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Persistence for a Class of Triangular Cross Diffusion Parabolic Systems

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Abstract

The purpose of this paper is to investigate the dynamics of a class of triangular parabolic systems given on bounded domains of arbitrary dimension. In particular, the existence of global attractors and the persistence property will be established.

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1 Introduction

In a recent work [16], we studied the global existence of a triangular cross diffusion parabolic systems of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla[(d_1 + \alpha_{11}u + \alpha_{12}v)\nabla u + \beta_{11}u\nabla v] + u(a_1 - b_1u - c_1v), \\ \dfrac{\partial v}{\partial t} = \nabla[(d_2 + \alpha_{21}u + \alpha_{22}v)\nabla v] + v(a_2 - b_2u - c_2v), \end{cases} \tag{1.1}$$

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which is supplied with the Neumann ($r_1 = r_2 = 0$) or Robin type boundary conditions

$$\frac{\partial u}{\partial n} + r_1(x)u = 0, \quad \frac{\partial v}{\partial n} + r_2(x)v = 0 \tag{1.2}$$

on the boundary $\partial\Omega$ of a bounded domain Ω in $\mathbb{R}^n$. Here r_1, r_2 are given nonnegative smooth functions on $\partial\Omega$. The initial conditions are described by $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$, $x \in \Omega$. Here $u_0, v_0 \in W^{1,p}(\Omega)$ for some $p > n$.

The system (1.1) has its origin from the Shigesada, Kawasaki and Teramoto model ([21])

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + \alpha'_{11}u + \alpha'_{12}v)u] + u(a_1 - b_1u - c_1v), \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + \alpha'_{21}u + \alpha'_{22}v)v] + v(a_2 - b_2u - c_2v), \end{cases} \tag{1.3}$$

in population dynamics, which has been recently investigated to study the competition of two species with cross diffusion effects. In the context of ecology, d_i's and α'_{ij}'s are the self and cross dispersal rates, a_i's represent growth rates, b_1, c_2 denote self-limitation rates, and c_1, b_2 are the interaction rates.

Many works have been done under the assumption that $\alpha'_{21} = 0$. In this case, our system (1.1) is a bit more general by having the term $\alpha_{21}u$ in the equation for v. Furthermore, the flux components in (1.1), when $\alpha_{12} \neq \beta_{11}$, do not have to be gradients of some functions as described in (1.3).

As far as we know, only global existence results were obtained for this system. In particular, one can find global existence results for a simplified version of (1.3) (when $\alpha'_{21} = 0$) in [4, 13, 16, 19], and a regularity result for the full system in [15].

A central issue in population dynamics is the long-term development of populations, and one finds terms such as uniform persistence, coexistence, and extinction describing important special types of asymptotic behavior of the solutions of associated model equations. If α_{ij}, β_{ij} are all zero, (1.1) reduced to the well known Lotka Volterra system, whose persistence property has been widely studied (see [8] for a good reference). However, to the best of our knowledge, this issue has not ever been addressed for cross diffusion cases. This is, of course, due to the presence of the cross diffusion terms making necessary a priori estimates extremely difficult.

In our previous results [13] and [19], we proved the existence of the global attractor for the system (1.1) with $\alpha_{21} = 0$. Global existence results for the case $\alpha_{21} > 0$ were established in [16]. Recently, in [17], we can only show that the L^∞ norms of solutions of (1.1) are ultimately uniformly bounded. We should remark that the presence of the term $\alpha_{21}u$ in the self-diffusion term in the equation of v makes the methods in [4, 13, 19] inapplicable. Furthermore, these methods require that the dimension n is less than 6. This restriction is not assumed in this current paper (and [16, 17]).

Steady state or coexistence problems for similar systems were also extensively studied (see [10] and the reference therein). However, whether these coexistence states are observable, that is their stability, is still yet to be determined. This question remains widely open even for the simpler Lotka-Volterra counterpart. Coexistence in the sense of uniform persistence would then be more appropriate and realistic. Roughly speaking, uniform persistence means that there are positive threshold levels below which time dependent solutions will never be for large t. In biological terms, this means that no species will be either wiped out or completely invaded by others.

Persistence theories for general dynamical systems have been available for some years (see [9] and the references therein). It is now well known that the first step needed to apply these theories to a concrete model is to establish the existence of the global attractor. For regular diffusion cases, by the smoothing effect of parabolic equations, this type of results is almost immediate as long as one can show that the L^∞ norms of the solutions are ultimately uniformly bounded. However, this is not the case for cross diffusion systems as one has to go further to show that the solutions are regular in higher norms, which are also uniformly bounded. To achieve this, more sophisticated PDE techniques will be needed.

Our first main result is to obtain uniform estimates in higher norms to establish the existence of an absorbing ball in the $W^{1,p}$ space as well as the compactness of the semiflow. Since u, v are population densities, only positive solutions are of interest in this paper. We then study the dynamics of the system on the positive cone of $W^{1,p}$, and prove the following theorem in Section 2.

Theorem 1.1 *Assume that $\alpha_{ij} \geq 0, d_i, \beta_{11} > 0, i, j = 1, 2$ and*

$$\alpha_{11} > \alpha_{21}, \quad \alpha_{22} > \alpha_{12}, \quad \textit{and} \quad \alpha_{22} \neq \alpha_{12} + \beta_{11}. \tag{1.4}$$

Then (1.1) defines a dynamical system on $W^{1,p}_+(\Omega)$, the positive cone of $W^{1,p}(\Omega)$, for some $p > n$.

This dynamical system possesses a global attractor in $W^{1,p}(\Omega)$. Furthermore, there exist $\nu > 1$ and a positive constant C_∞ independent of initial conditions such that

$$\|u(.,t)\|_{C^\nu}, \|v(.,t)\|_{C^\nu} \leq C_\infty \tag{1.5}$$

for sufficiently large t.

In population dynamics terms, the first two conditions in (1.4) means that self diffusion rates are stronger than cross diffusion ones. The third condition is a technical one. In fact, this condition was only used in [16, 17] to derive uniform estimates for L^∞ norms of the solutions via the existence of a Lyapunov function. The proof in this paper employs Morrey's estimates and imbedding theorems to achieve higher regularity. Once again, we should point out that the techniques in [4], in the absence of the term $\alpha_{21}u$, can only give that the $W^{1,p}$ norms do not blow up, and hence the global existence result. Meanwhile, [19, 16] do not provide uniform estimates like (1.5) for first order derivatives, which will be crucial for our proof of persistence below. Moreover, our technique works for more general systems and requires only uniform L^∞ estimates at the onset (see the assumptions (Q.1), (Q.2) in Section 2). Thus, Theorem 2.1 and Theorem 2.2 in Section 2 can apply to much more general settings, provided that L^∞ estimates are derived by other means.

Our next goal is to study the uniform persistence property of positive solutions of (1.1). We take advantage of the theory developed in [9] for dynamical systems (see Theorem 3.1), and apply it to our model. We will prove the following result.

Theorem 1.2 *Assume (1.4) holds, and that the principal eigenvalues of the following problems:*

$$\begin{cases} \lambda\psi = d_1\Delta\psi + a_1\psi, \\ \dfrac{\partial\psi}{\partial n} + r_1\psi = 0, \end{cases} \quad \text{and} \quad \begin{cases} \lambda\phi = d_2\Delta\phi + a_2\phi, \\ \dfrac{\partial\phi}{\partial n} + r_2\phi = 0, \end{cases} \tag{1.6}$$

are positive. Moreover, we assume that either

(P.1) $r_1 = r_2 \equiv 0$ *and*

$$\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2},$$

or

(P.2) $r_1, r_2 \neq 0$ *and*

$$\min\left\{\frac{b_1}{b_2}, \frac{\alpha_{11}}{2\alpha_{21}}\right\} > \frac{a_1}{a_2} > \max\left\{\frac{c_1}{c_2}, \frac{2\alpha_{12}}{\alpha_{22}}\right\},$$

and

(r.1) $\alpha_{12} > \beta_{11}$ *and* $d_1\alpha_{22} > 2d_2\beta_{11}$*;*

(r.2) *the quantities* α_{21}, $\alpha_{12} - \beta_{11}$, $|a_1d_2 - a_2d_1|$ *and* $\sup_{x\in\partial\Omega} |r_1(x) - r_2(x)|$ *are all sufficiently small.*

Then the system (1.1) is uniformly persistent, that is, there exists $\eta > 0$ *such that for any initial data* $u_0, v_0 \in W^{1,p}(\Omega)$ *with* $u_0, v_0 > 0$ *we have*

$$\liminf_{t\to\infty} \|u(.,t)\|_{C^1(\Omega)} \geq \eta, \quad \liminf_{t\to\infty} \|v(.,t)\|_{C^1(\Omega)} \geq \eta. \tag{1.7}$$

In the context of biology, this means that no species is completely invaded or wiped out by the other so that they coexist in time. From the structure of (1.1), the positivity of λ in (1.6) and the results of [2], it is known that the system possesses three trivial and semitrivial steady states $(0,0)$, $(0,v_*)$ and $(u_*,0)$. The trivial one describes the situation when both species are wiped out from the environment. The other two semitrivial solutions model the survival of one species while the other is completely invaded. The positivity of the principal eigenvalues in (1.6) gives the instability of the trivial steady state (see Proposition 3.1). Our conditions (P.1), (P.2) are essentially to guarantee that the two semitrivial steady states are unstable (or repelling) in their complement directions.

It is worth noticing that (P.1) is already well known for the Lotka-Volterra counterparts with homogeneous Neumann boundary conditions (see [2, 3, 8] and the references therein). It is not quite surprising to see that the cross diffusion parameters $(\alpha_{ij}, \beta_{11})$ do not manifest in this case as the semitrivial steady states u_*, v_* are being just constants. The situation will be more interesting when we consider (P.2) and the Robin boundary conditions in (1.1). Now, the semitrivial steady states are nonhomogeneous; and the cross diffusion (or gradient) effects will play an essential role.

The proof of this theorem will be presented at the end of Section 3. In fact, we will establish sufficient conditions for the uniform persistence of each component. That is to say when one species is not wiped out by the other (see Proposition 3.2 with Lemma 3.1, and Proposition 3.3 with Lemma 3.2).

Finally, we would like to remark that the uniform persistence property in this paper is established in the C^1 norm instead of the usual L^∞ norm widely used in literature of Lotka-Volterra systems. This is in part due to the setting of the phase space $W^{1,p}$ for strongly coupled parabolic systems (see [1]). So, our persistence result does not rule out the possibility that solutions might form spikes at some points but approach zero almost everywhere as $t \to \infty$. That type of behavior can be seen in some models for chemotaxis, which also involve a form of strong coupling, so it may be that the results presented here are optimal. However, it is naturally to ask if it is impossible for one species can survive in the sense that its density is going to be almost negligible (that is, the L^∞ norm goes to zero) while oscillating wildly to maintain the positivity of its C^1 norm. The answer to this question is still under investigation.

2 Estimates for the gradients

In this section we will establish the uniform bound (1.5) for the gradients and prove Theorem 1.1. In fact, we will consider a more general parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla[P(u,v)\nabla u + R(u,v)\nabla v] + f(u,v), \\ \dfrac{\partial v}{\partial t} = \nabla[Q(u,v)\nabla v] + g(u,v), \end{cases} \tag{2.1}$$

with Neumann or Robin boundary conditions. For the sake of simplicity, we will deal with the Neumann conditions $\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0$ in the proof below, and leave the Robin case to Remark 2.1.

In order to prove (1.5) for (2.1), we assume the following conditions on the parameters of the system and the uniform boundedness of the solutions.

(Q.1) There exists a positive constant d such that $P(u,v), Q(u,v) \geq d$. Moreover, there is a constant C such that $|R(u,v)| \leq C|u|$.

(Q.2) The solutions are uniformly bounded. That is

$$\limsup_{t\to\infty} \|u(.,t)\|_\infty, \limsup_{t\to\infty} \|v(.,t)\|_\infty \leq C_\infty \tag{2.2}$$

for some constant C_∞ independent of the initial data u_0, v_0.

Indeed, we proved in [12] that weak bounded solutions of triangular parabolic systems including (2.1) are Hölder continuous and therefore classical (see [1]). Moreover the C^α norms of solutions are ultimately bounded by a positive constant dependent only on their

L^∞ norms. Thus, (2.2) implies the existence of a constant $C_\infty(\alpha)$ independent of initial data such that

$$\limsup_{t\to\infty}\|u(.,t)\|_{C^\alpha}, \limsup_{t\to\infty}\|v(.,t)\|_{C^\alpha} \le C_\infty(\alpha),\ \forall\alpha\in(0,1). \tag{2.3}$$

Our main estimate of this section is the following.

Theorem 2.1 *Let (u,v) be a nonnegative solution of (2.1) satisfying (Q.1), (Q.2). For any $p\ge 1$, there exists a positive constant $C_{\infty,p}$ independent of the initial data such that*

$$\limsup_{t\to\infty}\|u(.,t)\|_{1,p} + \limsup_{t\to\infty}\|v(.,t)\|_{1,p} \le C_{\infty,p}. \tag{2.4}$$

Furthermore, the following stronger estimate also holds.

Theorem 2.2 *Let (u,v) be a nonnegative solution of (2.1) satisfying (Q.1) and (Q.2). There exist finite constants C_∞ and $\nu>1$ such that*

$$\limsup_{t\to\infty}\|u(.,t)\|_{C^\nu} + \limsup_{t\to\infty}\|v(.,t)\|_{C^\nu} \le C_\infty. \tag{2.5}$$

The main idea of the proof is to use the imbedding results for Morrey's spaces. We recall the definitions of the Morrey space $M^{p,\lambda}(\Omega)$ and the Sobolev-Morrey space $W^{1,(p,\lambda)}$. Let $B_R(x)$ denotes a cube centered at x with radius R in $\mathbb{R}^n$.

We say that $f\in M^{p,\lambda}(\Omega)$ if $f\in L^p(\Omega)$ and

$$\|f\|^p_{M^{p,\lambda}} := \sup_{x\in\Omega,\rho>0}\rho^{-\lambda}\int_{B_\rho(x)}|f|^p dy < \infty.$$

Moreover, f is in the Sobolev-Morrey space $W^{1,(p,\lambda)}$ if $f\in W^{1,p}(\Omega)$ and

$$\|f\|^p_{W^{1,(p,\lambda)}} := \|f\|^p_{M^{p,\lambda}} + \|\nabla f\|^p_{M^{p,\lambda}} < \infty.$$

If $\lambda<n-p$, $p\ge 1$, and $p_\lambda = \frac{p(n-\lambda)}{n-\lambda-p}$, we then have the following imbedding result (see Theorem 2.5 in [5])

$$W^{1,(p,\lambda)}(B)\subset M^{p_\lambda,\lambda}(B). \tag{2.6}$$

We then proceed by proving some estimates for the Morrey norms of the gradients of the solutions. In the sequel, the temporal variable t is always assumed to be sufficiently large such that (see (2.3))

$$\|u(.,t)\|_{C^\alpha}, \|v(.,t)\|_{C^\alpha} \le C_\infty(\alpha),\ \forall\alpha\in(0,1) \text{ and } t\ge T, \tag{2.7}$$

where T may depend on the initial data.

From now on, let us fix a point $(x,t)\in\overline{\Omega}\times(T,\infty)$. As far as no ambiguity can arise, we write $B_R = B_R(x)$, $\Omega_R = \Omega\bigcap B_R$, and $Q_R = \Omega_R\times[t-R^2,t]$.

We first have the following technical lemma.

Lemma 2.1 *For sufficiently small $R > 0$, we have the following estimate*

$$\int_{\Omega_R} (|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2)\ dx + \iint_{Q_R} [u_t^2 + v_t^2 + |\Delta u|^2 + |\Delta v|^2]\ dz \le CR^{n-2+2\alpha}.$$

For the proof, we will need two useful results from [11] by Ladyzhenskaya et al. for scalar functions. It is easy to see that they also hold for vector valued functions as we restate in the following lemmas.

Lemma 2.2 ([11, Lemma II.5.4]) *For any function u in $W^{1,2s+2}(\Omega, \mathbb{R}^m)$ and any smooth function ξ such that $\frac{\partial u}{\partial n}\xi$ vanishes on $\partial\Omega$, we have*

$$\int_{\Omega} |\nabla u|^{2s+2}\xi^2\ dx \le C osc^2\{u, \Omega\} \int_{\Omega} (|\nabla u|^{2s-2}|\Delta u|^2\xi^2 + |\nabla u|^{2s}|\nabla\xi|^2)\ dx. \quad (2.8)$$

Here, C is a constant depending on n, m, s.

Lemma 2.3 ([11, Lemma II.5.3]) *Let $\alpha > 0$ and v be a nonnegative function such that for any ball B_R and $\Omega_R = \Omega \bigcap B_R$ the estimate*

$$\int_{\Omega_R} v(x)\ dx \le CR^{n-2+\alpha}$$

holds. Then for any function ξ from $W_0^{1,2}(B_R)$ the inequality

$$\int_{\Omega_R} v(x)\xi^2\ dx \le CR^{\alpha} \int_{\Omega_R} |\nabla\xi|^2\ dx \quad (2.9)$$

is valid.

Proof of Lemma 2.1. Let $\xi(x,t)$ be a cut off function for Q_R and Q_{2R}. That is, $\xi = 1$ on Q_R and $\xi = 0$ outside Q_{2R}. Integration by parts in x gives

$$\iint_{Q_{2R}} v_t \Delta v \xi^2\ dz = \iint_{Q_{2R}} [-\frac{1}{2}\frac{\partial(|\nabla v|^2\xi^2)}{\partial t} + |\nabla v|^2\xi\xi_t - v_t\nabla v\xi\nabla\xi]\ dz.$$

We test the equation of v by $-\Delta v\xi^2$. Since $\xi(x, t - 2R^2) = 0$, the above and a simple use of the Young inequality yield

$$\begin{aligned}\int_{\Omega_R} |\nabla v(x,t)|^2\ dx &+ \iint_{Q_{2R}} |\Delta v|^2\xi^2\ dz \\ &\le \iint_{Q_2R} [\epsilon v_t^2\xi^2 + C(|\nabla u|^4 + |\nabla v|^4)\xi^2]\ dz \quad (2.10) \\ &+ C\iint_{Q_{2R}} |\nabla v|^2(|\xi_t| + |\nabla\xi|^2)\ dz + C\iint_{Q_{2R}} \xi^2\ dz.\end{aligned}$$

Here, we have used the fact that f, g are uniformly bounded thanks to (2.2). Also, because the solutions are classical, the integrals of $|\nabla u|^4, |\nabla v|^4$ make sense. Similarly, test the equation of u by $-\Delta u \xi^2$ to get

$$\begin{aligned}\int_{\Omega_R} |\nabla u(x,t)|^2\,dx &+ \iint_{Q_{2R}} |\Delta u|^2 \xi^2\,dz \\ &\leq \epsilon \iint_{Q_2R} u_t^2 \xi^2\,dz + C\iint_{Q_{2R}} (|\nabla u|^4 + |\nabla v|^4 + C|\Delta v|^2)\xi^2\,dz \\ &+ C\iint_{Q_{2R}} |\nabla u|^2(|\xi_t| + |\nabla \xi|^2)\,dz + C\iint_{Q_{2R}} \xi^2\,dz. \qquad (2.11)\end{aligned}$$

From the equations of (2.1), we also infer

$$u_t^2 + v_t^2 \leq C(|\Delta u|^2 + |\Delta v|^2 + |\nabla u|^4 + |\nabla v|^4 + |\nabla u|^2 + |\nabla v|^2 + 1). \qquad (2.12)$$

Using this in (2.10), (2.11) and adding them, we get

$$\begin{aligned}\int_{\Omega_R} (|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2)\,dx &+ \iint_{Q_{2R}} (|\Delta u|^2 + |\Delta v|^2)\xi^2\,dz \qquad (2.13) \\ &\leq C\iint_{Q_{2R}} (|\nabla u|^2 + |\nabla v|^2)(\xi^2 + |\xi_t| + |\nabla \xi|^2)\,dz \\ &+ C\iint_{Q_{2R}} (|\nabla u|^4 + |\nabla v|^4)\xi^2\,dz + C\iint_{Q_{2R}} \xi^2\,dz.\end{aligned}$$

Using Lemma 2.2, we have

$$\iint_{Q_{2R}} (|\nabla u|^4 + |\nabla v|^4)\xi^2\,dz \leq CR^\alpha \iint_{Q_{2R}} (|\Delta u|^2 + |\Delta v|^2)\xi^2 + (|\nabla u|^2 + |\nabla v|^2)|\nabla \xi|^2)\,dz.$$

Thus, for sufficiently small R, we see that the integrals of $|\nabla u|^4, |\nabla v|^4$ in (2.13) can be absorbed to the left. This shows that the quantity

$$\int_{\Omega_R} (|\nabla u|^2 + |\nabla v|^2)\,dx + \iint_{Q_{2R}} (|\Delta u|^2 + |\Delta v|^2)\xi^2\,dz$$

can be majorized by

$$C\iint_{Q_{2R}} [(|\nabla u|^2 + |\nabla v|^2)(\xi^2 + |\xi_t| + |\nabla \xi|^2) + \xi^2]\,dz. \qquad (2.14)$$

This fact and (2.12), together with another use of Lemma 2.2, show that the quantity

$$\int_{\Omega_R} (|\nabla u(x,t)|^2 + |\nabla v(x,t)|^2)\,dx + \iint_{Q_R} (u_t^2 + v_t^2)\,dz + \iint_{Q_R} (|\Delta u|^2 + |\Delta v|^2)\,dz$$

is also bounded by (2.14).

Finally, by testing equations of u and v in (2.1) with $(u - u_R)\xi^2$ and $(v - v_R)\xi^2$ respectively, with u_R, v_R being the averages of u, v over Q_R and ξ being the cut-off function for Q_R and Q_{3R}, we can easily prove that

$$\iint_{Q_{2R}} (|\nabla u|^2 + |\nabla v|^2)\, dz \le CR^{n+2\alpha}.$$

Putting this and the fact that $|\xi_t|, |\nabla\xi|^2 \le CR^{-2}$ into (2.14), we see that the claims in our lemma are established. ∎

The following lemma shows that $\nabla u, \nabla v$ are uniformly bounded in $W^{1,(2,n-4+2\alpha)}(\Omega_R)$ norms so that the imbedding (2.6) can be used.

Lemma 2.4 *For $R > 0$ sufficiently small, we have the following estimates:*

$$\int_{\Omega_R} (u_t^2 + v_t^2)\, dx \le CR^{n-4+2\alpha}, \tag{2.15}$$

and

$$\int_{\Omega_R} (|\Delta u|^2 + |\Delta v|^2)\, dx \le CR^{n-4+2\alpha}. \tag{2.16}$$

Proof. Again, let $\xi(x,t)$ be a cut off function for Q_R and Q_{2R}. We now test the equation of v with $-(v_t\xi^2)_t$. Integration by parts in t, x gives

$$\begin{aligned}\frac{1}{2}\iint_{Q_{2R}} \frac{\partial(v_t^2\xi^2)}{\partial t}\, dz \quad &- \quad \iint_{Q_{2R}} v_t^2\xi\xi_t\, dz + \iint_{Q_{2R}} (Q\nabla v)_t\nabla(v_t\xi^2)\, dz \\ &= \quad \iint_{Q_{2R}} g_t(u,v)v_t\xi^2\, dz.\end{aligned} \tag{2.17}$$

Note that, by the choice of ξ and the Neumann condition of v, $\xi\dfrac{\partial v}{\partial n} = 0$ on $\partial\Omega_{2R}$. Therefore the boundary integrals resulting in the integration by parts are all zero.

As

$$(Q\nabla v)_t\nabla(v_t\xi^2) = (Q\nabla v_t + Q_u u_t\nabla v + Q_v v_t\nabla v)(\nabla v_t\xi^2 + 2v_t\xi\nabla\xi),$$

we easily see that (2.17), the ellipticity condition (Q.1), (2.2) and the facts that $\xi(x, t - 2R^2) = 0$ and $|g_t(u,v)| \le C|u_t| + |v_t|$ give

$$\begin{aligned}\int_{\Omega_R} v_t^2\, dx \quad &+ \quad \iint_{Q_{2R}} |\nabla v_t|^2\xi^2\, dz \le C\iint_{Q_{2R}} (u_t^2 + v_t^2)(\xi^2 + |\xi_t|)\, dz + \\ &+ \quad C\iint_{Q_{2R}} [|v_t\nabla v_t\xi\nabla\xi| + |(|u_t| + |v_t|)\nabla v\nabla v_t\xi^2| + |(|u_t| + |v_t|)\nabla v v_t\xi\nabla\xi|]\, dz.\end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned}
|v_t\nabla v_t\xi\nabla\xi| &\le \epsilon|\nabla v_t|^2\xi^2 + C(\varepsilon)v_t^2|\nabla\xi|^2,\\
|(u_t+v_t)\nabla v\nabla v_t\xi^2| &\le \epsilon|\nabla v_t|^2\xi^2 + C(\varepsilon)(u_t^2+v_t^2)|\nabla v|^2\xi^2,\\
|(u_t+v_t)\nabla v v_t\xi\nabla\xi| &\le C[(u_t^2+v_t^2)|\nabla v|^2\xi^2 + v_t^2|\nabla\xi|^2].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\Omega_R} v_t^2\,dx &+ \iint_{Q_{2R}} |\nabla v_t|^2\xi^2\,dz \qquad (2.18)\\
&\le C\iint_{Q_{2R}} (u_t^2+v_t^2)|\nabla v|^2\xi^2\,dz + C\iint_{Q_{2R}} (u_t^2+v_t^2)(\xi^2+|\xi_t|+|\nabla\xi|^2)\,dz.
\end{aligned}$$

Next, we test the equation of u with $-(u_t\xi^2)_t$ and note that

$$(R\nabla v)_t\nabla(u_t\xi^2) = (R_u u_t\nabla v + R\nabla v_t)(\nabla u_t\xi^2 + 2u_t\xi\nabla\xi).$$

By Hölder inequality, we also have

$$\begin{aligned}
\iint_{Q_{2R}} |(R\nabla v)_t\nabla(u_t\xi^2)|\,dz &\le \epsilon\iint_{Q_{2R}} |\nabla u_t|^2\xi^2\,dz + C\iint_{Q_{2R}} |\nabla v_t|^2\xi^2\,dz\\
&+ \iint_{Q_{2R}} (u_t^2|\nabla v|^2\xi^2 + u_t^2|\nabla\xi|^2)\,dz.
\end{aligned}$$

Hence, arguing similarly as before, we also have

$$\begin{aligned}
\int_{\Omega_R} u_t^2\,dx &+ \iint_{Q_{2R}} |\nabla u_t|^2\xi^2\,dz \le C\iint_{Q_{2R}} |\nabla v_t|^2\xi^2\,dz \qquad (2.19)\\
&+ C\iint_{Q_{2R}} (u_t^2+v_t^2)(|\nabla u|^2+|\nabla v|^2)\xi^2 + (u_t^2+v_t^2)(\xi^2+|\xi_t|+|\nabla\xi|^2)\,dz.
\end{aligned}$$

By (2.18), the integral of $|\nabla v_t|^2\xi^2$ can be eliminated from the right hand side. The result and (2.18) together show that

$$\begin{aligned}
\int_{\Omega_R} u_t^2+v_t^2\,dx &+ \iint_{Q_{2R}} (|\nabla u_t|^2+|\nabla v_t|^2)\xi^2\,dz \le \qquad (2.20)\\
&+ C\iint_{Q_{2R}} (u_t^2+v_t^2)(|\nabla u|^2+|\nabla v|^2)\xi^2 + (u_t^2+v_t^2)(\xi^2+|\xi_t|+|\nabla\xi|^2)\,dz.
\end{aligned}$$

As we proved in Lemma 2.1, $\int_{\Omega_R} (|\nabla u|^2+|\nabla v|^2)\,dx \le cR^{n-2+2\alpha}$. This allows us to apply Lemma 2.3, with the function v replaced by $|\nabla u|^2+|\nabla v|^2$, to derive

$$\iint_{Q_{2R}} (|\nabla u|^2+|\nabla v|^2)v_t^2\xi^2\,dz \le cR^{2\alpha}\iint_{Q_{2R}} [|\nabla v_t|^2\xi^2 + v_t^2|\nabla\xi|^2]\,dz$$

and

$$\iint_{Q_{2R}} (|\nabla u|^2 + |\nabla v|^2) u_t^2 \xi^2 \, dz \le cR^{2\alpha} \iint_{Q_{2R}} [|\nabla u_t|^2 \xi^2 + u_t^2 |\nabla \xi|^2] \, dz.$$

Hence, for R sufficiently small, we obtain from (2.20) that

$$\int_{\Omega_R} u_t^2(x,t) + v_t^2(x,t) \, dx \le C \iint_{Q_{2R}} (u_t^2 + v_t^2)(\xi^2 + |\xi_t| + |\nabla \xi|^2) \, dz. \qquad (2.21)$$

Applying Lemma 2.1 and using the fact that $|\xi_t|, |\nabla \xi|^2 \le CR^{-2}$, we obtain the desired inequality (2.15). For (2.16), we solve Δu and Δv in terms of $u_t, v_t, \nabla u$, and ∇v and then integrate them over Ω_R to get

$$\int_{\Omega_R} (|\Delta u|^2 + |\Delta v|^2) \xi^2 \, dx \le C \int_{\Omega_R} (u_t^2 + v_t^2 + |\nabla u|^2 + |\nabla v|^2 + |\nabla u|^4 + |\nabla v|^4 + 1) \xi^2 \, dx.$$

We then use Lemma 2.2 again to absorb the term $|\nabla u|^4 + |\nabla v|^4$ to the left hand side. The result is

$$\int_{\Omega_R} (|\Delta u|^2 + |\Delta v|^2) \xi^2 \, dx \le C \int_{\Omega_R} [(u_t^2 + v_t^2) \xi^2 + (|\nabla u|^2 + |\nabla v|^2)(\xi^2 + |\nabla \xi|^2) + \xi^2] \, dx.$$

This, Lemma 2.1 and (2.15) give (2.16), and complete our proof. ∎

We are now ready to give

Proof of Theorem 2.1. Thanks to the above lemmas, the estimate

$$\int_{\Omega_R} [(u_t^2 + v_t^2) + (|\nabla u|^2 + |\nabla v|^2) + (|\nabla u|^4 + |\nabla v|^4) + (|\Delta u|^2 + |\Delta v|^2)] \, dx \le CR^{n-4+2\alpha}$$

holds for some constant C independent of the initial data if t is sufficiently large.

By rewriting the equations of u, v as $P\Delta u = \widetilde{F}$ and $Q\Delta v = \widetilde{G}$, with $\widetilde{F}, \widetilde{G}$ depending on the first order derivatives of u, v in x, t, and using the above estimates, we can apply [20, Lemma 4.1] to assert that the norms of ∇u and ∇v in $W^{1,(2,\lambda)}(\Omega_R)$, with $\lambda = n-4+2\alpha$, are uniformly bounded. Therefore, by the imbedding (2.6), and the fact that $M^{2_\lambda,\lambda} \subset L^{2_\lambda}$, we have $\|\nabla u(\bullet,t)\|_{L^{2_\lambda}(\Omega)}$ and $\|\nabla v(\bullet,t)\|_{L^{2_\lambda}(\Omega)}$, with $2_\lambda = \frac{2(4-2\alpha)}{2-2\alpha}$, are bounded by some constant C. Since α can be arbitrarily chosen in $(0,1)$, 2_λ can be as large as desired. This proves (2.4).

Regarding (2.5), we rewrite the equation of v as follows:

$$v_t = Q\Delta v + G$$

with $G = Q_u \nabla u \nabla v + Q_v |\nabla v|^2 + g$. Since u, v are Hölder continuous with uniformly bounded norms, we can regard Q as a Hölder continuous function in (x,t). Therefore, we can apply ii) of [13, Lemma 2.5] here to obtain

$$\|v(.,t)\|_{C^\mu} \le t^{-\beta}e^{-\delta t}\|v(.,\tau)\|_r + \int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}(\|\nabla u\|_{2r}^2 + \|\nabla v\|_{2r}^2 + \|g\|_r)ds \tag{2.22}$$

for any $0 < \tau < t$ and $2\beta > \mu + n/r$. Using (2.4) and (2.2), for sufficiently large t, τ, we have

$$\|v(.,t)\|_{C^\mu} \le C(r)t^{-\beta}e^{-\delta t} + C(r)\int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}ds \tag{2.23}$$

for some constant $C(r)$ independent of the initial data. The above integral is finite for all t if $\beta \in (0,1)$. Obviously, we can choose r sufficiently large and $\mu > 1$ such that $\beta < 1$, and therefore prove that $\|v(.,t)\|_{C^\mu}$ is uniformly bounded for large t. Finally, such Hölder estimate for ∇v allows us to follow the proof of Theorem 2.2 in [13] to get the uniform estimate for $\|u(.,t)\|_{C^\mu}$ as desired. ∎

Remark 2.1 The case of Robin boundary conditions can be reduced to the Neumann one by a simple change of variables. First of all, since our proof based on the local estimates of Lemma 2.1 and Lemma 2.4, we need only to study these inequalities near the boundary. As $\partial\Omega$ is smooth, we can locally flatten the boundary and assume that $\partial\Omega$ is the plane $\{x_n = 0\}$. Furthermore, we can take $\Omega_R = \{(x', x_n) : x_n > 0, |(x', x_n)| < r\}$. The boundary conditions become

$$\frac{\partial u}{\partial x_n} + \widetilde{r}_1(x')u = 0, \quad \frac{\partial v}{\partial x_n} + \widetilde{r}_2(x')v = 0.$$

We then introduce

$$U(x', x_n) = \exp(x_n\widetilde{r}_1(x'))u(x', x_n), \quad V(x', x_n) = \exp(x_n\widetilde{r}_2(x'))v(x', x_n).$$

Obviously, U, V satisfy the Neumann boundary condition on $x_n = 0$. Simple calculations also show that U, V verify a system similar to that for u, v, and the conditions (Q.1), (Q.2) are still valid. In fact, there will be some extra terms occurring in the divergence parts of the equations for U, V, but these terms can be handled by a simple use of Young's inequality so that our proof can go on with minor modifications. Thus Theorem 2.1 applies to U, V, and the estimates for u, v then follow.

We conclude this section by giving the proof of Theorem 1.1.

Proof of Theorem 1.1: In our recent works (see [16], [17]), we proved that nonnegative weak solutions of (1.1) are ultimately uniformly bounded in their L^∞ norms. Therefore, the conditions (Q.1), (Q.2) are verified by (1.1), and Theorem 2.1 applies here. The estimate (2.4) asserts the existence of an absorbing ball in $W^{1,p}(\Omega)$ attracting all solutions. The compactness of associated semiflow in $W^{1,p}(\Omega)$ comes from the estimate (1.5). The existence of the global attractor then follows (see [7]). ∎

3 Persistence results

In this section we shall consider the question of persistence and prove Theorem 1.2. Our proof mainly bases on a persistence result in [9] for general dynamical systems defined on metric spaces. In order to restate this result, let us first recall some definitions in the dynamical system theory. Let (X, d) be a metric space and Φ be a semiflow on X. A subset $A \subset X$ is said to be an **attractor** for Φ if A is nonempty, compact, invariant, and there exists some open neighborhood U of A in X such that $\lim_{t\to\infty} d(\Phi_t(u), A) = 0$ for all $u \in U$. Here, $d(x, A)$ is the usual Hausdorff distance from x to the set A. If A is an attractor which attracts every point in X, A is called **global attractor**. For a nonempty invariant set M, the set $W^s(M) := \{x \in X \ : \ \lim_{t\to\infty} d(\Phi_t(x), M) = 0\}$ is called the **stable** set of M. A nonempty invariant subset M of X is said to be **isolated** if it is the maximal invariant set in some neighborhood of itself.

Let A and B be two isolated invariant sets. A is said to be **chained** to B, denoted by $A \to B$, if there exists a globally defined trajectory $\Phi_t(x)$, $t \in (-\infty, \infty)$, through some $x \notin A \bigcup B$ whose range has compact closure such that the omega limit set $\omega(x) \subset B$ and the alpha limit set $\alpha(x) \subset A$. A finite sequence $\{M_1, M_2, ..., M_k\}$ of isolated invariant sets is called a **chain** if $M_1 \to M_2 \to ... \to M_k$. The chain is called a **cycle** if $M_k = M_1$.

Let $X_0 \subset X$ be an open set and $\partial X_0 = X \setminus X_0$. Assume that X_0 is positively invariant. Let $p(x) = d(x, \partial X_0)$, the distance from x to ∂X_0. Φ is said to be **uniformly persistent** with respect to $(X_0, \partial X_0, p)$ if there exists $\eta > 0$ such that

$$\liminf_{t\to\infty} p(\Phi_t(x)) \geq \eta$$

for all $x \in X_0$.

The following uniform persistence result is established in [9].

Theorem 3.1 (Theorem 4.3 in [9]) *Assume that*

(C1) *Φ has a global attractor A;*

(C2) *There exists a finite sequence $M = \{M_1, ..., M_k\}$ of pairwise disjoint, compact and isolated invariant sets in ∂X_0 with the following properties:*

- **(m.1)** $\bigcup_{x\in\partial X_0} \omega(x) \subset \bigcup_{i=1}^{k} M_i$,
- **(m.2)** *no set of M forms a cycle in ∂X_0,*
- **(m.3)** *M_i is isolated in X,*
- **(m.4)** *$W^s(M_i) \bigcap X_0 = \emptyset$ for each $i = 1, ..., k$.*

Then there exists $\delta > 0$ such that for any $x \in X_0$, the following inequality holds

$$\inf_{y\in\omega(x)} d(y, \partial X_0) > \delta.$$

We will apply this theorem to obtain the uniform persistence for the system (1.1). Let $C^1_+(\Omega) = \{u \in C^1(\Omega) \ : \ u(x) \geq 0 \, \forall x \in \Omega\}$. We define

$$X = C^1_+(\Omega) \times C^1_+(\Omega) \text{ and } X_0 = \{(u, v) \in X \ : \ u(x) > 0 \text{ and } v(x) > 0\},$$

with norm $\|(u,v)\|_X = \|u\|_{C^1(\Omega)} + \|v\|_{C^1(\Omega)}$. Thanks to Theorem 1.1, the solutions with initial data in $W^{1,p}(\Omega)$ will become C^1 smooth so that we need only to consider the system (1.1) and its associated semiflow Φ on X. That is, for any (u_0, v_0) in X, let $(u(\bullet,t), v(\bullet,t))$ be the solution to (1.1) and define $\Phi_t(u_0,v_0) = (u(\bullet,t), v(\bullet,t))$ for all $t \geq 0$.

Firstly, we will see that X_0 is positively invariant with respect to Φ. Indeed, let us rewrite the equation of u as follows:

$$u_t = \text{div}(a(x,t)\nabla u + b(x,t)u) + c(x,t),$$

where $a(x,t) = P(u(x,t), v(x,t)) \geq d > 0$, $b(x,t) = R_u \nabla v(x,t)$ and $c(x,t) = f(u(x,t), v(x,t))$. Here and throughout this section, we denote

$$P(u,v) = d_1 + \alpha_{11}u + \alpha_{12}v,\ R(u) = \beta_{11}u,\ Q(u,v) = d_2 + \alpha_{21}u + \alpha_{22}v,$$

and

$$f(u,v) = u(a_1 - b_1 u - c_1 v),\ g(u,v) = v(a_2 - b_2 u - c_2 v).$$

By virtue of Theorem 1.1, we see that a, b, c are bounded. Using the strong positivity result in [18], we see that $u(x,t) > 0$ for all t. Similar argument shows that the component v also stays positive. Hence, $\Phi_t(u_0,v_0) \in X_0$ for all t. Theorem 1.1 also asserts that Φ has a global attractor in X, and thus (C1) is verified.

Next, we consider the condition (C.2). It is clear that the "boundary" parts $u = 0$ or $v = 0$ of X_0 are also invariant with respect to Φ. On these boundaries, the dynamics of (1.1) is reduced to those of the following scalar parabolic equations.

$$u_t = \nabla(P(u,0)\nabla u) + f(u,0), \quad u(0) > 0, \tag{3.1}$$
$$v_t = \nabla(Q(0,v)\nabla v) + g(0,v), \quad v(0) > 0. \tag{3.2}$$

Investigating the dynamics of these equations leads us to the following steady state equations

$$\nabla(P(u_*,0)\nabla u_*) + f(u_*,0) = 0, \quad \nabla(Q(0,v_*)\nabla v_*) + g(0,v_*) = 0,$$

together with the boundary conditions as in (1.2). If the principal eigenvalues of (1.6) are positive, the above equations admit unique solutions, which are denoted respectively by u_* and v_*. Furthermore, the solutions $u(x,t), v(x,t)$ of (3.1), (3.2) converge to u_*, v_*, respectively, in the $C(\Omega)$ norm as t tends to infinity. Meanwhile, the trivial solution 0 is an unstable steady state for both equations. These claims are obtained by following closely the proof of [2, Corollary 2.4] or [3, Theorem 1.2], where the Dirichlet boundary condition was assumed.

Therefore, the sets $M_0 = (0,0)$, $M_1 = (u_*,0)$, and $M_2 = (0,v_*)$ are pairwise disjoint, compact and isolated invariant sets in ∂X_0 with respect to Φ. Moreover, no set of $\{M_i\}$ can form a cycle in ∂X_0; and $\bigcup_{x\in\partial X_0}\omega(x) \subset \bigcup_{i=0}^2 M_i$. We thus show that the conditions (m.1) and (m.2) are satisfied.

Checking (m.3) and (m.4) requires much more effort. The role of the parameters r_1, r_2 will play an important role here. Let us assume that the system (1.1) satisfies the Robin boundary condition (1.2) with $r_1, r_2 \neq 0$. The Neumann case is simpler, and will be discussed later in Remark 3.1.

We discuss first the property (m.4) at M_0. We will show below that the instability of M_0 is determined by the principal eigenvalue λ of (see (1.6))

$$\begin{cases} \lambda\phi = d_2\Delta\phi + a_2\phi, \\ \dfrac{\partial\phi}{\partial n} + r_2\phi = 0. \end{cases} \tag{3.3}$$

Proposition 3.1 *Assume that the principal eigenvalue λ of (3.3) is positive. There exists $\eta_0 > 0$ such that for any solution (u, v) of (1.1) with $(u_0, v_0) \in X_0$, we have*

$$\limsup_{t\to\infty} \|(u(.,t), v(.,t))\|_X \geq \eta_0.$$

Proof. Let ϕ be the positive eigenfunction associated to the principal eigenvalue λ of (3.3). By testing the equation of v by ϕ and (3.3) by v, we subtract the results to get

$$\frac{d}{dt}\int_\Omega v\phi\,dx = \lambda\int_\Omega v\phi\,dx + \int_\Omega [-Q_0\nabla v\nabla\phi + (g - a_2v)\phi]\,dx - \int_{\partial\Omega} Q_0r_2v\phi\,d\sigma. \tag{3.4}$$

Here, we denoted $Q_0 = Q - d_2 = \alpha_{21}u + \alpha_{22}v$. Integration by parts yields

$$-\int_\Omega Q_0\nabla v\nabla\phi\,dx = \int_\Omega v\nabla(Q_0\nabla\phi)\,dx + \int_{\partial\Omega} r_2Q_0v\phi\,d\sigma.$$

Putting this in (3.4), we infer

$$\frac{d}{dt}\int_\Omega v\phi\,dx = \lambda\int_\Omega v\phi\,dx + \int_\Omega v\phi\frac{\nabla(Q_0\nabla\phi)}{\phi}\,dx - \int_\Omega (b_2u + c_2v)v\phi\,dx.$$

Now, suppose that our claim was false. For any $\eta > 0$, there would be a solution u, v such that $\|(u(.,t), v(.,t))\|_X \leq \eta$ when t is large. This implies that the quantities $\frac{|\nabla(Q_0\nabla\phi)|}{\phi}$ and $(b_2u + c_2v)$ can be very small. Thus, if η is sufficiently small, then the above equation yields

$$\frac{d}{dt}\int_\Omega v\phi\,dx \geq \frac{\lambda}{2}\int_\Omega v\phi\,dx.$$

This shows that, as $t \to \infty$, $\int_\Omega v(.,t)\phi\,dx$ goes to infinity, contradicting the fact that $\|(u, v)\|_X$ is bounded. Our proof is complete. ∎

Next, we study M_1 and M_2. Our main assumption for (m.3) and (m.4) to hold is the instability of M_1, M_2 in their complement v, u directions, respectively. To this end, we consider the linearization of the system (1.1) at a general steady state point (u, v).

$$\begin{cases} \lambda\psi = \nabla[(P_u\psi + P_v\phi)\nabla u + P\nabla\psi + (R_u\psi + R_v\phi)\nabla v + R\nabla\phi] + f_u\psi + f_v\phi, \\ \lambda\phi = \nabla[(Q_u\psi + Q_v\phi)\nabla v + Q\nabla\phi] + g_u\psi + g_v\phi. \end{cases} \tag{3.5}$$

Here ψ (respectively, ϕ) satisfies the boundary condition of u (respectively, v) in (1.2).

Putting $(u, v) = (0, v_*)$ and $(\psi, \phi) = (\psi, 0)$, the instability of $M_2 = (0, v_*)$ in the direction u is determined by the sign of the principal eigenvalue of the following system.

$$\lambda\psi = \nabla(P(0, v_*)\nabla\psi + R_u\psi\nabla v_*) + f_u(0, v_*)\psi, \tag{3.6}$$

with v_* being the solution of

$$0 = \nabla(Q(0, v_*)\nabla v_*) + g(0, v_*). \tag{3.7}$$

We shall establish the following repelling property of $(0, v_*)$.

Proposition 3.2 *Suppose that the principal eigenvalue λ of (3.6) is positive. If $P_v - R_u = \alpha_{12} - \beta_{11}$ is positive and sufficiently small, then there exists $\eta_0 > 0$ such that for any solution (u, v) of (1.1) with $(u_0, v_0) \in X_0$, we have*

$$\limsup_{t\to\infty} \|(u(., t), v(., t)) - (0, v_*)\|_X \geq \eta_0.$$

Similarly, the instability of $M_1 = (u_*, 0)$ in the direction v is determined by the sign of the principal eigenvalue of the following system.

$$\lambda\phi = \nabla(Q(u_*, 0)\nabla\phi) + g_v(u_*, 0)\phi, \tag{3.8}$$

with u_* being the solution of

$$0 = \nabla(P(u_*, 0)\nabla u_*) + f(u_*, 0). \tag{3.9}$$

Proposition 3.3 *Suppose that the principal eigenvalue λ of (3.8) is positive. If $Q_u = \alpha_{21}$ is positive and sufficiently small, then there exists $\eta_0 > 0$ such that for any solution (u, v) of (1.1) with $(u_0, v_0) \in X_0$, we have*

$$\limsup_{t\to\infty} \|(u(., t), v(., t)) - (u_*, 0)\|_X \geq \eta_0.$$

An immediate consequence of these propositions is that $W^s(M_i) \bigcap X_0 = \emptyset, i = 0, 1, 2$ respectively. Otherwise, by the definition of $W^s(M_i)$, there exists $(u_0, v_0) \in X_0$ such that $d((u(t), v(t)), M_i) \to 0$ as $t \to \infty$, a contradiction to the above corresponding propositions.

Moreover, we also see that M_i is isolated in X. Indeed, consider a neighborhood of M_i in X_0, $V = \{(u, v) \in X_0 \ : \ d((u, v), M_i) < \eta_0/2\}$. For any $(u_0, v_0) \in X_0 \bigcap V$, the above proposition shows that $(u(t), v(t))$ will inevitably exits V. This means M_i is maximal in V, and isolated in X.

We now give the proof of Proposition 3.2 and Proposition 3.3.

Proof of Proposition 3.2. The proof is by contradiction. Assume that for any $\eta > 0$ there exists a solution (u, v) of (1.1) and $T > 0$ such that

$$\|u(., t)\|_{C^1(\Omega)}, \|v(., t) - v_*\|_{C^1(\Omega)} < \eta \tag{3.10}$$

for all $t > T$. Hereafter, we always consider $t > T$.

We denote $P_0 = P(0, v_*)$ and recall (3.5):

$$\lambda\psi = \nabla(P_0\nabla\psi + R_u\psi\nabla v_*) + f_u(0, v_*)\psi.$$

Set $\overline{P}(u, v) = \int_0^u P(s, v)ds$. We note that $\nabla\overline{P}(u, v) = P\nabla u + P_v u\nabla v$. Testing the above equation with $\overline{P}$, we obtain

$$\begin{aligned}\lambda\int_\Omega \psi\overline{P}(u, v)\, dx &= -\int_\Omega P_0 P\nabla\psi\nabla u\, dx - \int_\Omega P_0 P_v u\nabla\psi\nabla v\, dx \\ &- \int_\Omega R_u\psi\nabla v_*(P\nabla u + P_v u\nabla v)\, dx + \int_\Omega f_u((0, v_*))\psi\overline{P}\, dx \\ &+ \int_{\partial\Omega} (P_0\frac{\partial\psi}{\partial n} + R_u\psi\frac{\partial v_*}{\partial n})\overline{P}\, d\sigma. \qquad (3.11)\end{aligned}$$

Similarly, we test the equation of u in (1.1) with $P_0\psi$ ($\nabla(P_0\psi) = P_v\psi\nabla v_* + P_0\nabla\psi$), and get

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega P_0 u\psi\, dx &= -\int_\Omega PP_0\nabla u\nabla\psi\, dx - \int_\Omega PP_v\psi\nabla u\nabla v_*\, dx \\ &- \int_\Omega R\nabla v(P_0\nabla\psi + P_v\psi\nabla v_*)\, dx + \int_\Omega fP_0\psi\, dx \\ &+ \int_{\partial\Omega} (P\frac{\partial u}{\partial n} + R\frac{\partial v}{\partial n})P_0\psi\, d\sigma. \qquad (3.12)\end{aligned}$$

From (3.11), (3.12) and the boundary condition (1.2), we find

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega P_0 u\psi\, dx &= \lambda\int_\Omega \psi\overline{P}\, dx + (P_v - R_u)\int_\Omega [P_0 u\nabla v\nabla\psi - P\psi\nabla u\nabla v_*]\, dx \\ &+ \int_\Omega (fP_0 - f_u(0, v_*)\overline{P})\psi\, dx + I_\partial, \qquad (3.13)\end{aligned}$$

where $I_\partial = \int_{\partial\Omega} (\overline{P} - Pu)r_1\psi P_0 + (\overline{P}v_* - uvP_0)R_u r_2\psi\, d\sigma$.

Next, we shall show that the integrals on the right of (3.13) are either nonnegative or controlled by the first integral. From the definition of the parameters, we have

$$\begin{aligned}\overline{P}\psi &= (d_1 + \alpha_{12}v)u\psi + \frac{\alpha_{11}u^2}{2}\psi \geq P_0 u\psi + P_v(v - v_*)u\psi, \\ (fP_0 - f_u(0, v_*)\overline{P})\psi &= (c_1(v_* - v) - b_1 u)P_0 u\psi + f_u(\alpha_{12}(v_* - v) - \frac{\alpha_{11}u}{2})u\psi.\end{aligned}$$

Hence, if η in (3.10) is sufficiently small, the above gives

$$\int_\Omega \overline{P}\psi\, dx \geq \frac{3}{4}\int_\Omega P_0 u\psi\, dx, \quad \left|\int_\Omega (fP_0 - f_u(0, v_*)\overline{P})\psi\, dx\right| \leq \frac{\lambda}{4}\int_\Omega P_0 u\psi\, dx.$$

On the other hand, integrate by parts to get

$$
\begin{aligned}
-\int_\Omega P\psi\nabla u\nabla v_*\,dx &= \int_\Omega u\nabla(P\psi\nabla v_*)\,dx - \int_{\partial\Omega} uP\psi\frac{\partial v_*}{\partial n}\,d\sigma\\
&= \int_\Omega u\psi\frac{\nabla(P\psi\nabla v_*)}{\psi}\,dx + \int_{\partial\Omega} uP\psi r_2 v_*\,d\sigma.
\end{aligned}
$$

Thanks to (3.10) and the fact that $\psi > 0$ on $\bar{\Omega}$, the quantities $|\nabla v||\frac{\nabla\psi}{\psi}|$, $\frac{\nabla(P\psi\nabla v_*)}{P_0\psi}$ are bounded. Thus, if $P_v - R_u$ is positive and sufficiently small, then

$$
(P_v-R_u)\int_\Omega [P_0u\nabla v\nabla\psi - P\psi\nabla u\nabla v_*]\,dx \geq -\frac{\lambda}{4}\int_\Omega P_0u\psi\,dx + (P_v-R_u)\int_{\partial\Omega} Pr_2v_*u\psi\,d\sigma.
$$

Putting these facts in (3.13), we derive

$$
\frac{\partial}{\partial t}\int_\Omega P_0u\psi\,dx \geq \frac{\lambda}{4}\int_\Omega P_0u\psi\,dx + I_\partial + (P_v - R_u)\int_{\partial\Omega} uP\psi r_2v_*\,d\sigma.
$$

Finally, we study the boundary integrals. Straightforward calculations show

$$
I_\partial = \int_{\partial\Omega} [-\frac{\alpha_{11}}{2}ur_1P_0 + (d_1(v_* - v) + \frac{\alpha_{11}}{2}uv_*)R_ur_2]u\psi\,d\sigma.
$$

If η in (3.10) is small, then it is clear that the quantity in the brackets can be very small. Thus, I_∂ can be controlled by the positive boundary integral $(P_v - R_u)\int_{\partial\Omega} Pr_2v_*u\psi\,d\sigma$.

Therefore

$$
\frac{\partial}{\partial t}\int_\Omega P_0u\psi\,dx \geq \frac{\lambda}{4}\int_\Omega P_0u\psi\,dx. \tag{3.14}
$$

As $\lambda > 0$, this shows that $\int_\Omega P(0,v_*)u\psi\,dx$ goes to infinity as t does. This contradicts (3.10) and completes this proof. ∎

Proof of Proposition 3.3. Along the line of the proof of Proposition 3.2, we assume that for any $\eta > 0$ there exists a solution (u,v) of (1.1) and $T > 0$ such that

$$
\|v(.,t)\|_{C^1(\Omega)} + \|u(.,t) - u_*\|_{C^1(\Omega)} < \eta, \quad \text{for any } t > T. \tag{3.15}
$$

Consider the equation (3.8)

$$
\lambda\phi = \nabla(Q_0\nabla\phi) + g_v(u_*,0)\phi,
$$

where $Q_0 = Q(u_*,0)$. Set $\overline{Q}(u,v) = \int_0^v Q(u,s)ds$. Test the above equation with $\overline{Q}$ and the equation of v in (1.1) with $Q_0\phi$, we easily derive

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_\Omega Q_0v\phi\,dx &= \lambda\int_\Omega \phi\overline{Q}\,dx + Q_u\int_\Omega Q_0v\nabla u\nabla\phi\,dx - Q_u\int_\Omega Q\phi\nabla v\nabla u_*\,dx\\
&+ \int_\Omega (gQ_0 - g_v(u_*,0)\overline{Q})\phi\,dx + \int_{\partial\Omega}(\overline{Q} - Qv)r_2\phi Q_0\,d\sigma
\end{aligned} \tag{3.16}
$$

First, for sufficiently small η in (3.15), it is not difficult to see that

$$\left|\int_\Omega (gQ_0 - g_v(u_*,0)\overline{Q})\phi\, dx\right| \le \frac{\lambda}{4}\int_\Omega \phi\overline{Q}\, dx.$$

Next, by integration by parts and the boundary condition of u_*, we get

$$-\int_\Omega Q\phi\nabla v\nabla u_*\, dx = \int_\Omega v\nabla(Q\phi\nabla u_*)\, dx + \int_{\partial\Omega} Qv\phi r_1 u_*\, d\sigma.$$

As $|\nabla u||\nabla\phi|/|\phi|$ and $|\nabla(Q\phi\nabla u_*)|/\phi$ are bounded, if Q_u is sufficiently small, we also have

$$Q_u\int_\Omega Q_0 v\nabla u\nabla\phi\, dx \text{ and } Q_u\int_\Omega v\nabla(Q\phi\nabla u_*)\, dx \ge -\frac{\lambda}{4}\int_\Omega Q_0 v\phi\, dx.$$

Put these estimates in (3.16) to see that

$$\frac{\partial}{\partial t}\int_\Omega Q_0 v\phi\, dx \ge \frac{\lambda}{4}\int_\Omega Q_0 v\phi\, dx + \int_{\partial\Omega}(\overline{Q} - Qv)r_2\phi Q_0\, d\sigma + Q_u\int_{\partial\Omega} Qv\phi r_1 u_*\, d\sigma. \tag{3.17}$$

Concerning the boundary integrals, we note that $\overline{Q} - Qv = -\frac{\alpha_{22}}{2}v^2$. Therefore, if η in (3.15) is sufficiently small, then the sum of the boundary integral is positive. We then conclude that

$$\frac{\partial}{\partial t}\int_\Omega Q_0 v\phi\, dx \ge \frac{\lambda}{4}\int_\Omega Q_0 v\phi\, dx.$$

This inequality shows $\int_\Omega Q_0 v\phi\, dx$ tends to infinity, contradicting (3.15). Our proof is complete. ∎

Remark 3.1 If the boundary conditions are of Neumann type, then u_*, v_*, ψ, ϕ in the above proofs are just constant functions and our calculations will be much simpler. In fact, it is easy to see that the smallness condition for $P_v - R_u$ (respectively Q_u) in Proposition 3.2 (respectively Proposition 3.3) is no longer needed.

Next, we will present explicit and simple criteria on the parameters of (1.1) for the positivity of the principal eigenvalues of (3.6), (3.8).

Lemma 3.1 *Assume that either* $r_1 = r_2 \equiv 0$ *and* $a_1/a_2 > c_1/c_2$, *or*

$$\frac{a_1}{a_2} > \max\left\{\frac{c_1}{c_2}, \frac{2\alpha_{12}}{\alpha_{22}}\right\}, \tag{3.18}$$

and

a) $\alpha_{12} > \beta_{11}$;

b) $d_1\alpha_{22} \ge 2d_2\beta_{11}$; $\sup_{x\in\partial\Omega}(r_1(x) - r_2(x))_+$ *and* $(a_2d_1 - a_1d_2)_+$ *are sufficiently small.*

Then λ in (3.6) is positive.

Proof. Set $P_0 = P(0, v_*)$, $Q_0 = Q(0, v_*)$. We test (3.6) with $\overline{Q} = \int_0^{v_*} Q(0,s)ds$ and test (3.7) with $P(0, v_*)\psi$. Together, we get

$$\begin{aligned}
\lambda\int_\Omega \psi\overline{Q}\,dx &= -\int_\Omega P_0Q_0\nabla\psi\nabla v_*\,dx - \int_\Omega R_uQ_0\psi|\nabla v_*|^2\,dx \\
&+ \int_{\partial\Omega}(P_0\frac{\partial\psi}{\partial n} + R_u\psi\frac{\partial v_*}{\partial n})\overline{Q}\,d\sigma + \int_\Omega f_u\psi\overline{Q}\,dx \\
&= (P_v - R_u)\int_\Omega Q_0\psi|\nabla v_*|^2\,dx + \int_\Omega (f_u\overline{Q} - g_0P_0)\psi\,dx \\
&+ \int_{\partial\Omega}(P_0\frac{\partial\psi}{\partial n} + R_u\psi\frac{\partial v_*}{\partial n})\overline{Q}\,d\sigma - \int_{\partial\Omega} Q_0P_0\psi\frac{\partial v_*}{\partial n}\,d\sigma.
\end{aligned}$$

We need only show that the right hand side is positive. Since $P_v = \alpha_{12} > \beta_{11} = R_u$, the first term on the right is nonnegative. For the second integral, we note that

$$f_u\overline{Q} - g_0P_0 = v_*[(a_1 - c_1v_*)(d_2 + \frac{\alpha_{22}}{2}v_*) - (a_2 - c_2v_*)(d_1 + \alpha_{12}v_*)].$$

We study the quantity in the brackets by considering the quadratic

$$\begin{aligned}
F(X) &= (a_1 - c_1X)(d_2 + \frac{\alpha_{22}}{2}X) - (a_2 - c_2X)(d_1 + \alpha_{12}X) \\
&= (c_2\alpha_{12} - \frac{1}{2}c_1\alpha_{22})X^2 + (\frac{1}{2}a_1\alpha_{22} - a_2\alpha_{12} + c_2d_1 - c_1d_2)X + a_1d_2 - a_2d_1.
\end{aligned}$$

First of all, by a simple use of maximum principles, we can easily show that $0 < v_*(x) \le a_2/c_2$ for all $x \in \overline{\Omega}$. Let $\mu = \inf_\Omega v_*(x) > 0$.

We will show that $F(v_*) > 0$. Firstly, due to (3.18),

$$F(0) = a_1d_2 - a_2d_1 \text{ and } F(a_2/c_2) = (a_1 - \frac{a_2c_1}{c_2})(d_2 + \frac{a_2\alpha_{22}}{2c_2}) > 0.$$

Consider the case when the coefficient of X^2 in $F(X)$ is negative. If $F(0) \ge 0$ then $F(v_*) > 0$ because $0 < \mu \le v_*(x) \le a_2/c_2$. If $F(0) < 0$, then $F(X) = 0$ has two positive roots X_1, X_2 with $X_2 > a_2/c_2$. Hence, if $|F(0)|$ is sufficiently small then $\mu > X_1$ and therefore $F(v_*) > 0$.

Otherwise, by (3.18), we have $F(v_*) \ge (\frac{1}{2}a_1\alpha_{22} - a_2\alpha_{12} + c_2d_1 - c_1d_2)v_* + F(0)$. If $(c_2d_1 - c_1d_2) \ge 0$, the last quantity is obviously positive when either $F(0) \ge 0$ or $F(0) < 0$ but $|F(0)|$ is small. Or else, because $v_* \le a_2/c_2$ we have

$$F(v_*) \ge (c_2d_1 - c_1d_2)\frac{a_2}{c_2} + a_1d_2 - a_2d_1 = a_2d_2(\frac{a_1}{a_2} - \frac{c_1}{c_2}) > 0.$$

In all cases, $F(v_*) > 0$. Thus, the second integral is also positive. It remains to consider the boundary integrals. In view of (1.2), they are

$$\int_{\partial\Omega}(r_2 - r_1)P_0\overline{Q}\psi\,d\sigma + \int_{\partial\Omega} r_2\psi v_*^2\left(\frac{\alpha_{22}}{2}d_1 - \beta_{11}d_2 + \frac{\alpha_{22}}{2}(P_v - R_u)v_*\right)\,d\sigma$$

The last integrand is positive due to the first condition in b). Therefore the above sum is nonnegative if either $r_2 \geq r_1$ or $r_1 - r_2 > 0$ but sufficiently small. Therefore, under the stated assumptions in the lemma, λ is positive. ∎

Similarly, we have the following result.

Lemma 3.2 *Assume that either* $r_1 = r_2 \equiv 0$ *and* $a_2/a_1 > b_2/b_1$*, or*

$$\frac{a_2}{a_1} > \max\left\{\frac{b_2}{b_1}, \frac{2\alpha_{21}}{\alpha_{11}}\right\}, \tag{3.19}$$

and $(a_2d_1 - a_1d_2)_-$, $\sup_{x\in\partial\Omega}(r_2(x) - r_1(x))_+$ *are sufficiently small.*

Then λ *in (3.8) is positive.*

Proof. Following the previous proof, we test (3.8) with $\overline{P} = \int_0^{u_*} P(s,0)ds$, (3.9) with $Q_0\phi$ ($Q_0 = Q(u_*,0)$, $P_0 = P(u_*,0)$), to get

$$\lambda\int_\Omega \phi\overline{P}\,dx = \int_\Omega [Q_uP_0\phi|\nabla u_*|^2 + (g_v\overline{P} - fQ_0)\phi]\,dx + \int_{\partial\Omega}(r_1P_0u_* - r_2\overline{P})Q_0\phi\,d\sigma.$$

We note that $r_1P_0u_* - r_2\overline{P} = (r_1 - r_2)P_0u_* + \frac{P_u}{2}r_2u_*^2$, and

$$g_v\overline{P} - fQ_0 = (a_2 - b_2u_*)(d_1 + \frac{\alpha_{11}}{2}u_*)u_* - (a_1 - b_1u_*)(d_2 + \alpha_{21}u_*)u_*.$$

The proof is then similar to that of Lemma 3.1. We omit the details. ∎

We conclude this paper by giving the proof of Theorem 1.2.

Proof of Theorem 1.2. It is clear that the stated conditions (P.1) or (P.2) satisfy those of our propositions and lemmas of this section. The theorem then follows from Theorem 3.1. ∎

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