega))$ for any $p > 4$ (see the proof of Lemma 3.1 for details about the renormalized theory).

Uniqueness. The uniqueness part now follows from the celebrated proof of Yudovich [13]. Let v_1 and v_2 be two weak global solutions for the same initial v^0. We introduce $\tilde{v} := v_1 - v_2$. As $\tilde{v}$ belongs to $W^{1,p}$ for any $p \in (4,\infty)$, we get from the velocity formulation some estimates for $\partial_t \tilde{v}$. This allows us to replace the test function by $b\tilde{v} = \mathbb{P}_\Omega(b\tilde{v}) \in C^1([0,T]; L^{\frac{5}{4}}(\Omega))$. As $\tilde{v} \in C\left(\mathbb{R}_+, L^5(\mathbb{R}^2)\right)$, we get for all $T \in \mathbb{R}^+$

$$\|\sqrt{b}\tilde{v}(T)\|^2_{L^2(\Omega)} = 2\int_0^T \langle \partial_t(b\tilde{v}), \tilde{v}\rangle_{L^{\frac{5}{4}} \times L^5}\, ds \leq 2\int_0^T \int_\Omega |\sqrt{b}\tilde{v}(s,x)||\nabla v_2(s,x)||\sqrt{b}\tilde{v}(s,x)|\, dxds$$

where we have used that $\operatorname{div} bv_1 = \operatorname{div} b\tilde{v} = 0$. Next, we use the Calderón-Zygmund inequality (2.28) on ∇v_2 to infer by interpolation that

$$\|\sqrt{b}\tilde{v}(T,\cdot)\|^2_{L^2} \leq 2Cp\int_0^T \|\sqrt{b}\tilde{v}\|^{2-2/p}_{L^2}\, dt.$$

Together with a Gronwall-like argument, this implies

$$\|\sqrt{b}\tilde{v}(T,\cdot)\|^2_{L^2} \leq (2CT)^p, \qquad \forall p \geq 2.$$

Letting p tend to infinity, we conclude that $\|\sqrt{b}\tilde{v}(T,\cdot)\|_{L^2} = 0$ for all $T < 1/(2C)$. Finally, we consider the maximal interval of $[0,\infty)$ on which $\|\sqrt{b}\tilde{v}(T,\cdot)\|_{L^2} \equiv 0$, which is closed by continuity of $\|\sqrt{b}\tilde{v}(T,\cdot)\|_{L^2}$. If it is not equal to the whole of $[0,\infty)$, we may repeat the above proof, which leads to a contradiction by maximality. Therefore uniqueness holds on $[0,\infty)$, and this concludes the proof of well-posedness.

Constant circulation. If the domain is not simply connected, we have proved in the first subsection that the vorticity alone is not sufficient to determine the velocity uniquely, and that we need to fix the generalized circulation to derive the Biot-Savart law. In the following section, the main idea is to prove compactness in each terms in this Biot-Savart law. Therefore, it is crucial to establish Kelvin's theorem in our case, namely that the generalized circulations are conserved. Fortunately, this is valid in great generality following Proposition 2.13 as follows.

Proposition 2.13. *Let (Ω, b) be a lake satisfying (H1)-(H2) with $b \in W^{1,\infty}_{\rm loc}(\Omega)$. Let v be a global interior weak solution of the velocity formulation and a global weak solution of the vorticity formulation. Then for each $k = 1, \cdots, N$, the generalized circulation γ^k defined as in (2.12) is independent of t.*

Proof. Let $l(t) \in C^\infty_c([0,\infty))$, note that since $\nabla^\perp \chi^k \equiv 0$ in a neighborhood of the boundary, then $l(t)\nabla^\perp\chi^k(x)$ is a test function for which the velocity equation (1.8) is verified. As χ^k is constant in each neighborhood of the boundary, $l(t)\chi^k(x)$ is a test function for which the vorticity equation (1.11) holds. Then, we can compute

$$\begin{aligned}\int_{\mathbb{R}} \gamma^k(t)\frac{d}{dt}l(t)dt - \gamma^k(0)l(0) &= -\int_{\mathbb{R}}\int_\Omega \frac{d}{dt}\Big[l(t)\nabla^\perp\chi^k\Big]\cdot v\,dxdt - \int_{\mathbb{R}}\int_\Omega \frac{d}{dt}\Big[l(t)\chi^k\Big]\,b\omega\,dxdt - \gamma^k(0)l(0)\\ &= \int_{\mathbb{R}}\int_\Omega \Big\{(bv\otimes v):\nabla\Big[\frac{1}{b}\nabla^\perp\chi^k\Big] + \nabla\chi^k\cdot v\,\mathrm{curl}(v)\Big\}\,l(t)\,dxdt\\ &\quad + l(0)\int_\Omega \Big\{v^0\cdot\nabla^\perp\chi^k + \chi^k\mathrm{curl}(v^0)\Big\}\,dx - \gamma^k(0)l(0)\\ &= \int_{\mathbb{R}}\int_\Omega \Big\{(bv\otimes v):\nabla\Big[\frac{1}{b}\nabla^\perp\chi^k\Big] + \nabla\chi^k\cdot v\,\mathrm{curl}(v)\Big\}\,l(t)\,dxdt.\end{aligned}$$

Using the fact that $\mathrm{div}(bv) = 0$ and $\nabla^\perp\chi^k \equiv 0$ in a neighborhood of the boundary, we may integrate by parts and use (A.1) to have

$$\int_{\mathbb{R}} \gamma^k(t)\frac{d}{dt}l(t)dt - \gamma^k(0)l(0) = -\int_{\mathbb{R}} l(t)\int_\Omega \nabla^\perp\chi^k\cdot\nabla\frac{|v|^2}{2}\,dxdt.$$

Now, we let $\widetilde{\chi}^k$ be a smooth function, compactly supported inside Ω and such that $\widetilde{\chi}^k\nabla\chi^k = \nabla\chi^k$. Integrating by parts, we then find that

$$\int_\Omega \nabla^\perp\chi^k\cdot\nabla\frac{|v|^2}{2}\,dx = \int_\Omega \nabla^\perp\chi^k\cdot\nabla\Big[\widetilde{\chi}^k\frac{|v|^2}{2}\Big]\,dx = -\int_\Omega \mathrm{div}\Big\{\nabla^\perp\chi^k\Big\}\,\widetilde{\chi}^k\frac{|v|^2}{2}\,dx = 0.$$

This finishes the proof. □

3. Proof of the convergence

In this section, we shall prove our main result (Theorem 1.5). Here, we recall our main assumption that (Ω_n, b_n) converges to the lake (Ω, b) as $n \to \infty$ in the sense of Definition 1.4. Let us denote by D a large open ball such that D contains Ω and Ω_n, and extend the bottom functions b and b_n to zero on the sets $D \setminus \Omega$ and $D \setminus \Omega_n$, respectively.

We prove the main theorem via several steps. First, from the velocity equation, it is relatively easy to obtain an a priori bound on $\sqrt{b_n}v_n$ in $L^\infty(\mathbb{R}_+; L^2)$ (here one needs uniform estimates on $\sqrt{b_n}v_n^0$ in $L^2(\Omega_n)$). Unfortunately, such a bound is too weak to give any reasonable information on the possible limiting velocity solution v. To obtain sufficient compactness, we derive estimates on the stream function ψ_n, defined by

$$v_n = \frac{1}{b_n}\nabla^\perp\psi_n. \tag{3.1}$$

The Biot-Savart law (2.18) which is established in Proposition 2.10 gives

$$\psi_n(t,x) = \psi_n^0(t,x) + \sum_{k=1}^N \alpha_n^k(t)\psi_n^k(x), \tag{3.2}$$

where for each n, ψ_n^0 solves $\mathrm{div}\,(b_n^{-1}\nabla\psi_n^0) = b_n\omega_n$ with the Dirichlet boundary condition on $\partial\Omega_n$, and the so-called b-harmonic functions ψ_n^k solve $\mathrm{div}\,(b_n^{-1}\nabla\psi_n^k) = 0$ and have their circulations equal to δ_{jk} around each island $\mathcal{C}_n^j$, $j = 1, \cdots, N$. The real numbers $\alpha_n^k(t)$ are given by

$$\alpha_n^k(t) = \gamma_n^k + \int_\Omega b_n(x)\omega_n(t,x)\varphi_n^k(x)\,dx$$

where $\gamma_n^k = \gamma^k(v_n)$ is the circulation of v_n around each $\mathcal{C}_n^k$ introduced as in (2.12), which is constant in time (see Proposition 2.13).

3.1. **Vorticity estimates.** We begin by deriving some basic estimates on the vorticity ω_n.

Lemma 3.1. *For each n, the L^p norm of $b_n^{1/p}\omega_n$ is conserved in time and uniformly bounded for all $p \geq 1$, that is,*

$$\|b_n^{1/p}\omega_n(t)\|_{L^p(\Omega_n)} = \|b_n^{1/p}\omega_n^0\|_{L^p(\Omega_n)} \lesssim \|\omega_n^0\|_{L^\infty} \lesssim 1, \qquad \forall t \geq 0.$$

In addition, ω_n is bounded in $L^\infty_{x,t}$, uniformly in n.

Proof. We recall that the vorticity ω_n solves (1.3) in the distributional sense and belongs to $L^\infty(\mathbb{R}_+ \times \Omega_n)$. Thanks to Proposition 2.12 we deduce that the velocity is regular enough to apply the renormalized theory in the sense of DiPerna-Lions: let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$|f'(s)| \leq C(1 + |s|^p), \qquad \forall t \in \mathbb{R},$$

for some $p \geq 0$, then $f(\omega_n)$ is a solution of the transport equation (1.3) (in the sense of distribution) with initial datum $f(\omega_0)$.

By smooth approximation of $s \mapsto |s|^p$ for $1 \leq p < \infty$, the renormalized solutions yields

$$\frac{d}{dt}(b_n|\omega_n|^p) = -b_n v_n \nabla|\omega_n|^p = -\mathrm{div}\,(b_n v_n|\omega_n|^p).$$

Integrating this identity over Ω_n and using the Stokes theorem, we get

$$\frac{d}{dt}\int_{\Omega_n} b_n(x)|\omega_n|^p(t,x)\,dx = -\int_{\partial\Omega_n} b_n v_n \cdot \nu|\omega_n|^p d\sigma_n(x),$$

where the boundary term vanishes due to the boundary condition on the velocity (see (1.1)). The lemma is proved for $1 \leq p < \infty$.

The case $p = \infty$ is easily obtained by taking f a function vanishing on the interval $[-\|\omega_0\|_{L^\infty}, \|\omega_0\|_{L^\infty}]$ and strictly positive elsewhere. Indeed, it shows that the L^∞ norm cannot increase, and by time reversibility that it is constant. ☐

Lemma 3.1 in particular yields that the vorticity ω_n is bounded in $L^\infty(\Omega_n)$ and, after extending ω_n by 0 in $D \setminus \Omega_n$, by the Banach-Alaoglu theorem, we can extract a subsequence such that

$$\begin{aligned} b_n^{1/p}\omega_n &\rightharpoonup b^{1/p}\omega \quad \text{weakly-}* \text{ in} \quad L^\infty(\mathbb{R}_+; L^p(D)) \\ \omega_n &\rightharpoonup \omega \quad \text{weakly-}* \text{ in} \quad L^\infty(\mathbb{R}_+ \times D). \end{aligned}$$

3.2. **b-harmonic functions: Dirichlet case.** We now derive estimates for the b-harmonic solutions φ_n^k, $k = 1, \cdots, N$. We recall that φ_n^k vanishes on the outer boundary $\partial\tilde{\Omega}_n$ and solves

$$\begin{cases} \mathrm{div}\left[\dfrac{1}{b_n}\nabla\varphi_n^k\right] = 0, & \text{in} \quad \Omega_n \\ \varphi_n^k = \delta_{jk}, & \text{on} \quad \partial\mathcal{C}_n^j, \qquad j = 1, \cdots, N. \end{cases} \tag{3.3}$$

The existence and uniqueness of φ_n^k was established in Proposition 2.5. We obtain the following.

Lemma 3.2. *The sequence $b_n^{-1/2}\nabla\varphi_n^k$ converges strongly to $b^{-1/2}\nabla\varphi^k$ in $L^2(D)$. In particular, φ_n^k is uniformly bounded in $H^1(D)$ and $b_n^{-1/2}\nabla\varphi_n^k$ is uniformly bounded in $L^2(D)$.*

In this statement and in all the sequel, $b_n^{-1/2}\nabla\varphi_n^k$ is extended by zero on $D \setminus \Omega_n$.

Proof. We first prove the boundedness and obtain convergence as a result of the convergence of the norm. As before, it is convenient to write, as in Proposition 2.5,

$$\varphi_n^k = \tilde{\varphi}_n^k + \chi^k, \qquad k = 1, \cdots, N.$$

Here, $\tilde{\varphi}_n^k \in X_{b_n}$ and χ^k denote the cut-off functions in $C_c^\infty(\Omega)$ such that χ^k is supported in a neighborhood of $\mathcal{C}^k$ and is identically equal to one on a smaller neighborhood of $\mathcal{C}^k$. Since $\mathcal{C}_n^k$ converges to $\mathcal{C}^k$, without loss of generality we can further assume that the same assumptions hold for $\mathcal{C}_n^k$ uniformly in $n \geq 0$. We then obtain $\tilde{\varphi}_n^k$ by solving

$$\operatorname{div}\Big[\frac{1}{b_n}\nabla\tilde{\varphi}_n^k\Big] = -\operatorname{div}\Big[\frac{1}{b_n}\nabla\chi^k\Big], \qquad \text{in } \Omega_n, \qquad \tilde{\varphi}_n^k = 0 \quad \text{on} \quad \partial\Omega_n. \tag{3.4}$$

Multiplying this equation by $\tilde{\varphi}_n^k$ and integrating the result over Ω_n, we readily obtain an a priori estimate:

$$\begin{aligned}\int_{\Omega_n} \frac{1}{b_n}|\nabla\tilde{\varphi}_n^k|^2\,dx = -\int_{\Omega_n}\frac{1}{b_n}\nabla\tilde{\varphi}_n^k\nabla\chi^k\,dx &\leq \frac12\int_{\Omega_n}\frac{1}{b_n}|\nabla\tilde{\varphi}_n^k|^2\,dx + \frac12\int_{\Omega_n}\frac{1}{b_n}|\nabla\chi^k|^2\,dx,\\ &\leq \int_{\Omega_n}\frac{1}{b_n}|\nabla\chi^k|^2\,dx.\end{aligned}$$

Here, we have used the Dirichlet boundary condition on $\tilde{\varphi}_n^k$. Now, remark that $\nabla\chi^k$ vanishes identically on a neighborhood of the boundary $\partial\Omega_n$ and b_n are bounded above and below away from $\partial\Omega_n$. The last integral on the right-hand side of the above estimate is therefore uniformly bounded in n, k.

This proves the boundedness and the weak convergence of $b_n^{-1/2}\nabla\tilde{\varphi}_n^k$ in $L^2(D)$ (with zero extension on $D \setminus \Omega_n$). Therefore, $\nabla\tilde{\varphi}_n^k$ is uniformly bounded in $L^2(D)$. The H^1 boundedness of $\tilde{\varphi}_n^k$ follows at once by the standard Poincaré inequality.

Consequently, solutions φ_n^k to (3.3) converge weakly in $H^1(D)$ to $\varphi^k \in H_0^1(D)$ verifying (in the sense of distributions):

$$\operatorname{div}\Big[\frac{1}{b}\nabla\varphi^k\Big] = 0, \qquad \text{in} \quad \Omega.$$

Without assuming that Ω_n is an increasing sequence, the difficulty could be to prove that φ^k satisfies the right boundary conditions. The tool to get the boundary conditions is the γ-convergence. Namely, as Ω_n converges in the Hausdorff topology to Ω and as $\mathbb{R}^2 \setminus \Omega_n$ has $N+1$ connected components, then Proposition B.2 states that Ω_n γ-converges to Ω. Hence, we can apply Proposition B.3 to $\tilde{\varphi}_n^k$ and infer that $\tilde{\varphi}^k$ belongs to $H_0^1(\Omega)$. Therefore, we have the right boundary conditions:

$$\varphi^k = \delta_{jk}, \qquad \text{on} \quad \partial\mathcal{C}^j, \qquad j = 1, \cdots, N.$$

Now, from the boundedness of $b_n^{-1/2}\nabla\tilde{\varphi}_n^k$ in L^2, we obtain at once the integrability of $b^{-1/2}\nabla\tilde{\varphi}^k$. Thus, by definition, $\tilde{\varphi}^k \in X$.

From the equation (3.4), the weak convergence obtained above, the fact that $\tilde{\varphi}^k \in H_0^1(\Omega)$ and that $b_n^{-1} \to b^{-1}$ in $L^2(\operatorname{supp} \nabla\chi^k)$ (by Definition 1.4), we have that

$$\int_{\Omega_n}\frac{1}{b_n}|\nabla\tilde{\varphi}_n^k|^2\,dx = -\int_{\Omega_n}\frac{1}{b_n}\nabla\tilde{\varphi}_n^k\nabla\chi^k\,dx \to -\int_{\Omega}\frac{1}{b}\nabla\tilde{\varphi}^k\nabla\chi^k\,dx = \int_{\Omega}\frac{1}{b}|\nabla\tilde{\varphi}^k|^2\,dx.$$

This proves the strong convergence as claimed. □

3.3. **b-harmonic functions: constant circulation.** We next derive the convergence for the b-harmonic solutions ψ_n^k. We recall that ψ_n^k vanishes on the outer boundary $\partial\tilde{\Omega}_n$ and solves

$$\begin{cases}\operatorname{div}\Big[\dfrac{1}{b_n}\nabla\psi_n^k\Big] = 0, & \text{in} \quad \Omega_n\\ \gamma_n^j\Big(\dfrac{1}{b_n}\nabla^\perp\psi_n^k\Big) = \delta_{jk}, & j = 1, \cdots, N.\end{cases} \tag{3.5}$$

where the circulation around $\mathcal{C}^k$ defined in (2.12) verifies

$$\gamma_n^j\Big(\frac{1}{b_n}\nabla^\perp\psi_n^k\Big) = -\int_{\Omega_n} \operatorname{div}\Big[\frac{1}{b_n}\chi^j\nabla\psi_n^k\Big]\,dx = -\int_{\Omega_n} \operatorname{div}\Big[\frac{1}{b_n}\varphi_n^j\nabla\psi_n^k\Big]\,dx,$$

for φ_n^j defined in the previous subsection. In the last equality above, we have used that $\chi^j - \varphi_n^j \in X_{b_n}$ and hence be approximated by compactly supported smooth functions, an argument already used in the proof of Proposition 2.10 (see (2.20)).

Now, since $\{\varphi_n^k\}_{k=1,\cdots,N}$ forms a basis (see Proposition 2.5), we can write

$$\psi_n^k = \sum_{j=1}^N a_n^{(k,j)}\varphi_n^j. \tag{3.6}$$

Thus, by (3.5), we have

$$\delta_{jk} = -\int_{\Omega_n}\frac{1}{b_n}\nabla\psi_n^k\cdot\nabla\varphi_n^j\,dx = -\sum_{l=1}^N a_n^{(k,l)}\int_{\Omega_n}\frac{1}{b_n}\nabla\varphi_n^l\cdot\nabla\varphi_n^j\,dx.$$

Let A_n be the $N\times N$ matrix with components $a_n^{(j,k)}$ and Φ_n the matrix formed by $\int_{\Omega_n}\frac{1}{b_n}\nabla\varphi_n^k\cdot\nabla\varphi_n^j\,dx$. By Lemma 3.2, Φ_n is well-defined and is uniformly bounded in n. We also let A and Φ be the matrix obtained from A_n and Φ_n by replacing b_n by b and φ_n^j by φ^j. The above identity yields that $-I = A_n\Phi_n$ and Lemma 3.2 implies that $\Phi_n \to \Phi$. To get that $A_n \to A$ we need to prove that Φ is invertible. If (Ω, b) is a smooth lake, then it is obvious because we also have $I = -A\Phi$. Concerning non-smooth lake, the invertible property comes from the positive capacity of islands (see [4, Sub. 2.2] for all details).

The expansion (3.6) then yields the following lemma.

Lemma 3.3. *For each $1 \le k \le N$, $b_n^{-1/2}\nabla\psi_n^k$ converges strongly in $L^2(D)$ to $b^{-1/2}\nabla\psi^k$ as $n \to +\infty$.*

3.4. **Estimates of α_n^k.** As the circulation is conserved $\gamma_n^k(v_n) = \gamma_n^k$ (see Proposition 2.13), it is easy to get from the uniform bound of $\sqrt{b_n}\omega_n$ in L^2 (see Lemma 3.1), of $\sqrt{b_n}$ (see Definition 1.4), of φ_n^k in L^2 (see Lemma 3.2), that α_n^k is uniformly bounded in time and in n. From the boundedness, we deduce directly that

$$\alpha_n^k \text{ converges weak-}* \text{ in } L^\infty(\mathbb{R}_+) \text{ to } \alpha^k(t) = \gamma^k + \int_\Omega b\omega\varphi^k dx.$$

3.5. **Kernel part with Dirichlet condition.** Let us next deal with the kernel part

$$\operatorname{div}\Big[\frac{1}{b_n}\nabla\psi_n^0\Big] = b_n\omega_n, \quad \psi_n^0|_{\partial\Omega_n} = 0. \tag{3.7}$$

Lemma 3.4. *ψ_n^0 converges weakly-$*$ in $W^{1,\infty}(\mathbb{R}_+; H^1(D))$ to ψ^0, which is the solution of*

$$\operatorname{div}\Big[\frac{1}{b}\nabla\psi^0\Big] = b\omega, \quad \psi^0|_{\partial\Omega} = 0.$$

Furthermore, there holds the strong convergence

$$\frac{1}{\sqrt{b_n}}\nabla\psi_n^0 \to \frac{1}{\sqrt{b}}\nabla\psi^0 \qquad \text{strongly in} \quad L^2((0,T)\times D) \text{ for any } T > 0.$$

Proof. Multiplying (3.7) by ψ_n^0, we get

$$\int_{\Omega_n}\frac{1}{b_n}|\nabla\psi_n^0|^2\,dx = -\int_{\Omega_n} b_n\omega_n\psi_n^0\,dx \le \|\sqrt{b_n}\omega_n\|_{L^2}\|\sqrt{b_n}\psi_n^0\|_{L^2}, \tag{3.8}$$

in which $\|\sqrt{b_n}\omega_n\|_{L^2}$ is bounded thanks to Lemma 3.1. Using the Poincaré inequality on D with Definition 1.4, we obtain that

$$\|\sqrt{b_n}\psi_n^0\|_{L^2(\Omega_n)} \le \sqrt{M}\|\psi_n^0\|_{L^2(\Omega_n)} \le c_0\sqrt{M}\|\nabla\psi_n^0\|_{L^2(\Omega_n)} \le c_0 M\|\frac{1}{\sqrt{b_n}}\nabla\psi_n^0\|_{L^2(\Omega_n)},$$

hence $\frac{1}{\sqrt{b_n}}\nabla\psi_n^0$ and $\nabla\psi_n^0$ are uniformly bounded in $L^2(D)$, which implies that ψ_n^0 is uniformly bounded $H_0^1(D)$.

Putting together all the uniform bounds obtained in this section, we finally see that

$$\sqrt{b_n}v_n^0 = b_n^{-\frac{1}{2}}\nabla\psi_n^0 + \sum_{k=1}^N \alpha_n^k(0)b_n^{-\frac{1}{2}}\nabla\psi_n^k \text{ is uniformly bounded in } L^2(D).$$

It is now possible to state that $\sqrt{b_n}v_n$ is uniformly bounded in $L^\infty(\mathbb{R}_+;L^2(D))$ by the standard energy estimate, which is useful in the following estimate.

Similarly, $\partial_t\psi_n^0$ solves

$$\mathrm{div}\left[\frac{1}{b_n}\nabla\partial_t\psi_n^0\right] = \partial_t(b_n\omega_n) = -\mathrm{div}\,(b_nv_n\omega_n), \quad \partial_t\psi_n^0|_{\partial\Omega_n} = 0,$$

from which we obtain in the same way that

$$\Big\|\frac{1}{\sqrt{b_n}}\partial_t\nabla\psi_n^0\Big\|_{L^2(\Omega_n)} \le \|\sqrt{b_n}v_n\|_{L^2}\|b_n\omega_n\|_{L^\infty} \lesssim 1.$$

It follows that $\frac{1}{\sqrt{b_n}}\nabla\psi_n^0$ belongs to $W^{1,\infty}(\mathbb{R}_+;L^2(D))$ and ψ_n^0 is in $W^{1,\infty}(\mathbb{R}_+;H_0^1(D))$. Consequently, up to some subsequence, there holds that

$$\frac{1}{\sqrt{b_n}}\nabla\psi_n^0 \rightharpoonup \frac{1}{\sqrt{b}}\nabla\psi^0 \qquad \text{weakly-}*\ \text{in} \quad L^\infty(\mathbb{R}_+;L^2(D)), \tag{3.9}$$

and

$$\psi_n^0 \to \psi^0 \qquad \text{weak-}*\text{ in} \quad W^{1,\infty}(\mathbb{R}_+;H_0^1(D)) \text{ and strongly in } C(\mathbb{R}_+;L^2(D)).$$

By Mosco's convergence (see Proposition B.3), it follows that $\psi^0 \in H_0^1(\Omega)$. Furthermore, it follows easily that ψ^0 solves

$$\mathrm{div}\left[\frac{1}{b}\nabla\psi^0\right] = b\omega, \quad \psi^0|_{\partial\Omega} = 0, \tag{3.10}$$

in the distributional sense.

Next, by using the weak convergence of $\sqrt{b_n}\omega_n$ and strong convergence of $\sqrt{b_n}\psi_n^0$ in L^2, the identity (3.8) then yields

$$\int_0^T\int_{\Omega_n}\frac{1}{b_n}|\nabla\psi_n^0|^2\,dxdt = -\int_0^T\int_{\Omega_n}b_n\omega_n\psi_n^0\,dxdt \to -\int_0^T\int_\Omega b\omega\psi^0\,dxdt = \int_0^T\int_\Omega\frac{1}{b}|\nabla\psi^0|^2\,dxdt,$$

in which the last identity follows from the equation (3.10). Thus, the convergence (3.9) is indeed a strong convergence in $L^2((0,T)\times D)$. This proves the lemma. □

3.6. **Convergence of α_n^k.** In view of (3.2), we next study the convergence of $\alpha_n^k(t)$. We have already obtained a uniform bound in $L^\infty(\mathbb{R}_+)$. Using the boundary condition $b_nv_n\cdot\nu_n = 0$ in the last identity below, it follows that

$$\partial_t\alpha_n^k(t) = -\int_{\Omega_n}\partial_t(b_n\omega_n)\varphi_n^k\,dx = \int_{\Omega_n}\mathrm{div}\,(b_nv_n\omega_n)\varphi_n^k\,dx = -\int_{\Omega_n}b_n\omega_n\sqrt{b_n}v_n\cdot\frac{1}{\sqrt{b_n}}\nabla\varphi_n^k\,dx,$$

which is again bounded by L^2 estimates. The strong convergence of $\alpha_n^k(t)$ to $\alpha^k(t)$ in $L^2((0,T))$ for any $T>0$ thus follows from this bound in $W^{1,\infty}(\mathbb{R}_+)$.

3.7. **Passing to the limit in the lake equation.** It is now easy by (3.1) and the expression (3.2) to construct the limiting solution. Indeed, we recall from (3.2) that

$$\psi_n(t,x) = \psi_n^0(t,x) + \sum_{k=1}^N\alpha_n^k(t)\psi_n^k(x)$$

with ψ_n^0 constructed as in (3.7) and ψ_n^k as in (3.5). Lemmas 3.3 and 3.4 together with the convergence of $\alpha_n^k(t)$ then yield that the limiting function ψ satisfies

$$\psi(t,x) = \psi^0(t,x) + \sum_{k=1}^{N} \alpha^k(t)\psi^k(x).$$

We then introduce the limiting velocity through

$$v := \frac{1}{b}\nabla^\perp\psi.$$

It follows clearly that $\sqrt{b_n}v_n \to \sqrt{b}v$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(D))$. For any test function $\varphi \in C_c^\infty([0,\infty)\times\Omega)$, by the Hausdorff convergence, there exists N_φ such that for any $n \geq N_\varphi$, $\varphi(t,\cdot)$ is compactly supported in Ω_n for all t. As (v_n,ω_n) is a global interior solution in the sense of Definition 1.2, hence the vorticity equation (1.11) holds for any $n \geq N_\varphi$, and for the limit. In addition, the divergence-free and boundary conditions follow at once from Lemma 2.6 and our construction of the approximate solutions: $\psi_n^k = \sum_{j=1}^N a_n^{(k,j)}\varphi_n^j$ with $\varphi_n^k = \tilde\varphi_n^k + \chi^k$ (see (3.6)).

Therefore, the limit v enjoys a Biot-Savart decomposition, and passing to the limit in the circulation definition (2.12) we obtain that the circulations of v are conserved.

One notices that all the convergence results hold up to a subsequence extraction. However, if the limit lake is smooth, i.e. $(\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega})$ verifying assumptions (H1)-(H3), then we have proved that (v,ω) is a global weak interior solution in the vorticity formulation for the lake (Ω,b), with constant circulations, hence (v,ω) is also a global weak solution in the vorticity formulation (see Proposition A.5) and a global weak solution in the velocity formulation (see Proposition A.4). The uniqueness result implies that the whole sequence converges to the unique solution of the lake equations. This ends the proof of Theorem 1.5.

Remark 3.5. In the previous proof, we never use that the islands are simply connected, hence we can relax this condition by assuming that $\mathcal{C}^i$ is a connected compact subset of Ω. Indeed, in [4, Proposition 1], it is proved that any connected compact set $\mathcal{C}^i$ can be approximated, in the Hausdorff topology, by smooth simply-connected compact set. Therefore, the case of a smooth simply-connected island which closes on itself (giving at the limit an annulus) is included in our analysis (see [4, Section 5.1] for pictures).

4. Non-smooth lakes

Let (Ω,b) be a lake satisfying (H1)-(H2). We assume that the H^1 capacity of all the islands is positive: cap $(\mathcal{C}^k) > 0$ for all $1 \leq k \leq N$. Here, we assume no regularity on the boundary $\partial\Omega$.

Domain approximation. Without assuming any regularity on Ω, we infer that Ω verifying (H1) is the Hausdorff limit of a sequence

$$\Omega_n \;:=\; \widetilde{\Omega}_n \;\setminus\; \left(\cup_{i=1}^k \overline{O_n^i}\right),$$

where $\widetilde{\Omega}_n$ and O_n^i's are smooth Jordan domains, and such that $\widetilde{\Omega}_n$, resp. $\overline{O_n^i}$, converges in the Hausdorff sense to $\widetilde{\Omega}$, resp. $\mathcal{C}^i$. Such a property is a consequence of the Hausdorff topology and a proof can be found in [4, Proposition 1]. Moreover, therein, the sequence Ω_n can be constructed to be increasing thanks to the assumption that the obstacles $\mathcal{C}^i$ are simply connected[2].

Bottom approximation. We assume that b is a bounded positive function on Ω. It follows that there is a sequence $b_n \in C^\infty(\Omega_n)$ with $M+1 \geq b_n \geq \theta_n > 0$ on Ω_n such that b_n converges strongly to b in $L^p_{\mathrm{loc}}(\Omega)$ for any $p \in [1,\infty)$. For instance, we may construct b_n via convolution:

$$b_n := (\rho_n * b)_{|\Omega_n} + \tfrac{1}{n},$$

in which ρ_n are some compactly supported, C^∞ smooth functions in Ω_n.

[2] If $\mathcal{C}^i$ is a simply connected compact set, there exists a Riemann mapping $\mathcal{T}$ from $(\mathcal{C}^i)^c$ to the exterior of the unit disk. Then, $O_n^i := (\mathcal{T}^{-1}(B(0,1+1/n)^c))^c$ is a smooth Jordan domain such that $\mathcal{C}^i \subset O_{n+1}^i \subset O_n^i$.

Hence, the lake (Ω_n, b_n) is a smooth lake with a non-vanishing topography ($\theta_n = \frac{1}{n}$). Moreover, since for any compact set $K \subset \Omega$ there exists $\theta_K > 0$ such that $b(x) \geq \theta_K$ on K, then there exists $n_0(K)$ such that for all $n \geq n_0(K)$ we have $b_n(x) \geq \theta_K/2$ on K. It then follows that the lake (Ω_n, b_n) converges to (Ω, b) in the sense of Definition 1.4.

Moreover, if $b \in W^{1,\infty}_{\text{loc}}(\Omega)$ then our approximation b_n converges weakly to b in $W^{1,\infty}_{\text{loc}}(\Omega)$.

Initial data approximation. For a function u defined on a subset U of D, we define $\underline{u}$ by $\underline{u}(x) = u(x)$ if $x \in U$ and $\underline{u}(x) = 0$ if $x \in D \setminus U$. If $\omega^0 \in L^\infty(\Omega)$ and $\gamma \in \mathbb{R}^N$ is given, then we consider v_n^0 such that

$$\operatorname{div}(b_n v_n^0) = 0, \quad \operatorname{curl} v_n^0 = b_n \underline{\omega^0}|_{\Omega_n}, \quad (b_n v_n^0) \cdot \nu|_{\partial\Omega_n} = 0,$$

with its circulation around each $\overline{O_n^k}$ equal to γ^k, for all $k = 1, \cdots, N$.

Existence result. Similarly to the analysis of (v_n, ω_n) done in Section 3, we get, up to extraction of a subsequence, that

$$\sqrt{b_n} v_n \to \sqrt{b} v \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}_+; L^2(D)), \qquad \omega_n \rightharpoonup \omega \text{ weakly in } L^\infty(\mathbb{R}_+ \times D),$$

for some limiting pair (v, ω). It also follows that (v, ω) is a global weak interior solution in the vorticity formulation of the lake equations on the lake (Ω, b) with initial vorticity ω^0 and initial circulation $\gamma \in \mathbb{R}^N$. Furthermore, this constructed solution also enjoys a Biot-Savart decomposition, with constant circulations. For this part, we do not use any regularity on b.

To prove that (v, ω) is a global weak solution in the vorticity formulation, we take an arbitrary $\varphi \in C^\infty_c([0, \infty) \times \overline{\Omega})$ and verify (1.11) for (v, ω). Indeed, since Ω_n is increasing, $\varphi|_{\Omega_n} \in C^\infty_c([0, \infty) \times \overline{\Omega_n})$ is a test function for which (1.11) holds for (v_n, ω_n) (see Remark A.6). It is now easy to pass to the limit in (1.11), which gives at once that (v, ω) is a global weak solution in the vorticity formulation, even for test functions which are not constant on the boundary.

As for *interior* solution in the velocity formulation, we have to consider $b \in W^{1,\infty}_{\text{loc}}(\Omega)$. In this case, we have stronger convergence of b_n to b, which allows us to pass to the limit in the velocity equations (1.8).

This completes the proof of Theorem 1.6.

Remark on initial velocity. If the initial data is given in terms of $v^0 \in L^1_{\text{loc}}(\Omega)$ such that $\omega^0 := \frac{\operatorname{curl} v^0}{b} \in L^\infty(\Omega)$, then the generalized circulation of a vector field v around $\mathcal{C}^k$ is well defined:

$$\gamma^k(v^0) = -\int_\Omega \left(\nabla^\perp \chi^k \cdot v^0 + \chi^k \operatorname{curl} v^0\right) dx.$$

Hence we can consider v_n^0 such that

$$\operatorname{div}(b_n v_n^0) = 0, \quad \operatorname{curl} v_n^0 = b_n \underline{\frac{\operatorname{curl}(v^0)}{b}}|_{\Omega_n}, \quad (b_n v_n^0) \cdot \hat{n}|_{\partial\Omega_n} = 0,$$

with its circulation around each $\overline{O_n^k}$ equal to $\gamma^k(v^0)$, for all $k = 1, \cdots, N$. Therefore, the previous compactness argument gives a solution with an initial velocity $\tilde{v}^0$ which has the same properties as v^0 (namely, same vorticity, circulations, and the same divergence and tangency condition). Nevertheless, it is not clear that $\tilde{v}^0 = v^0$, even for b lipschitz, because we need that the lake is smooth to apply Lemma 2.8.

Remark on uniqueness. As written in the introduction, the uniqueness is still open for non-smooth lake. To prove uniqueness in Section 2, we need that the velocity belongs to $W^{1,p}$ for any $p < \infty$ with good bounds, which follows from the Calderón-Zygmund inequality. However, such an inequality is only true for smooth lake (e.g. $(\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega})$) and the first author shows in [8] that the velocity for Euler equations blows up near an obtuse corner. Therefore, the uniqueness result seems challenging for non-smooth domains. Even if the first author obtained a uniqueness result for the Euler equations adding some assumptions (namely, ω^0 is assumed to be compactly supported with definite sign, and Ω is a simply connected bounded open set which is smooth except in a finite number of points), it is not clear how to adapt those techniques to the lake equations (e.g. to have an explicit formula for the Green kernel when the bottom is not flat).

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Appendix A. Equivalence of the various weak formulation

The goal of this section is to link together the various formulations of weak solutions to the lake equations in Definition 1.1 and Definition 1.2 and in the comments just following them. First, it is obvious that

- a global weak solution of the velocity formulation is a global *interior* weak solution of the velocity formulation;
- a global weak solution of the vorticity formulation is a global *interior* weak solution of the vorticity formulation;

Next, for any vector field v such that $\operatorname{div} bv = 0$, we can compute

$$\operatorname{div}(bv\otimes v) = bv\cdot\nabla v = (bv)^\perp \operatorname{curl} v + \frac{b}{2}\nabla|v|^2. \tag{A.1}$$

This equality is the key of the following propositions.

Proposition A.1. *Let (Ω, b) be a lake satisfying (H1)-(H2) with $b\in W^{1,\infty}_{\rm loc}(\Omega)$. Then a global* interior *weak solution of the velocity formulation is a global* interior *weak solution of the vorticity formulation.*

Proof. Let us fix a test function $\varphi\in C^\infty_c([0,\infty)\times\Omega)$, then $\Phi := \nabla^\perp\varphi$ is divergence free and belongs to $C^\infty_c([0,\infty)\times\Omega)$. As v is a global interior weak solution of the velocity formulation, there holds that

$$\begin{aligned}0 &= \int_0^\infty\int_\Omega \Big[\Phi_t\cdot v + (bv\otimes v) : \nabla\Big(\frac{\Phi}{b}\Big)\Big]\,dxdt + \int_\Omega \Phi(0,x)\cdot v^0(x)\,dx\\ &= \int_0^\infty\int_\Omega \Big[\Phi_t\cdot v - \operatorname{div}(bv\otimes v)\cdot\Big(\frac{\Phi}{b}\Big)\Big]\,dxdt + \int_0^\infty\int_{\partial\Omega}\Phi(v\otimes v)\nu\,d\tau dt + \int_\Omega \Phi(0,x)\cdot v^0(x)\,dx.\end{aligned}$$

The boundary term vanishes because the support of test function does not intersect the boundary. Next, by elliptic regularity on the support of φ we can use (A.1). Integrating by parts the linear terms gives

$$0 = \int_0^\infty\int_\Omega \Big[\varphi_t \operatorname{curl} v + (\operatorname{curl} v)v^\perp\cdot\nabla^\perp\varphi + \frac12\nabla|v|^2\cdot\nabla^\perp\varphi\Big]\,dxdt + \int_\Omega \varphi(0,x)\operatorname{curl} v^0(x)\,dx.$$

Integrating by part the third integral and setting $\omega := b^{-1}\operatorname{curl}(v)$, we then find

$$0 = \int_0^\infty\int_\Omega \Big[\varphi_t b\omega + b\omega v\cdot\nabla\varphi\Big]\,dxdt + \int_\Omega \varphi(0,x)b\omega^0(x)\,dx,$$

which is the vorticity equation (1.11). This ends the proof. □

The following concerns the converse of Proposition A.1 in the case the domain is simply connected.

Proposition A.2. *Let (Ω, b) be a lake satisfying (H1)-(H2), with $b\in W^{1,\infty}_{\rm loc}(\Omega)$ and with $N = 0$, i.e. we assume that Ω is simply connected. Then a global interior weak solution of the vorticity formulation is a global interior weak solution of the velocity formulation.*

Proof. Let us fix a divergence free test function $\Phi\in C^\infty_c([0,\infty)\times\Omega)$, then there exists a stream function φ such that $\Phi = \nabla^\perp\varphi$. As Φ is compactly supported, we infer that φ is constant in a neighborhood of the boundary. If Ω is simply connected, there is only one connected component of $\partial\Omega$, and as φ can be chosen up to a constant, then we can consider φ vanishing in the neighborhood of $\partial\Omega$. The conclusion follows from the same computations as in the previous proposition. □

Concerning solutions up to the boundary, we need more regularity in order to justify the velocity equation (1.8) and the boundary terms in the integrations by parts. First, we show the following technical lemma which will be useful for the next proposition.

Lemma A.3. *Assume that Ω is a C^3-domain, that $v \in W^{1,p}(\Omega)$ for some $1 \le p \le \infty$ satisfies $v \cdot \nu = 0$ in $\partial\Omega$. Let $d(x) = dist(x, \partial\Omega)$ and let $\mathcal{N}$ be a neighborhood of $\partial\Omega$ where d is C^3. Then*

$$v'_d := v \cdot \frac{\nabla d}{d} \in L^p(\mathcal{N}), \qquad \text{and} \qquad \|v'_d\|_{L^p(\mathcal{N})} \lesssim \|\nabla v\|_{L^p(\Omega)}$$

Proof. We extend ν into a vector field on $\mathcal{N}$ by setting $\nu(x) = -\nabla d(x)$. For $x \in \mathcal{N}$, we introduce $\gamma_s(x)$ the solution of

$$\frac{d}{ds}\gamma_s(x) = -\nabla d(\gamma_s(x)), \quad \gamma_0(x) = x$$

and let $\overline{x} = \lim_{s \to d(x)} \gamma_s(x) \in \partial\Omega$. Then, we simply remark that

$$v(x) = v(\overline{x}) - \int_0^{d(x)} \nu(\gamma_s(x)) \cdot \nabla v(\gamma_s(x)) ds$$
$$\nu(\gamma_s(x)) = \nu(x) = \nu(\overline{x})$$

and therefore, we see that

$$v(x) \cdot \nu(x) = v(\overline{x}) \cdot \nu(\overline{x}) - d(x) \int_0^1 [\nu \otimes \nu : \nabla v] (\gamma_{sd(x)}(x)) ds.$$

Since for any $0 \le s \le 1$, the mapping $x \mapsto \gamma_{sd(x)}(x)$ has Jacobian uniformly bounded (this can be seen from the fact that $x \mapsto (d(x), \overline{x})$ has Jacobian uniformly bounded), we see that

$$v'_d(x) = -\int_0^1 [\nu \otimes \nu : \nabla v] (\gamma_{sd(x)}(x)) ds$$

is bounded in L^p.

□

Finally, let us give the equivalence between the formulations when the circulations around $\mathcal{C}^k$ for all k are independent of the time.

Proposition A.4. *Let (Ω, b) be a lake verifying (H1)-(H3) and $(\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega})$. Then a global weak solution of the vorticity formulation, whose the circulations are constant in time, is a global weak solution of the velocity formulation. Conversely a global weak solution of the velocity formulation, whose the circulations are constant in time, is also a global weak solution of the vorticity formulation.*

Proof. In a smooth lake, then the global weak solution v is more regular, namely thanks to the points i) and ii), the Calderón-Zygmund inequality (see Proposition 2.12) implies that v belongs to $L^\infty(\mathbb{R}_+, W^{1,p}(\Omega))$ for any $p \in (4, \infty)$ and that $v \cdot \nu = 0$ on $\partial\Omega$.

First, let us check that each term in the velocity equation (1.8) indeed makes sense. Note in particular that in the case of a boundary with vanishing topography, the term $b^{-1}\Phi$ can be unbounded. To fix this problem, we observe that

$$(bv \otimes v) : \nabla\Big(\frac{\Phi}{b}\Big) = v \otimes v : \nabla\Phi - \frac{v \cdot \nabla b}{b} v \cdot \Phi.$$

The first term creates no difficulty. For the second, we remark that given our assumption (1.2), we have that, on $\mathcal{O}^k$,

$$\frac{\nabla b(x)}{b(x)} = a_k \frac{\nabla d(x)}{d(x)} + \frac{\nabla c(x)}{c(x)}.$$

Once again, the second term does not create any problem, while for the first, we may use Lemma A.3 to see that $v \cdot \frac{\nabla d(x)}{d(x)}$ in fact belongs to $L^\infty(L^p)$ for $p \in (4, \infty)$, which ends the justification of the velocity equation (1.8).

To prove the equivalence of the two formulations, we note that the fact that $\Phi \in C^\infty(\Omega)$ is tangent to the boundary implies that there exists $\varphi \in C^\infty(\Omega)$ constant on each connected component of $\partial\Omega$

such that $\Phi = \nabla^\perp \varphi$, and conversely $\varphi \in C^\infty(\Omega)$ with $\partial_\tau \varphi|_{\partial\Omega} = 0$ implies that $\Phi := \nabla^\perp \varphi$ is divergence free and tangent to the boundary. Therefore, it suffices to check that the boundary terms vanish in every integration by parts, following the proof in Proposition A.1:

- in the first integral, we have $\Phi(v \otimes v)\nu = (\Phi \cdot v)(v \cdot \nu)$ which is equal to zero because v is tangent to the boundary (see Proposition 2.12);
- for the linear terms, we compute that

$$\begin{aligned}\int_0^\infty \int_\Omega \Phi_t \cdot v \, dxdt + \int_\Omega \Phi(0,x) \cdot v^0(x)\, dx &= \int_\Omega \left\{ \int_0^\infty \nabla^\perp \varphi_t \cdot v \, dt + \nabla^\perp \varphi(0,x) \cdot v^0(x) \right\} dx \\ &= -\int_\Omega \left\{ \int_0^\infty \varphi_t \operatorname{curl} v \, dt + \varphi(0,x) \operatorname{curl} v^0(x) \right\} dx \\ &\quad + \int_{\partial\Omega} \left\{ \int_0^\infty \varphi_t (v \cdot \tau) dt + \varphi(0,x)(v^0(x) \cdot \tau) \right\} d\sigma\end{aligned}$$

  Thanks to the regularity of v, the notions of generalized circulation and classical circulation are equivalent: using the fact that $\varphi(t,\cdot)$ is constant on each connected component of the boundary and the conservation of the circulations, we can rewrite the last integral as

$$\int_{\partial\Omega} \left\{ \int_0^\infty \varphi_t (v \cdot \tau) dt + \varphi(0,x)(v^0 \cdot \tau) \right\} d\sigma = \sum_{k=1}^N \gamma_k(v) \left[\int_0^\infty \partial_t \varphi_{|\mathcal{C}^k} dt + \varphi(0,\cdot)_{|\mathcal{C}^k} \right] = 0.$$

  In this computation, we have assumed that $\varphi|_{\partial\widetilde{\Omega}} \equiv 0$, which is the general convention (because we always consider φ up to a constant), however we could also state that the circulation on $\partial\widetilde{\Omega}$ is constant because the Stokes formula gives $\oint_{\partial\widetilde{\Omega}} v(x) \cdot \tau d\sigma = \sum_{k=1}^N \gamma_k(v) + \int_\Omega \operatorname{curl} v$, where the last integral is independent of time from the transport nature (1.3).
- in the last integration by parts, the boundary term is $\int_0^\infty \int_{\partial\Omega} |v|^2 (\nu \cdot \nabla^\perp \varphi)\, dxdt$, which is equal to zero because we assume that $\partial_\tau \varphi \equiv 0$.

This ends the proof. □

We note in the last two bullets of the previous proof why we have assumed that Φ is tangent to the boundary in the sense of Definition 1.1 and that φ is constant on the boundary in Definition 1.2.

Finally, for smooth lakes, we prove that it is sufficient to prove existence of an interior solution.

Proposition A.5. *Let (Ω, b) be a lake verifying (H1)-(H3) and $(\partial\Omega, b) \in C^3 \times C^3(\overline{\Omega})$. Then a global interior weak solution of the vorticity formulation is a global weak solution of the vorticity formulation.*

Proof. Let $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega})$ and let

$$\eta_\varepsilon(x) = \eta(\varepsilon^{-1} d(x)), \quad \eta_\varepsilon \in C^3(\Omega)$$

where $\eta \in C_c^\infty(\mathbb{R})$ is equal to one in $[-1/2, 1/2]$ and vanishes outside of $[-1,1]$. As in the previous proof, we infer by Proposition 2.12 and Lemma A.3 that

$$\|v \cdot \nabla \eta_\varepsilon\|_{L^p(\Omega)} \lesssim \|v \cdot d^{-1} \nabla d\|_{L^p(\partial\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{1/(2p)}),$$

where $\partial\Omega_\varepsilon$ is a ε-neighborhood of $\partial\Omega$ as in (2.2). Again, this yields that $\|v \cdot \nabla \eta_\varepsilon\|_{L^p} \to 0$ when $\varepsilon \to 0$, for any $p \in [1,\infty)$.

Since $\eta_\varepsilon \varphi$ is now known to be $C_c^\infty([0,\infty) \times \Omega)$, the identity (1.11) holds for φ replaced by $\eta_\varepsilon \varphi$

$$\int_0^\infty \int_\Omega \Big[\eta_\varepsilon \varphi_t b\omega + \varphi(\nabla \eta_\varepsilon \cdot v) b\omega + \eta_\varepsilon (\nabla \varphi \cdot v) b\omega \Big] dxdt + \int_{\mathbb{R}^2} \eta_\varepsilon \varphi(0,x) b\omega^0(x)\, dx = 0.$$

Thanks to the convergence of $v \cdot \nabla \eta_\varepsilon$ and the estimates on v and ω, we are now able to pass to the limit $\varepsilon \to 0$. This proves that (v, ω) is a global weak solutions of the vorticity formulation, even for test functions in $C_c^\infty([0,\infty) \times \overline{\Omega})$. □

Remark A.6. We note in the previous proof that we do not need the condition "φ constant on the boundary". Therefore, for smooth lake, we could avoid this condition in Definition 1.2: indeed, a global weak solution in the sense of this definition (with the condition $\partial_\tau \varphi|_{\partial\Omega} \equiv 0$) is a global weak interior solution, and the previous proposition states that it is a global weak solution for any test functions in $C_c^\infty([0,\infty)\times\overline{\Omega})$ (without boundary condition).

We can also prove the following for any $\varphi \in C_c^\infty([0,\infty)\times\overline{\Omega})$:

$$\int_0^\infty \int_\Omega \varphi_t \omega \, dxdt + \int_0^\infty \int_\Omega (\nabla\varphi \cdot v)\omega \, dxdt + \int_{\mathbb{R}^2} \varphi(0,x)\omega^0(x)\, dx = 0.$$

Indeed, by assumption (H2), we can divide by b if $\varphi \in C_c^\infty([0,\infty)\times\Omega)$, and thus passing to the limit holds in the same manner.

Appendix B. γ-convergence of open sets

Let D be a bounded open set. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets included in D. One says that $(\Omega_n)_{n\in\mathbb{N}}$ γ-converges to $\Omega \subset D$ if for any $f \in H^{-1}(D)$, the sequence of solutions $\psi_n \in H^1_0(\Omega_n)$ of

$$-\Delta\psi_n = f \text{ in } \Omega_n, \quad \psi_n|_{\partial\Omega_n} = 0, \quad \psi_n \equiv 0 \text{ on } D\setminus\Omega_n$$

converges in $H^1_0(D)$ to the solution $\psi \in H^1_0(\Omega)$ of

$$-\Delta\psi = f \text{ in } \Omega, \quad \psi|_{\partial\Omega} = 0.$$

In this definition, $H^1_0(\Omega)$ and $H^1_0(\Omega_n)$ are seen as subsets of $H^1_0(D)$, through extension by zero. In a dual way, $H^{-1}(D)$ is seen as a subset of $H^{-1}(\Omega_n)$ and $H^{-1}(\Omega)$. As for the Hausdorff convergence of open sets, the definition of γ-convergence does not depend on the choice of the confining set D.

The notion of γ-convergence is extensively discussed in [6]. The basic example of γ-convergence is given by increasing sequences:

Proposition B.1. *If $(\Omega_n)_{n\in\mathbb{N}}$ is an increasing sequence in D, it γ-converges to $\Omega = \cup\Omega_n$. More generally, if $(\Omega_n)_{n\in\mathbb{N}}$ is included in Ω and converges to Ω in the Hausdorff sense, then it γ-converges to Ω.*

In general, Hausdorff converging sequences are not γ-converging. We refer to [6] for counterexamples, with domains Ω_n that have more and more holes as n goes to infinity. This kind of counterexamples, reminiscent of homogenization problems, is the only one in dimension 2, as proved by Sverak [11]:

Proposition B.2. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^2$, included in D. Assume that the number of connected components of $D\setminus\Omega_n$ is bounded uniformly in n. If $(\Omega_n)_{n\in\mathbb{N}}$ converges in the Hausdorff sense to Ω, it γ-converges to Ω.*

This result is a crucial ingredient in the convergence proofs.

One can characterize the γ-convergence in terms of the Mosco-convergence of $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$. Namely:

Proposition B.3. *$(\Omega_n)_{n\in\mathbb{N}}$ γ-converges to Ω if and only if the following two conditions are satisfied:*

1. *For all $\psi \in H^1_0(\Omega)$, there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $H^1_0(\Omega_n)$ that converges strongly to ψ.*
2. *For any sequence $(\psi_n)_{n\in\mathbb{N}}$ with ψ_n in $H^1_0(\Omega_n)$, weakly converging to ψ in $H^1_0(D)$, $\psi \in H^1_0(\Omega)$.*

One can also characterize γ-convergence with capacity, see [6, Proposition 3.5.5 page 114]. Let us finally mention that the notion of γ-convergence of open sets is related to the more standard Γ-convergence of Di Giorgi. Loosely speaking, Ω_n γ-converges to Ω if the corresponding Dirichlet energy functional J_{Ω_n} Γ-converges to J_Ω: see [6, section 7.1.1] for all details.

References

[1] Bresch D. and Métivier G., Global existence and uniqueness for the Lake equations with vanishing topography : elliptic estimates for degenerate equations, *Nonlinearity* 19 (2006), no. 3, 591-610.

[2] Camassa R., Holm D. D. and Levermore, C. D., Long-time effects of bottom topography in shallow water. Nonlinear phenomena in ocean dynamics (Los Alamos, NM, 1995), *Physica D* 98 (1996), no. 2-4, 258-286.

[3] Galdi G. P., *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Second edition*, Springer Monographs in Mathematics. Springer, New York, 2011.

[4] Gérard-Varet D. and Lacave C., The Two Dimensional Euler Equation on Singular Domains, *Arch. Ration. Mech. Anal.*, 209(1):131–170, 2013.

[5] Greenspan H.P., *The theory of rotating fluids*, Cambridge University Press, London, (1968).

[6] Henrot A. and Pierre M., *Variation et optimisation de formes. Une analyse géométrique* (French) [Shape variation and optimization. A geometric analysis], Mathématiques & Applications 48, Springer, Berlin, 2005.

[7] Iftimie D., Lopes Filho M. C., Nussenzveig Lopes H. J. and Sueur F., Weak vorticity formulation for the incompressible 2D Euler equations in domains with boundary, preprint 2013. arXiv:1305.0905.

[8] Lacave C., Uniqueness for Two Dimensional Incompressible Ideal Flow on Singular Domains, preprint 2011. arXiv:1109.1153.

[9] Levermore C. D., Oliver M. and Titi E. S., Global well-posedness for models of shallow water in a basin with a varying bottom, *Indiana Univ. Math. J.* 45 (1996), no. 2, 479-510.

[10] Levermore C. D., Oliver M. and Titi E. S., Global well-posedness for the lake equation, *Physica D*, (1996), 492-509.

[11] Sverák V., On optimal shape design, *J. Math. Pures Appl.* (9) 72 (1993), no. 6, 537-551.

[12] Wolibner W., Un théorème sur l'existence du mouvement plan d'un fluide parfait homogène, incompressible, pendant un temps infiniment long, *Math. Z.* **37** (1933), no. 1, 698-726.

[13] Yudovich V.I., Non-stationary flows of an ideal incompressible fluid, *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032-1066, 1963.

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