

# Global attractors and uniform persistence for cross diffusion parabolic systems

DUNG LE AND TOAN T. NGUYEN<sup>1</sup>

## Abstract

A class of cross diffusion parabolic systems given on bounded domains of  $\mathbb{R}^n$ , with arbitrary  $n$ , is investigated. We show that there is a global attractor with finite Hausdorff dimension which attracts all solutions. The result will be applied to the generalized Shigesada, Kawasaki and Teramoto (SKT) model with Lotka-Volterra reactions. In addition, the persistence property of the SKT model will be studied.

## 1 Introduction

In a recent work [9], we studied the global existence of a cross diffusion parabolic systems of the type

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla[(d_1 + a_{11}u + a_{12}v)\nabla u + b_{11}u\nabla v] + F(u, v), \\ \frac{\partial v}{\partial t} = \nabla[b_{22}v\nabla u + (d_2 + a_{21}u + a_{22}v)\nabla v] + G(u, v), \end{cases} \quad (1.1)$$

which is supplied with the Neumann or Robin type boundary conditions

$$\frac{\partial u}{\partial n} + r_1(x)u = 0, \quad \frac{\partial v}{\partial n} + r_2(x)v = 0, \quad (1.2)$$

on the boundary  $\partial\Omega$  of a bounded domain  $\Omega$  in  $\mathbb{R}^n$ . Here  $r_1, r_2$  are given nonnegative functions on  $\partial\Omega$ . The initial conditions are described by  $u(x, 0) = u_0(x)$  and  $v(x, 0) = v_0(x)$ ,  $x \in \Omega$ . Here  $u_0, v_0 \in W^{1,p}(\Omega)$  for some  $p > n$ .

System (1.1) has its origin from the Shigesada, Kawasaki and Teramoto model ([16])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta[(d_1 + a'_{11}u + a'_{12}v)u] + u(a_1 - b_1u - c_1v), \\ \frac{\partial v}{\partial t} = \Delta[(d_2 + a'_{21}u + a'_{22}v)v] + v(a_2 - b_2u - c_2v), \end{cases} \quad (1.3)$$

in population dynamics, which has been recently investigated to study the competition of two species with cross diffusion effects. In the context of ecology,  $d_i$ 's and  $a'_{ij}$ 's are the self and cross dispersal rates,  $a_i$  represent growth rates,  $b_1, c_2$  denote self-limitation rates,

---

<sup>1</sup>Department of Applied Mathematics, University of Texas at San Antonio, 6900 North Loop 1604 West, San Antonio, TX 78249. Email: [dle@math.utsa.edu](mailto:dle@math.utsa.edu); [tnguyen@math.utsa.edu](mailto:tnguyen@math.utsa.edu)

*Mathematics Subject Classifications:* 35K65, 35B65.

*Key words:* Strongly coupled (Cross diffusion) systems; Uniform estimates; Global Attractors; Uniform persistence.

and  $c_1, b_2$  are the interaction rates. Since  $u, v$  are population densities, only nonnegative solutions are of interest in this paper.

In our previous results [8, 12, 11], we proved the existence of the global attractor for system (1.3) with  $a'_{21} = 0$ . However, to the best of our knowledge, the case  $a'_{21} > 0$  has never been addressed. Obviously, the Shigesada, Kawasaki and Teramoto model (1.3) is a special case of (1.1) when  $b_{11} = a_{12}$  and  $b_{22} = a_{21}$ . In this work, we will concentrate on the generalized model (1.1). Global existence results for this case were established in [9]. In this paper, we go further to show that there exists a global attractor for system (1.1). To achieve this, more sophisticated PDE techniques will be needed.

The first main result of this paper is to obtain uniform estimates in higher norms to establish the existence of an absorbing ball in the  $W^{1,p}$  space as well as the compactness of the semiflow.

We obtain the following result whose proof is given in Section 3.

**Theorem 1.1** *Assume that  $a_{ij} \geq 0, d_i, b_{11}, b_{22} > 0, i, j = 1, 2$ , and*

$$a_{11} - a_{21} > b_{22}, \quad a_{22} - a_{12} > b_{11}. \quad (1.4)$$

*In addition, there exist positive constants  $K_0$  and  $K_1$  such that if either  $u \geq K_0$  or  $v \geq K_0$ , then*

$$F(u, v) \leq -K_1 u, \quad G(u, v) \leq -K_1 v. \quad (1.5)$$

*Then (1.1) and (1.2) define a dynamical system on  $W_+^{1,p}(\Omega)$ , the positive cone of  $W^{1,p}(\Omega)$ , for some  $p > n$ . And this dynamical system possesses a global attractor set.*

*Furthermore, let  $(u, v)$  be a nonnegative solution to (1.1). Then there exist  $\nu > 1$  and  $C_\infty > 0$  independent of initial data such that*

$$\limsup_{t \rightarrow \infty} \|u(\bullet, t)\|_{C^\nu(\Omega)} + \limsup_{t \rightarrow \infty} \|v(\bullet, t)\|_{C^\nu(\Omega)} \leq C_\infty. \quad (1.6)$$

In population dynamics terms, condition (1.4) means that self diffusion rates are stronger than cross diffusion ones. In fact, the assumptions of this theorem are needed only to establish that the Hölder norms of weak solutions are uniformly bounded in time (see [9] and Section 3). In Section 2, we will show that the estimate for the gradients like (1.6) still holds for much more general systems (of more than two equations) if a priori estimates for  $C^\alpha$  norms ( $\alpha \in (0, 1)$ ) are given.

In Section 4, we study the uniform persistence property of nonnegative solutions of (1.1) in the space  $X = \{(u, v) \in C^1(\Omega) \times C^1(\Omega) : u \geq 0, v \geq 0\}$ . We assume that reaction terms are of competitive Lotka-Volterra type that is commonly hypothesized in mathematical biology contexts, that is,

$$F(u, v) = u(a_1 - b_1 u - c_1 v), \quad G(u, v) = v(a_2 - b_2 u - c_2 v), \quad (1.7)$$

where  $a_i, b_i, c_i$  for  $i = 1, 2$  are positive constants. We also denote

$$\begin{aligned} P_u &= d_1 + a_{11}u + a_{12}v, & P_v &= b_{11}u, \\ Q_v &= d_2 + a_{21}u + a_{22}v, & Q_u &= b_{22}v. \end{aligned} \quad (1.8)$$

Let  $u_*, v_*$  be the unique positive solutions (see [3]) to

$$0 = \nabla(P_u(u_*, 0)\nabla u_*) + f(u_*, 0), \quad 0 = \nabla(Q_v(0, v_*)\nabla v_*) + g(0, v_*),$$

and boundary condition (1.2).

We consider the eigenvalue problems

$$\lambda\psi = d_1\Delta\psi + a_1\psi, \quad \text{and} \quad \lambda\phi = d_2\Delta\phi + a_2\phi, \quad (1.9)$$

$$\lambda\psi = \nabla[P_u(0, v_*)\nabla\psi + \partial_u P_v(0, v_*)\psi\nabla v_*] + \partial_u f(0, v_*)\psi, \quad (1.10)$$

$$\lambda\phi = \nabla[Q_v(u_*, 0)\nabla\phi + \partial_v Q_u(u_*, 0)\phi\nabla u_*] + \partial_v g(u_*, 0)\phi \quad (1.11)$$

with the boundary conditions  $\frac{\partial\psi}{\partial n} + r_1\psi = \frac{\partial\phi}{\partial n} + r_2\phi = 0$ .

Our persistence result reads as follows.

**Theorem 1.2** *Assume as in Theorem 1.1. Furthermore, suppose that the principal eigenvalues of (1.9), (1.10) and (1.11) are positive. If Robin boundary conditions are considered, we also assume further that the two quantities  $a_{12} - b_{11}$  and  $a_{21} - b_{22}$  are positive and sufficiently small.*

*Then system (1.1), with (1.2) and (1.7), is uniformly persistent. That is, there exists  $\eta > 0$  such that any its solution  $(u, v)$ , whose initial data  $u_0, v_0 \in W^{1,p}(\Omega)$  are positive, satisfies*

$$\liminf_{t \rightarrow \infty} \|u(\bullet, t)\|_{C^1(\Omega)} \geq \eta, \quad \liminf_{t \rightarrow \infty} \|v(\bullet, t)\|_{C^1(\Omega)} \geq \eta. \quad (1.12)$$

Thanks to [13, Theorem 4.5], our result implies that there exists at least a positive steady state solution of system (1.1) and (1.2) in  $W_+^{1,p}(\Omega)$ .

The positivity of the principal eigenvalues means that the trivial steady state  $(0, 0)$  is repelling in the  $(u, 0), (0, v)$  directions, and the semitrivial steady states  $(u_*, 0), (0, v_*)$  are unstable in their complementary directions. In the context of biology, (1.12) asserts that no species is completely invaded or wiped out by the other so that they coexist in time. We also remark that the uniform persistence property in this theorem is established in the  $C^1$  norm instead of the usual  $L^\infty$  norm widely used in literature of Lotka-Volterra systems. This is in part due to the setting of the phase space  $W^{1,p}$  for strongly coupled parabolic systems (see [1]). So, our persistence result does not rule out the possibility that solutions might form spikes at some points but approach zero almost everywhere as  $t \rightarrow \infty$ . That type of behavior can be seen in some models for chemotaxis, which also involve a form of strong coupling, so it may be that the results presented here are optimal.

At the end of the paper, we also present explicit conditions on the parameters of (1.1) that guarantee the positivity of the principal eigenvalues assumed in the above theorem.

## 2 Uniform estimates for higher norms.

In this section, we shall consider the following parabolic system for a vector-valued unknown  $u = (u_i)_i^m$

$$u_t = \operatorname{div}(a(u)\nabla u) + f(u, \nabla u), \quad (2.1)$$

which is supplied with the Neumann or Robin type boundary conditions. For the sake of simplicity, we will deal with the Neumann conditions  $\frac{\partial u}{\partial n} = 0$  in the proof below, and leave the Robin case to Remark 2.8.

Here,  $a(u)$  is a  $m \times m$  matrix. We need the following assumption on parameters of the system: there exist a positive constant  $\lambda$  and a continuous function  $C(|u|)$  such that for any  $\xi \in \mathbb{R}^m$

$$|f(u, \xi)| + |f_u(u, \xi)| \leq C(|u|)(1 + |\xi|^2), \quad |f_\xi(u, \xi)| \leq C(|u|)(1 + |\xi|), \quad (2.2)$$

$$\lambda|\xi|^2 \leq a_{ij}(u)\xi_i\xi_j \leq C(|u|)|\xi|^2. \quad (2.3)$$

Our main results in this section are the following estimates for higher order norms of solutions. We first establish uniform estimates in  $W^{1,p}$  norms of solutions to prove the existence of an absorbing ball in the  $\bigotimes_{i=1}^m W^{1,p}(\Omega)$  space. This is a crucial step of proving the existence of the global attractor set.

**Theorem 2.1** *Let  $u = (u_i)$  be a nonnegative solution of (2.1). Suppose that there exists a positive constant  $C_\infty(\alpha)$  independent of initial data such that*

$$\limsup_{t \rightarrow \infty} \|u_i(\bullet, t)\|_{C^\alpha(\Omega)} \leq C_\infty(\alpha) \quad (2.4)$$

for all  $\alpha \in (0, 1)$  and  $i = 1, \dots, m$ .

*Then there exists a positive constant  $C_\infty(p)$  independent of the initial data such that*

$$\limsup_{t \rightarrow \infty} \|u_i(\bullet, t)\|_{W^{1,p}(\Omega)} \leq C_\infty(p) \quad (2.5)$$

for any  $p > 1$  and  $i = 1, \dots, m$ .

We would remark here that estimate (2.4) for solutions to general system (2.1) is extremely difficult and not addressed yet. In the present paper, we shall show that estimate (2.4) is valid for a special case, system (1.1), of system (2.1) (see Section 3).

In the case that the matrix  $a(u)$  is triangular, Amann established (2.5) in [1] for some  $p > n$ . However, his argument can not be extended to the case when  $a(u)$  is a full matrix as considered here. On the other hand, it is well known that the  $\nabla u$  is Hölder continuous if  $u$  is (see [5]). However, as far as we are aware of, the following uniform estimate for Hölder norms of  $\nabla u$  has not existed yet in literature.

**Theorem 2.2** *Assume as in Theorem 2.1. Then there exist  $\nu > 1$  and a positive constant  $C_\infty$  independent of the initial data such that*

$$\limsup_{t \rightarrow \infty} \|u_i(\bullet, t)\|_{C^\nu(\Omega)} \leq C_\infty. \quad (2.6)$$

*Moreover, for  $p > n \geq 2$ , let  $K$  be a closed bounded subset in  $\bigotimes_{i=1}^m W^{1,p}(\Omega)$ . We consider solutions  $u$  with their initial data  $u_0 \in K$ . Then the image of  $K$  under solution flow  $K_t := \{u(\bullet, t) : u_0 \in K\}$  is a compact subset of  $\bigotimes_{i=1}^m W^{1,p}(\Omega)$ .*

The proof of these theorems will be based on several lemmas. The main idea to prove the above theorems is to use the imbedding results for Morrey's spaces. We recall the definitions of the Morrey space  $M^{p,\lambda}(\Omega)$  and the Sobolev-Morrey space  $W^{1,(p,\lambda)}$ . Let  $B_R(x)$  denote a cube centered at  $x$  with radius  $R$  in  $\mathbb{R}^n$ .

We say that  $f \in M^{p,\lambda}(\Omega)$  if  $f \in L^p(\Omega)$  and

$$\|f\|_{M^{p,\lambda}}^p := \sup_{x \in \Omega, \rho > 0} \rho^{-\lambda} \int_{B_\rho(x)} |f|^p dy < \infty.$$

Also,  $f$  is said to be in Sobolev-Morrey space  $W^{1,(p,\lambda)}$  if  $f \in W^{1,p}(\Omega)$  and

$$\|f\|_{W^{1,(p,\lambda)}}^p := \|f\|_{M^{p,\lambda}}^p + \|\nabla f\|_{M^{p,\lambda}}^p < \infty.$$

If  $\lambda < n - p$ ,  $p \geq 1$ , and  $p_\lambda = \frac{p(n-\lambda)}{n-\lambda-p}$ , we then have the following imbedding result (see Theorem 2.5 in [4])

$$W^{1,(p,\lambda)}(B) \subset M^{p_\lambda,\lambda}(B). \quad (2.7)$$

We then proceed by proving some estimates for the Morrey norms of the gradients of the solutions. In the sequel, the temporal variable  $t$  is always assumed to be sufficiently large such that

$$\|u(\cdot, t)\|_{C^\alpha} \leq C_\infty(\alpha), \quad \forall \alpha \in (0, 1) \text{ and } t \geq T, \quad (2.8)$$

where  $T$  may depend on the initial data.

From now on, let us fix a point  $(x, t) \in \bar{\Omega} \times (T, \infty)$ . As far as no ambiguity can arise, we write  $B_R = B_R(x)$ ,  $\Omega_R = \Omega \cap B_R$ , and  $Q_R = \Omega_R \times [t - R^2, t]$ . In the proofs, we will always use  $\xi(x, t)$  as a cut off function between  $B_R \times [t - R^2, t]$  and  $B_{2R} \times [t - 4R^2, t]$ , that is,  $\xi$  is a smooth function that  $\xi = 1$  in  $B_R \times [t - R^2, t]$  and  $\xi = 0$  outside  $B_{2R} \times [t - 4R^2, t]$ .

We first have the following technical lemma.

**Lemma 2.3** *For sufficiently small  $R > 0$ , we have the following estimate*

$$\int_{\Omega_R} |\nabla u|^2 dx + \iint_{Q_R} [|u_t|^2 + |\Delta u|^2] dz \leq CR^{n-2+2\alpha}.$$

Here  $\Delta u = (\Delta u_1, \Delta u_2, \dots, \Delta u_m)$ .

In the proof below, we will need two following useful results by Ladyzhenskaya *et al.* [7], which are stated for scalar functions. One can easily see that they are still true for *vector-valued* functions.

**Lemma 2.4** ([7, Lemma II.5.4]) *For any function  $u$  in  $W^{1,2s+2}(\Omega, \mathbb{R}^m)$  and  $\eta$  is a smooth function such that  $\frac{\partial u}{\partial n} \eta$  vanishes on  $\partial\Omega$  we have*

$$\int_{\Omega} |\nabla u|^{2s+2} \eta^2 dx \leq \text{osc}\{u, \Omega\} \text{Cont.} \int_{\Omega} (|\nabla u|^{2s-2} |\Delta u|^2 \eta^2 + |\nabla u|^{2s} |\nabla \eta|^2) dx. \quad (2.9)$$

**Lemma 2.5** ([7, Lemma II.5.3]) *Let  $\alpha > 0$  and  $v$  be a nonnegative function such that for any ball  $B_R$  and  $\Omega_R = \Omega \cap B_R$  the estimate*

$$\int_{\Omega_R} v(x) dx \leq CR^{n-2+\alpha}$$

*holds. Then for any function  $\eta$  from  $W_0^{1,2}(B_R)$  the inequality*

$$\int_{\Omega_R} v(x)\eta^2 dx \leq CR^\alpha \int_{\Omega_R} |\nabla\eta|^2 dx \quad (2.10)$$

*is valid.*

**Proof of Lemma 2.3:** Rewrite (2.1) as follows

$$u_t = a(u)\Delta u + (a_{u_i}\nabla u_i)\nabla u + f(u) \quad (2.11)$$

and test this by  $-\Delta u\xi^2$ . Integration by parts gives

$$\iint_{Q_{2R}} u_t\Delta u\xi^2 dz = -\frac{1}{2}\iint_{Q_{2R}} \frac{\partial(|\nabla u|^2\xi^2)}{\partial t} dz + \iint_{Q_{2R}} [|\nabla u|^2\xi\xi_t - u_t\nabla u\xi\nabla\xi] dz.$$

Note that we have used  $\xi\frac{\partial u}{\partial n} = 0$  on  $\partial Q_{2R}$  that is due to the choice of  $\xi$  and the Neumann condition of  $u$ . Therefore the boundary integrals resulting in the integration by parts are all zero.

Since  $a(u)\Delta u\Delta u \geq \lambda|\Delta u|^2$  (see (2.3)), we obtain

$$\begin{aligned} \int_{\Omega_R} |\nabla u(x,t)|^2 dx + \iint_{Q_{2R}} |\Delta u|^2\xi^2 dz &\leq C \iint_{Q_{2R}} |\nabla u|^2(\xi|\xi_t| + \xi^2 + \xi^2|\Delta u|) dz \\ &+ C \iint_{Q_{2R}} [|u_t||\nabla u|\xi|\nabla\xi| + |f||\Delta u|\xi^2] dz. \end{aligned}$$

By Young's inequality and the facts that  $|\xi_t|, |\nabla\xi|^2 \leq C/R^2$ , we derive

$$\begin{aligned} \int_{\Omega_R} |\nabla u(x,t)|^2 dx + \iint_{Q_{2R}} |\Delta u|^2\xi^2 dz &\leq \epsilon \iint_{Q_{2R}} |u_t|^2\xi^2 dz \\ &+ C \iint_{Q_{2R}} \left( |\nabla u|^4\xi^2 + \frac{1}{R^2}|\nabla u|^2 \right) dz + CR^{n+2}. \end{aligned} \quad (2.12)$$

From (2.11), we get

$$\iint_{Q_{2R}} |u_t|^2\xi^2 dz \leq \iint_{Q_{2R}} (|\Delta u|^2 + |\nabla u|^4 + |\nabla u|^2 + 1)\xi^2 dz.$$

Using Lemma 2.4 with  $s = 1$ , we then have

$$\iint_{Q_{2R}} |\nabla u|^4\xi^2 dz \leq CR^\alpha \iint_{Q_{2R}} (|\Delta u|^2\xi^2 + |\nabla u|^2|\nabla\xi|^2) dz. \quad (2.13)$$

We then choose  $R, \epsilon$  sufficiently small in (2.12) to derive that

$$\int_{\Omega_R} |\nabla u(x, t)|^2 dx + \iint_{Q_R} (|u_t|^2 + |\Delta u|^2) dz \leq \frac{C}{R^2} \iint_{Q_{2R}} |\nabla u|^2 dz + CR^{n+2}. \quad (2.14)$$

On the other hand, by testing (2.1) with  $(u - u_R)\xi^2$ , which  $u_R$  is the average of  $u$  over  $Q_R$ , one can easily get

$$\iint_{Q_{2R}} |\nabla u|^2 dz \leq CR^{n+2\alpha}.$$

This and (2.14) complete the proof of this lemma. ■

The following lemma shows that  $\nabla u$  is uniformly bounded in  $W^{1,(2,n-4+2\alpha)}(\Omega_R)$  so that imbedding (2.7) can be employed.

**Lemma 2.6** *For  $R > 0$  sufficiently small, we have the following estimates*

$$\int_{\Omega_R} (u_t^2 + |\Delta u|^2) dx \leq CR^{n-4+2\alpha}. \quad (2.15)$$

**Proof:** We now test (2.1) with  $-(u_t\xi^2)_t$ . Integration by parts in  $t$  gives

$$\begin{aligned} -\frac{1}{2} \frac{\partial}{\partial t} \iint_{Q_{2R}} u_t^2 \xi^2 dz + \iint_{Q_{2R}} u_t^2 \xi \xi_t dz + \iint_{Q_{2R}} (a(u)\nabla u)_t \nabla(u_t \xi^2) dz \\ = -\iint_{Q_{2R}} f_t(u, \nabla u) u_t \xi^2 dz. \end{aligned} \quad (2.16)$$

We again note that the boundary integrals resulting in the integration by parts are all zero. We consider the term

$$(a(u)\nabla u)_t \nabla(u_t \xi^2) = (a(u)\nabla u_t + a_u(u)u_t \nabla u)(\nabla u_t \xi^2 + 2u_t \xi \nabla \xi).$$

Using assumptions (2.2), (2.3), and Young's inequality, we have the following inequalities:  $a(u)\nabla u_t \nabla u_t \geq \lambda |\nabla u_t|^2$ , and

$$\begin{aligned} |u_t \nabla u_t \xi \nabla \xi| &\leq \epsilon |\nabla u_t|^2 \xi^2 + C(\epsilon) u_t^2 |\nabla \xi|^2, \\ |u_t \nabla u \nabla u_t \xi^2| &\leq \epsilon |\nabla u_t|^2 \xi^2 + C(\epsilon) u_t^2 |\nabla u|^2 \xi^2, \\ |u_t^2 \nabla u \xi \nabla \xi| &\leq u_t^2 |\nabla u|^2 \xi^2 + u_t^2 |\nabla \xi|^2, \\ |f_t(u, \nabla u) u_t \xi^2| &\leq \epsilon |\nabla u_t|^2 \xi^2 + C(\epsilon) u_t^2 |\nabla u|^2 \xi^2 + C(\epsilon) u_t^2 \xi^2. \end{aligned}$$

These inequalities and (2.16) yield

$$\int_{\Omega_R} |u_t|^2 dx + \iint_{Q_{2R}} |\nabla u_t|^2 \xi^2 dz \leq C \iint_{Q_{2R}} |u_t|^2 (|\nabla u|^2 \xi^2 + \xi^2 + |\nabla \xi|^2 + |\xi|) dz. \quad (2.17)$$

As we have shown in Lemma 2.3,  $\int_{\Omega_R} |\nabla u|^2 dx \leq cR^{n-2+\alpha}$ . This allows us to apply Lemma 2.5, with the function  $v$  being  $|\nabla u|^2$ , to derive

$$\iint_{Q_{2R}} |\nabla u|^2 u_t^2 \xi^2 dz \leq cR^{2\alpha} \iint_{Q_{2R}} [|\nabla u_t|^2 \xi^2 + u_t^2 |\nabla \xi|^2] dz.$$

Hence, for  $R$  sufficiently small, we obtain from the above and (2.17) that

$$\int_{\Omega_R} |u_t|^2 dx + \iint_{Q_R} |\nabla u_t|^2 dz \leq C \iint_{Q_{2R}} |u_t|^2 (\xi^2 + |\nabla \xi|^2 + |\xi_t|) dz. \quad (2.18)$$

Applying Lemma 2.3 and using the fact that  $|\xi_t|, |\nabla \xi|^2 \leq CR^{-2}$ , we obtain the desired inequality  $u_t$ . In order of the estimate of  $\Delta u$ , we solve  $\Delta u$  in terms of  $u_t$  and  $\nabla u$ , and then integrate them over  $\Omega_R$  to get

$$\int_{\Omega_R} |\Delta u|^2 dx \leq C \int_{\Omega_R} (u_t^2 + |\nabla u|^2 + 1)\xi^2 dx + C \int_{\Omega_R} |\nabla u|^4 \xi^2 dx.$$

The last integral can be absorbed into the left hand side by using (2.13) for sufficiently small  $R$ . This results in

$$\int_{\Omega_R} |\Delta u|^2 dx \leq C \int_{\Omega_R} (u_t^2 \xi^2 + |\nabla u|^2 \xi^2 + |\nabla u|^2 |\nabla \xi|^2 + |\xi|^2) dx.$$

Using Lemma 2.3, (2.18), and the fact that  $|\nabla \xi| \leq C/R$ , we conclude the proof. ■

We are now ready to give

**Proof of Theorem 2.1:** Thanks to the above lemmas, estimates

$$\int_{\Omega_R} |\nabla u|^2 dx, \quad \int_{\Omega_R} |\Delta u|^2 dx \leq CR^{n-4+2\alpha}$$

hold for some constant  $C$  independent of the initial data if  $t$  is sufficiently large.

By rewriting the equations of  $u$  as  $a(u)\Delta u = \tilde{F}$ , with  $\tilde{F}$  depending on the first order derivatives of  $u$  in  $x, t$ , and using the above estimates, we can apply [14, Lemma 4.1] to assert that the norms of  $\nabla u$  in  $W^{1,(2,\lambda)}(\Omega_R)$ ,  $\lambda = n - 4 + 2\alpha$ , are uniformly bounded. Therefore, by the imbedding (2.7) and the fact that  $M^{2\lambda,\lambda} \subset L^{2\lambda}$ , we have  $\|\nabla u(\bullet, t)\|_{L^{2\lambda}(\Omega)}$  with  $2\lambda = \frac{2(4-2\alpha)}{2-2\alpha}$  bounded by some constant  $C$ . Since  $\alpha$  is arbitrarily chosen in  $(0, 1)$ ,  $2\lambda$  can be as large as we wish. This proves that there exists a positive constant  $C_\infty(p)$  such that  $\|u(\bullet, t)\|_{W^{1,p}(\Omega)} \leq C_\infty(p)$ , for any  $p > 1$  and  $t \geq T$ .  $T$  is in (2.8). The proof of Theorem 2.1 is complete. ■

We now turn to the proof of Theorem 2.2. To this end we will need the following Schauder estimate by Schlag in [15].

**Lemma 2.7** *Let  $u \in C^{2,1}(Q_T)$  be a solution of (2.1). Then, for  $1 < q < \infty$ , there exists a constant  $C(q, T)$  such that*

$$\|D^2 u\|_{L^q(Q_T)} \leq C(q, T) [\|f\|_{L^q(Q_T)} + \|u\|_{L^q(Q_T)}], \quad (2.19)$$

where  $Q_T = \Omega \times [0, T]$ .

In fact, this result was proven in [15] under the assumption that  $a$  is a symmetric tensor, that is,  $a = (a_{ij}^{\alpha\beta})$  with  $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$ . In our case,  $a$  is a matrix  $a = (a_{ij})$  and it is not necessary

symmetric. However, the above estimate is still in force as we will discuss the necessary modifications in the argument of [15] at the end of this section after the proof of our main theorem.

**Proof of Theorem 2.2:** For each  $i$ , we rewrite each equation for  $u_i$  as follows

$$u_t^i = \Delta u_i + F_i$$

where  $F_i = \sum_{i,j} (a_{ij}(u) - \delta_{ij}) \Delta u_j + a_u(u) \nabla u \bullet \nabla u + f(u, \nabla u)$ , where  $\delta_{ij}$  is the Kronecker delta. We now apply *ii*) of [8, Lemma 2.5] here to obtain

$$\|u(\bullet, t)\|_{C^\nu(\Omega)} \leq C \|u(\bullet, \tau)\|_{L^r(\Omega)} + C_\beta \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \|F(\bullet, s)\|_{L^r(\Omega)} ds$$

for any  $t > T + 1$ ,  $\tau = t - 1$  and  $\beta \in (0, 1)$  satisfying  $2\beta > \nu + n/r$ , and for some fixed constants  $C, \delta, C_\beta > 0$ . By Hölder's inequality, we have

$$\int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \|F(\bullet, s)\|_{L^r(\Omega)} ds \leq \|F\|_{L^r(Q_{\tau,t})} \left[ \int_\tau^t (t-s)^{-\beta r'} e^{-\delta(t-s)r'} ds \right]^{1/r'}$$

Here  $r' = \frac{r}{r-1}$ . The last integral is bounded by a constant  $C(\beta, r, \delta)$  as long as  $\beta r' \in (0, 1)$  or  $r$  is sufficiently large. On the other hand, since  $\|u(\bullet, t)\|_{L^\infty(\Omega)}$  is uniformly bounded for large  $t$ ,  $|F(\bullet, t)| \leq C(|\Delta u| + |\nabla u|^2)$ . This, (2.5) (with  $p = 2r$ ) and Schauder estimate (2.19) imply that there is a constant  $C_r$  such that

$$\|F\|_{L^r(Q_{\tau,t})} \leq C_r, \quad \forall t > T.$$

Putting these facts together, we now choose  $r$  sufficiently large and  $\beta < 1$  such that  $\nu > 1$ . We then see that  $\|u_i(\bullet, t)\|_{C^\nu(\Omega)}$  is uniformly bounded for large  $t$ . This proves (2.6).

Concerning the compactness, let  $p > n \geq 2$  be given and  $K$  be a bounded subset in  $\bigotimes_{i=1}^m W^{1,p}(\Omega)$ . We consider solutions  $u$  with their initial data  $u_0 \in K$ . Estimate (2.6) shows that  $K_t$  is a bounded subset of  $\bigotimes_{i=1}^m C^\nu(\Omega)$ . By using the well-known compact imbedding  $C^\nu(\Omega) \subset W^{1,p}(\Omega)$ ,  $K_t$  is a compact subset of  $\bigotimes_{i=1}^m W^{1,p}(\Omega)$ . The proof of this theorem is complete. ■

**Remark 2.8** The case of Robin boundary conditions can be reduced to the Neumann one by a simple change of variables. First of all, since our proof is based on the local estimates of Lemmas 2.3 and 2.6, we need only to study these inequalities near the boundary. As  $\partial\Omega$  is smooth, we can locally flatten the boundary and assume that  $\partial\Omega$  is the plane  $\{x_n = 0\}$ . Furthermore, we can take  $\Omega_R = \{(x', x_n) : x_n > 0, |(x', x_n)| < R\}$ . The boundary conditions become

$$\frac{\partial u_i}{\partial x_n} + \tilde{r}_i(x') u_i = 0.$$

We then introduce  $U(x', x_n) = (U_1(x', x_n), \dots, U_m(x', x_n))$  with

$$U_i(x', x_n) = \exp(x_n \tilde{r}_i(x')) u_i(x', x_n)$$

Obviously,  $U$  satisfies the Neumann boundary condition on  $x_n = 0$ . Simple calculations also show that  $U$  verifies a system similar to that for  $u$ , and conditions (Q.1), (Q.2) are

still valid. In fact, there will be some extra terms occurring in the divergence parts of the equations for  $U$ , but these terms can be handled by a simple use of Young's inequality so that our proof is still in force. Thus Theorem 2.2 applies to  $U$ , and the estimates for  $u$  then follow.

Finally, we conclude this section by a brief discussion of Lemma 2.7. A careful reading of [15] reveals that the only place where the symmetry of  $a(u)$  is needed is the proof of [15, Lemma 1]. This lemma concerns the estimates for solutions to homogeneous systems with constant coefficients

$$v_t^i - A_{ij}\Delta v^j = 0 \quad \text{in } Q_R \quad (2.20)$$

which  $v = 0$  on  $\partial B_R^+ \cap \{x_n > 0\} \times [-R^2, 0]$  and on  $B_R^+ \times \{-R^2\}$  and  $\frac{\partial v}{\partial n} = 0$  on  $B_R^+ \cap \{x_n = 0\} \times [-R^2, 0]$ .

The lemma is stated as follows.

**Lemma 2.9** *Let  $0 < r \leq R$ . Then any smooth solution  $v$  of (2.20) satisfies*

**a.**

$$\iint_{Q_{r/2}} |v_t|^2 dz \leq Cr^{-2} \iint_{Q_r} |\nabla v|^2 dz. \quad (2.21)$$

**b.** *for  $k = 1, 2, 3, \dots$*

$$\iint_{Q_{r/2}} |\nabla^k v|^2 dz \leq C_k r^{-2k} \iint_{Q_r} |v|^2 dz. \quad (2.22)$$

**c.** *for any  $0 < \rho < r \leq R$ ,*

$$\iint_{Q_\rho} |v|^2 dz \leq C \left(\frac{\rho}{r}\right)^2 \iint_{Q_r} |v|^2 dz. \quad (2.23)$$

Thus, Lemma 2.7 is proven if we can relax the symmetry assumption in this lemma.

**Proof:** Let  $v = (v^i)$  be a solution of (2.20), that is,

$$\iint_{Q_{2R}} v_t^i \phi + A_{ij} \nabla v^j \nabla \phi dz = 0, \quad (2.24)$$

where  $\phi \in C^1(Q_R)$  such that  $\phi = 0$  on  $\partial B_R^+ \cap \{x_n > 0\} \times [-R^2, 0]$  and on  $B_R^+ \times \{-R^2\}$ .

Let  $\eta$  be a cut-off function in  $Q_r$  such that  $\eta = 1$  in  $Q_{r/2}$ ,  $\eta(\cdot, -r^2) = 0$ , and  $\eta$  vanishes on  $\partial B_r \cap \{x_n > 0\} \times [-r^2, 0]$ .

By squaring equations (2.20) and summing up the results, we have

$$\iint_{Q_r} |v_t|^2 \eta dz \leq C \iint_{Q_r} |\Delta v|^2 \eta dz. \quad (2.25)$$

Now, by choosing  $\phi = \Delta v^i \eta^2$  in (2.24), one can easily see that

$$\iint_{Q_r} v_t^i \Delta v^i \eta^2 dz - \iint_{Q_r} A_{ij} \Delta v^j \Delta v^i \eta^2 dz = 0.$$

Thanks to the ellipticity and integrations by parts, we obtain

$$\begin{aligned} \lambda \iint_{Q_r} |\Delta v|^2 \eta^2 dz &\leq \iint_{Q_r} v_t^i \Delta v^i \eta^2 dz \\ &= -\frac{1}{2} \frac{\partial}{\partial t} \iint_{Q_r} |\nabla v^i|^2 \eta^2 dz - \iint_{Q_r} (\nabla v^i v_t^i \nabla \eta - |\nabla v^i|^2 \eta_t) \eta dz \\ &\leq \epsilon \iint_{Q_r} |v_t|^2 \eta dz + C \iint_{Q_r} (|\eta_t| + |\nabla \eta|^2) |\nabla v|^2 dz. \end{aligned}$$

Using this, (2.25), and the fact that  $|\eta_t|, |\nabla \eta|^2 \leq Cr^{-2}$ , we easily get (2.21).

In order to prove (2.22) for  $k = 1$ , we choose  $\phi = v^i \eta^2$  in (2.20). It is now standard (see [2]) to see that

$$\iint_{Q_{r/2}} |\nabla v|^2 dz \leq Cr^{-2} \iint_{Q_r} |v|^2 dz.$$

From this point on, we can follow [15] to complete the proof. ■

### 3 Global attractors for the generalized SKT model.

In this section, we shall show that Theorem 2.2 can apply to (1.1), and therefore give the proof of Theorem 1.1. It is now clear that we need only establish the following uniform estimates for the Hölder norms of solutions.

**Lemma 3.1** *Assume as in Theorem 1.1. Let  $(u, v)$  be a nonnegative solution to (1.1). Then there exists a constant  $C_\infty(\alpha)$  independent of initial data such that*

$$\limsup_{t \rightarrow \infty} \|u(\bullet, t)\|_{C^\alpha} + \limsup_{t \rightarrow \infty} \|v(\bullet, t)\|_{C^\alpha} \leq C_\infty(\alpha), \quad \forall \alpha \in (0, 1). \quad (3.1)$$

In our recent work [9], we presented some sufficient conditions on the parameters of (1.1) for its bounded weak solutions to be Hölder continuous. Furthermore, the proof of [9, Theorem 1.1] also shows that the Hölder norm of a solution depends only on the bound of its  $L^\infty$  norm. Our present condition (1.4) in Theorem 1.1 clearly satisfies (1.5) and ii) of [9, Theorem 1.1]. Hence, in order to obtain uniform estimate (3.1), it suffices to show that the  $L^\infty$  norms of the solution are ultimately uniformly bounded. That is to say that we need only find a positive constant  $C_\infty$  independent of the initial data such that

$$\limsup_{t \rightarrow \infty} \|u(\bullet, t)\|_\infty + \limsup_{t \rightarrow \infty} \|v(\bullet, t)\|_\infty \leq C_\infty. \quad (3.2)$$

The proof of this fact will largely base on the analysis in [9, Section 4.2] where we proved the existence of a  $C^2$  function  $H(u, v)$  defined on  $\mathbb{R}_+^2$  such that the below conditions are satisfied.

**(H.0)**  $H(u, v) = \exp(\mu g(u, v))$  for some sufficiently large  $\mu > 0$  (depending only on the parameters of the system) and  $g$  is a solution of  $g_u = f(u, v)g_v$ , with  $f(u, v)$  being the positive solution of (see (1.8))

$$F(f) := -P_v f^2 + (P_u - Q_v)f + Q_u = 0. \quad (3.3)$$

**(H.1)** There exists a constant  $K_0$  such that  $(H_u F + H_v G)(H - K)_+ \leq 0$  for any  $(u, v) \in \Gamma_0$  and  $K \geq K_0$ .

**(H.2)** Let  $\mathcal{P} = P_u \nabla u + P_v \nabla v$  and  $\mathcal{Q} = Q_u \nabla u + Q_v \nabla v$ . There exists  $\lambda > 0$  such that

$$H_u \mathcal{P} + H_v \mathcal{Q} \geq \lambda |\nabla H|^2, \quad (3.4)$$

$$\mathcal{P} \nabla H_u + \mathcal{Q} \nabla H_v \geq 0, \quad (3.5)$$

for any  $(u, v) \in F \cap \{(u, v) : H(u, v) \geq K_0\}$ , with  $K_0$  given in (H.1).

**(H.2)** If  $(u, v) \rightarrow \infty$  in  $\mathbb{R}^2$ , then  $H(u, v) \rightarrow \infty$ .

In addition, we also have the following property.

**Lemma 3.2** *Assume as in Theorem 1.1. There exists a positive constant  $C$  such that*

$$H_u F + H_v G \leq -CH \quad (3.6)$$

*if either  $u \geq K_0$  or  $v \geq K_0$ , with  $K_0$  being given in (H.1).*

**Proof:** Without loss of generality, we can assume that  $d_1 \leq d_2$ . Substituting  $f = \frac{a_{11} - a_{21}}{b_{11}} > 0$  in the quadratic on the left hand side of (3.3) and simplifying the result, we get

$$-\frac{[(a_{11} - a_{21})(a_{22} - a_{12}) - b_{11}b_{22}]v + (d_2 - d_1)(a_{11} - a_{21})}{b_{11}}.$$

By (1.4) and the fact that  $d_1 \leq d_2$ , the above is negative. Since leading coefficient  $-P_v$  is negative and  $f(u, v)$  is the positive root, we must have that  $f(u, v)$  is bounded by  $(a_{11} - a_{21})/b_{11}$  for all  $u, v \geq 0$ . This and [9, Lemma 4.3] imply that there exists a positive constant  $C$  such that

$$g_v \geq \frac{C}{f(u, v)} \geq \frac{b_{11}C}{a_{11} - a_{21}}.$$

Now, from  $H = \exp(\mu g)$  and assumption (1.5) on the reaction terms  $F$  and  $G$ , we easily obtain

$$H_u F + H_v G = \mu H g_v (fF(u, v) + G(u, v)) \leq -\mu K_1 H g_v (fu + v) \leq -C_1 H$$

if either  $u \geq K_0$  or  $v \geq K_0$ . The proof of this lemma is complete. ■

The following technical lemma is a simple consequence of Moser's iteration technique and we omit its proof (see [10, Lemma 2.1]).

**Lemma 3.3** *For  $T_1 > T > T_0$ , let  $U$  be a function on  $\Omega \times [T_0, T_1]$  such that*

$$\sup_{\tau \in [t, T_1]} \int_{\Omega} U^q(x, \tau) dx + \int_t^{T_1} \int_{\Omega} |\nabla U^{q/2}|^2 dx d\tau \leq \frac{Cq^\nu}{t-s} \int_s^{T_1} \int_{\Omega} U^q dx d\tau \quad (3.7)$$

*for all  $q \geq p$ , some  $\nu > 0$ , and  $T_0 < s < t < T_1$ . Then there exists a constant  $C_0$ , depending on  $T - T_0$ , such that*

$$\sup_{\Omega \times [T, T_1]} U(x, t) \leq C_0 \left( \int_{T_0}^{T_1} \int_{\Omega} U^p dx \right)^{1/p}. \quad (3.8)$$

We are now ready to give the proof of our main theorem.

**Proof of Theorem 1.1:** For any positive test function  $\phi$ , we test the equations of  $u, v$  respectively with  $H_u\phi, H_v\phi$  and add the results. By using (3.5), we easily obtain

$$\int_{\Omega} H_t\phi \, dx + \int_{\Omega} [H_u\mathcal{P} + H_v\mathcal{Q}] \nabla\phi \, dx \leq \int_{\Omega} (H_uF + H_vG)\phi \, dx. \quad (3.9)$$

Let  $T > 1$  and  $s < t$  be two numbers in  $(T - 1, T)$ . We consider a  $C^1$  function  $\eta : (0, \infty) \rightarrow [0, 1]$  that satisfies

$$\eta(\tau) = \begin{cases} 0 & \text{if } \tau < s, \\ 1 & \text{if } \tau > s \end{cases} \quad \text{and} \quad |\eta'| \leq \frac{1}{t-s}. \quad (3.10)$$

Set  $U = (H - K_1)_+$ . For  $q \geq 2$  and  $T$  sufficiently large, we substitute  $\phi$  in (3.9) by  $U^{q-1}\eta$  and use (H.1) and (H.2) to obtain

$$\iint_Q \frac{1}{q} \frac{\partial U^q}{\partial t} \eta \, dz + \lambda(q-1) \iint_Q U^{q-2} |\nabla U|^2 \, dz \leq 0.$$

By (3.10), this implies

$$\iint_Q \frac{\partial(U^q\eta)}{\partial t} \, dz + \lambda \iint_Q |\nabla U^{q/2}|^2 \, dz \leq \frac{Cq}{t-s} \iint_Q U^q \, dz.$$

We then apply Lemma 3.3 to assert that

$$\sup_{Q_T} U(x, t) \leq C_0 \left( \int_{T-1}^{T+1} \int_{\Omega} U^2 \, dx \right)^{1/2}. \quad (3.11)$$

On the other hand, we choose  $\phi = U$  in (3.9) and use (3.6), (3.5) to get

$$Y' \leq -CY, \quad \text{with} \quad Y(t) = \int_{\Omega} U^2(x, t) \, dx.$$

This shows that  $\limsup_{t \rightarrow \infty} Y(t)$  is bounded by some constant independent of  $Y(0)$  or  $u_0, v_0$ . Hence, (3.11) shows that

$$\limsup_{t \rightarrow \infty} \|H(u(\bullet, t), v(\bullet, t))\|_{\infty} \leq C_{\infty}. \quad (3.12)$$

As  $\lim_{(u,v) \rightarrow \infty} H(u, v) = \infty$  (see (H.2)), the above also gives the estimate for  $L^{\infty}$  norms (3.2). By our earlier discussion, Lemma 3.1 holds and completes our proof of Theorem 1.1.  $\blacksquare$

## 4 Persistence property

We conclude our paper by a discussion of Theorem 1.2. In fact, once uniform estimates for gradients like (1.6) are established, the proof of persistence property follows the lines in [11] where we dealt with triangular systems. Therefore, we will restrain ourself from giving the details of the calculations here but only sketch the main steps.

Let us recall the parameters of our system:

$$F(u, v) := u(a_1 - b_1u - c_1v), \quad G(u, v) := v(a_2 - b_2u - c_2v),$$

and

$$\begin{aligned} P_u &= d_1 + a_{11}u + a_{12}v, & P_v &= b_{11}u, \\ Q_v &= d_2 + a_{21}u + a_{22}v, & Q_u &= b_{22}v. \end{aligned}$$

Denote  $X = C^1(\Omega) \times C^1(\Omega)$  and its positive cone  $X_0 = \{(u, v) \in X : u > 0 \text{ and } v > 0\}$ . Then  $(X, d)$ , with  $d(x, y) = \|x - y\|_{C^1(\Omega)}$  is a complete metric space. The boundary of  $X_0$  consists of  $I_u := \{(u, 0) : u \geq 0\}$  and  $I_v := \{(0, v) : v \geq 0\}$ . Thanks to Theorem 1.1, we can define the semiflow on  $X$  as follows: for any initial data  $(u_0, v_0)$  in  $X$ , define  $\Phi_t(u_0, v_0) = (u(\bullet, t), v(\bullet, t))$  for all  $t \geq 0$ . Estimate (1.6) also gives that  $\Phi$  is a compact semiflow and possesses a global attractor in  $X$ . A simple application of maximum principles for scalar parabolic equations shows that  $X_0, I_u, I_v$  are positively invariant under  $\Phi$ .

Let  $M_0 = (0, 0)$ ,  $M_1 = (u_*, 0)$ , and  $M_2 = (0, v_*)$ , where  $u_*, v_*$  are the unique positive solutions to

$$0 = \nabla(P_u(u_*, 0)\nabla u_*) + F(u_*, 0), \quad 0 = \nabla(Q_v(0, v_*)\nabla v_*) + G(0, v_*)$$

and boundary condition (1.2).

It is clear that  $\{M_0, M_1, M_2\}$  are pairwise disjoint, compact and isolated invariant sets in  $\partial X_0 = I_u \cup I_v$ . Moreover, as  $M_0$  is repelling in both  $u, v$  directions, from [3] we know that  $M_1$  (respectively  $M_2$ ) is globally attracting in  $I_u$  (respectively  $I_v$ ). This implies that no subset of  $\{M_i\}$  can form a cycle (see [6]) in  $\partial X_0$ .

The above facts allow us to apply a result [6, Theorem 4.3] on the persistence property for general dynamical systems to our setting. According to this theorem, what is left is to show that  $M_i$ , for  $i = 1, 2$ , is isolated in  $X$  and its stable set

$$W^s(M_i) := \{x \in X : \lim_{t \rightarrow \infty} d(\Phi_t(x), M_i) = 0\}$$

does not intersect  $X_0$ .

If  $(u, v)$  is a steady state solution of (1.1), we consider the eigenvalue problems associated to the linearization of (1.1) at  $(u, v)$

$$\lambda\psi = \nabla [(\partial_u P_u \psi + \partial_v P_u \phi)\nabla u + P_u \nabla \psi + (\partial_u P_v \psi + \partial_v P_v \phi)\nabla v + P_v \nabla \phi] + \partial_u F \psi + \partial_v F \phi,$$

$$\lambda\phi = \nabla [(\partial_u Q_u \psi + \partial_v Q_u \phi)\nabla u + Q_u \nabla \psi + (\partial_u Q_v \psi + \partial_v Q_v \phi)\nabla v + Q_v \nabla \phi] + \partial_u G \psi + \partial_v G \phi.$$

At  $(u, v) = (0, v_*)$  and  $(\psi, \phi) = (\psi, 0)$ , this reads

$$\lambda\psi = \nabla [P_u(0, v_*)\nabla \psi + \partial_u P_v(0, v_*)\psi \nabla v_*] + \partial_u F(0, v_*)\psi. \quad (4.1)$$

While at  $(u, v) = (u_*, 0)$  and  $(\psi, \phi) = (0, \phi)$ , we have

$$\lambda\phi = \nabla[Q_v(u_*, 0)\nabla\phi + \partial_v Q_u(u_*, 0)\phi\nabla u_*] + \partial_v G(u_*, 0)\phi. \quad (4.2)$$

The uniform estimates for  $\nabla u, \nabla v$  in (1.6) allow us to repeat the argument in the proof of [11, Proposition 3.3] to assert the followings.

**Proposition 4.1** *Assume that the principal eigenvalue  $\lambda$  of (4.1) is positive. There exists  $\eta_0 > 0$  such that for any solution  $(u, v)$  of (1.1) with  $(u_0, v_0) \in X_0$ ,*

$$\limsup_{t \rightarrow \infty} \|(u(\bullet, t), v(\bullet, t)) - (0, v_*)\|_X \geq \eta_0.$$

**Proposition 4.2** *Assume that the principal eigenvalue  $\lambda$  of (4.2) is positive. There exists  $\eta_0 > 0$  such that for any solution  $(u, v)$  of (1.1) with  $(u_0, v_0) \in X_0$ ,*

$$\limsup_{t \rightarrow \infty} \|(u(\bullet, t), v(\bullet, t)) - (u_*, 0)\|_X \geq \eta_0.$$

An immediately consequence of the above propositions is that  $W^s(M_i) \cap X_0 = \emptyset$  and the  $M_i$ 's are isolated. Our Theorem 1.2 now follows at once.

We conclude this section by presenting explicit conditions on the parameters of (1.1) for the positivity of the principal eigenvalues of (4.1) and (4.2). The proof of the following lemmas follow closely that of [11, Lemma 3.1].

**Lemma 4.3** *Assume that either  $r_1 = r_2 \equiv 0$  and*

$$a_1/a_2 > c_1/c_2,$$

*or  $r_1, r_2 \neq 0$  and*

$$\frac{a_1}{a_2} > \max \left\{ \frac{c_1}{c_2}, \frac{2a_{12}}{a_{22}} \right\}, \quad (4.3)$$

*and*

- a)  $a_{12} > b_{11}$  and  $d_1 a_{22} \geq 2d_2 b_{11}$ ;
- b)  $\sup_{\partial\Omega} (r_1(x) - r_2(x))_+$  and  $(a_2 d_1 - a_1 d_2)_+$  are sufficiently small.

*Then the principal eigenvalue of (4.1) is positive.*

**Lemma 4.4** *Assume that either  $r_1 = r_2 \equiv 0$  and*

$$b_1/b_2 > a_1/a_2,$$

*or  $r_1, r_2 \neq 0$  and*

$$\frac{a_1}{a_2} < \min \left\{ \frac{b_1}{b_2}, \frac{a_{11}}{2a_{21}} \right\},$$

*and*

- a)  $a_{21} > b_{22}$  and  $d_2 a_{11} \geq 2d_1 b_{22}$ ;
- b)  $\sup_{\partial\Omega} (r_2(x) - r_1(x))_+$  and  $(a_1 d_2 - a_2 d_1)_+$  are sufficiently small.

*Then the principal eigenvalue of (4.2) is positive.*

Putting these together, we obtain sufficient conditions for the uniform persistence of (1.1) and therefore the existence of the positive steady state.

## References

- [1] H. Amann, *Dynamic theory of quasilinear parabolic systems-III. global existence*, Math. Z., 202(1989), pp. 219-250.
- [2] S. Campanato,  *$L^p$  regularity for weak solutions of parabolic systems*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser. 4,7(1980), pp. 65–85.
- [3] R. S. Cantrell and C. Cosner, *Diffusive logistic equations with indefinite weights: population models in disrupted environments. II*, SIAM J. Math. Anal. 22(1991), no. 4, pp. 1043-1064.
- [4] G. Cupini and R. Petti, *Morrey spaces and local regularity of minimizers of variational integrals*, Rend. Mat. Appl., 21(2001), no. 1-4, pp. 121-141.
- [5] M. Giaquinta and M. Struwe. *On the partial regularity of weak solutions of nonlinear parabolic systems*. Math. Z. 179(1982), pp. 437–451.
- [6] M. W. Hirsch, H. L. Smith, and X.Q. Zhao, *Chain transitivity, attractivity, and strong repellers for semidynamical systems*, J. Dyn. Diff. Eq., 13(2001), no. 1, pp. 107-131.
- [7] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, AMS Transl. Monographs, vol. 23, 1968.
- [8] D. Le, *Cross diffusion systems on  $n$  spatial dimensional domains*, Indiana Univ. Math. J. 51(2002), no. 3, pp. 625-643.
- [9] D. Le, *Global existence for a class of strongly coupled parabolic systems*, Annali di Matematica Pura ed Applicata, 185(2006), no. 1, pp. 133-154.
- [10] D. Le and T. Nguyen, *Uniform boundedness of solutions for triangular parabolic system*, submitted.
- [11] D. Le and T. Nguyen, *Persistence for a class of triangular cross diffusion parabolic systems*, Adv. Non. Studies, 5(2005), 509-530.
- [12] D. Le, L. Nguyen, and T. Nguyen, *Shigesada-Kawasaki-Teramoto Model on Higher Dimensional Domains*, E. J. Diff. Eq., 2003(2003), no. 72, pp. 1-12.
- [13] P. Magal and X.-Q. Zhao. *Global attractors and steady states for uniformly persistent dynamical systems*, SIAM. J. Math. Anal., 37(2005), no. 1, pp. 251-275.
- [14] D. Palagachev, M. Ragusa, and L. Softova, *Regular oblique derivative problem in Morrey spaces*, E. J. Diff. Eq., 39(2000), pp. 1–17.
- [15] W. Schlag, *Schauder and  $L^p$  estimates for parabolic systems via Campanato spaces*, Comm. in PDEs, 21(1996), no. 7&8, pp. 1141-1175.
- [16] N. Shigesada, K. Kawasaki and E. Teramoto. *Spatial segregation of interacting species*, J. Theoretical Biology, 79(1979), pp. 8399.