

Nonlinear instability of Vlasov-Maxwell systems in the classical and quasineutral limits

Daniel Han-Kwan* Toan T. Nguyen[†]

January 5, 2017

Abstract

We study the instability of solutions to the relativistic Vlasov-Maxwell systems in two limiting regimes: the classical limit when the speed of light tends to infinity and the quasineutral limit when the Debye length tends to zero. First, in the classical limit $\varepsilon \rightarrow 0$, with ε being the inverse of the speed of light, we construct a family of solutions that converge initially polynomially fast to an homogeneous solution μ of Vlasov-Poisson in arbitrarily high Sobolev norms, but become of order one away from μ in arbitrary negative Sobolev norms within time of order $|\log \varepsilon|$. Second, we deduce the invalidity of the quasineutral limit in L^2 in arbitrarily short time.

1 Introduction

We study the relativistic Vlasov-Maxwell system

$$\begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \frac{1}{c} \hat{v} \times B) \cdot \nabla_v f = 0, \\ \frac{1}{c} \partial_t B + \nabla_x \times E = 0, & \nabla_x \cdot E = \rho - 1, \\ -\frac{1}{c} \partial_t E + \nabla_x \times B = \frac{1}{c} j, & \nabla_x \cdot B = 0, \end{cases} \quad (1.1)$$

describing the evolution of an electron distribution function $f(t, x, v)$ at time $t \geq 0$, position $x \in \mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$, momentum $v \in \mathbb{R}^3$ and relativistic velocity

$$\hat{v} = \frac{v}{\sqrt{1 + \frac{|v|^2}{c^2}}}.$$

In these equations, the parameter c is the speed of light. Here, \mathbb{T}^3 is equipped with the normalized Lebesgue measure so that $\text{Leb}(\mathbb{T}^3) = 1$. The electric and magnetic fields $E(x, t), B(x, t)$ are three-dimensional vector fields, satisfying the classical Maxwell equations, with sources given by

$$\rho(t, x) = \int_{\mathbb{R}^3} f \, dv, \quad j(t, x) = \int_{\mathbb{R}^3} \hat{v} f \, dv,$$

which denote the usual charge density and current of electrons. The background ions are assumed to be homogeneous with a constant charge density equal to one.

*CNRS & École polytechnique, Centre de Mathématiques Laurent Schwartz UMR 7640, 91128 Palaiseau Cedex, France.
Email: daniel.han-kwan@polytechnique.edu

[†]Department of Mathematics, Pennsylvania State University, State College, PA 16802, USA.
Email: nguyen@math.psu.edu

We focus in the regime where $c \rightarrow +\infty$, that is known as the *classical (or non-relativistic) limit of the Vlasov-Maxwell system*. The formal limit is the classical Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ \nabla_x \times E = 0, \quad \nabla_x \cdot E = \rho - 1, \end{cases} \quad (1.2)$$

where $\rho(t, x) = \int_{\mathbb{R}^3} f dv$. This limit was justified on finite intervals of time in the independent and simultaneous works of Asano-Ukai [2], Degond [6], and Schaeffer [36]. We shall prove in this paper that *in the classical limit, the Vlasov-Maxwell system can develop instabilities in time $\log c$, due to instabilities of the limiting Vlasov-Poisson system*.

We are also interested in the following non-dimensional Vlasov-Maxwell system:

$$\begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f + (E + \alpha \hat{v} \times B) \cdot \nabla_v f = 0, \\ \alpha \partial_t B + \nabla_x \times E = 0, \quad \varepsilon^2 \nabla_x \cdot E = \rho - 1, \\ -\alpha \varepsilon^2 \partial_t E + \nabla_x \times B = \alpha j, \quad \nabla_x \cdot B = 0, \end{cases} \quad (1.3)$$

with

$$\hat{v} = \frac{v}{\sqrt{1 + \alpha \varepsilon^2 |v|^2}}, \quad \rho(t, x) = \int_{\mathbb{R}^3} f dv, \quad j(t, x) = \int_{\mathbb{R}^3} \frac{v}{\sqrt{1 + \alpha \varepsilon^2 |v|^2}} f dv.$$

In physical units, the parameters α, ε are given by

$$\alpha = \sqrt{\frac{r_0}{\varepsilon_0}}, \quad \varepsilon = \sqrt{\frac{\varepsilon_0}{r_0 c^2}},$$

where r_0 denotes the classical electron radius, ε_0 is the vacuum dielectric constant, and c is still the speed of light. The parameter ε corresponds to the classical *Debye length* of the electrons; see [35] for further discussions. In this work, we are interested in the regime where $\alpha \sim 1$ and $\varepsilon \rightarrow 0$, a limit in which the charge density of ions and electrons are formally equal. We shall thus refer to this problem as the *quasineutral limit of the Vlasov-Maxwell system*. Note that α is equal to the ratio between $\frac{1}{\varepsilon}$ and c , so that this means that we consider that the inverse of the Debye length is of the same order as the speed of light. For simplicity, throughout the paper, we set $\alpha = 1$.

The quasineutral limit of the Vlasov-Maxwell system has been studied previously by Brenier, Mauser and Puel [5] and Puel and Saint-Raymond [35] in the case where the initial density distribution converges in some weak sense to a monokinetic distribution (that is, a Dirac delta function in velocity). The convergence to monokinetic distributions can be interpreted from the physical point of view as *vanishing temperature*: therefore, this is sometimes referred to as the *cold electrons limit*. The work [35] furthermore describes the propagation of time oscillating waves, which turn out to be absent in the well-prepared framework considered in [5].

For each fixed $\varepsilon > 0$, the global-in-time Cauchy theory for smooth solutions of (1.3) remains an outstanding open problem. However, there are local strong solutions and continuation conditions ([37, 38, 1, 11, 9, 10, 3, 26, 30], among others) or global weak solutions ([8]). In this paper, we shall construct particular solutions of (1.3) that are sufficiently smooth. This also pertains to system (1.1), for each fixed $c > 0$.

Formally, in the limit $\varepsilon \rightarrow 0$, it is straightforward to obtain the expected formal limit of (1.3), a system we shall call the *kinetic eMHD* system:

$$\begin{cases} \partial_t f^0 + v \cdot \nabla_x f^0 + (E^0 + v \times B^0) \cdot \nabla_v f^0 = 0, \\ \partial_t B^0 + \nabla_x \times E^0 = 0, \quad \rho^0 = 1, \\ \nabla_x \times B^0 = j^0, \quad \nabla_x \cdot B^0 = 0, \end{cases} \quad (1.4)$$

in which

$$\rho^0(t, x) = \int_{\mathbb{R}^3} f^0 dv, \quad j^0(t, x) = \int_{\mathbb{R}^3} v f^0 dv.$$

By imposing that the distribution function is monokinetic, that is considering the ansatz

$$f^0(t, x, v) = \rho^0(t, x) \delta_{v=u^0(t, x)},$$

where δ stands for the Dirac measure, it follows that f^0 is a solution in the sense of distribution to (1.4) if and only if (ρ^0, u^0) satisfies the following hydrodynamic equations:

$$\begin{cases} \partial_t u^0 + \nabla \cdot (u^0 \otimes u^0) = E^0 + u^0 \times B^0, \\ \partial_t B^0 + \nabla \times E^0 = 0, \quad \rho^0 = 1, \\ \nabla \times B^0 = u^0, \quad \nabla \cdot B^0 = 0. \end{cases} \quad (1.5)$$

This system is known in the literature as the *electron Magneto-Hydro-Dynamics* equations (eMHD); see, for instance, [25, 5]. This motivates the choice of the name *kinetic eMHD* for (1.4). The work [5], and then [35], justify the eMHD system from (1.3) in the above monokinetic situation via the so-called modulated energy (or relative entropy) method devised by Brenier in [4]. In this paper, we rather focus on the question of *validity of (1.4) in the quasineutral limit*.

The kinetic eMHD system (1.4) can be studied as follows. It is common to express E^0 and B^0 in terms of electromagnetic potentials (ϕ, A) . Precisely, write

$$E^0 = -\nabla \phi^0 - \partial_t A^0, \quad B^0 = \nabla \times A^0, \quad (1.6)$$

in which ϕ^0 is a scalar function and A^0 is a divergence-free vector potential. The equations satisfied by the electromagnetic field are then simply reduced to the elliptic equation

$$-\Delta A^0 = j^0, \quad (1.7)$$

together with the constraint $\rho^0 = 1$. On the other hand, the potential ϕ^0 is determined from the Vlasov equation. Indeed, there holds the conservation of charge density and current

$$\begin{aligned} \partial_t \rho^0 + \nabla \cdot j^0 &= 0, \\ \partial_t j^0 + \nabla \cdot \int f^0 v \otimes v dv &= E^0 + j^0 \times B^0. \end{aligned}$$

From the neutrality condition $\rho^0 = 1$, we conclude that the first moment j^0 is divergence-free. Hence, taking the divergence of the conservation of current, we end up with the following elliptic problem for ϕ^0 :

$$-\Delta \phi^0 = \nabla \cdot \left(\nabla \cdot \int f^0 v \otimes v dv \right) - \nabla \cdot (j^0 \times B^0). \quad (1.8)$$

The equations (1.6)-(1.8), together with the Vlasov equation in (1.4), form a complete set of equations for solutions of the kinetic eMHD system. However, the elliptic problem (1.8) reveals a loss of one x -derivative in E^0 , or precisely in the curl part of E^0 , as compared to f^0 . One can expect this loss to be the source of some *ill-posedness* for (1.4). This feature is a first strong indication of the singularity of the quasineutral limit. Note finally that this loss can only occur for (1.4), while the eMHD equations are well-posed in the classical sense as shown in [5].

Lately, there have been many works [12, 13, 14, 4, 31, 24, 21, 22, 23] that were devoted to the study of the quasineutral limit for Vlasov-Poisson systems, that is, the framework where there is no magnetic field, $B \equiv 0$. In that case, the formal limit is also straightforward to identify and corresponds to a kinetic version of the incompressible Euler equations. The limiting kinetic system also displays a loss of derivatives, through

the electric field $E^0 = -\nabla\phi^0$, exactly as in (1.8). In the paper [24], using ideas originating from [14], it is shown in particular that instabilities such as *two-stream instabilities* (see [34, 18]) for the Vlasov-Poisson system give rise to instabilities in the quasineutral limit. These instabilities have a destabilizing effect and the formal limit is not true in general, even on very short intervals of time $[0, T_\varepsilon]$, with $T_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

One purpose of this work is to extend these ideas to the Vlasov-Maxwell system (1.3). As in the Vlasov-Poisson case [24], the effect of instabilities of the Vlasov-Maxwell system in the quasineutral limit can be observed in the high spatial frequency regime. More precisely, we introduce the hyperbolic change of variables:

$$(t, x, v) \mapsto (s, y, v) := \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v \right) \quad (1.9)$$

and set $E^\varepsilon = \varepsilon E$, $B^\varepsilon = B$, and $f^\varepsilon = f$. The Vlasov-Maxwell system (1.3) then becomes

$$\begin{cases} \partial_s f^\varepsilon + \hat{v} \cdot \nabla_y f^\varepsilon + (E^\varepsilon + \varepsilon \hat{v} \times B^\varepsilon) \cdot \nabla_v f^\varepsilon = 0, \\ \varepsilon \partial_s B^\varepsilon + \nabla_y \times E^\varepsilon = 0, & \nabla_y \cdot E^\varepsilon = \rho^\varepsilon - 1, \\ -\varepsilon \partial_s E^\varepsilon + \nabla_y \times B^\varepsilon = \varepsilon j^\varepsilon, & \nabla_y \cdot B^\varepsilon = 0, \end{cases} \quad (1.10)$$

where

$$\begin{aligned} \hat{v} &= \frac{v}{\sqrt{1 + \varepsilon^2 |v|^2}}, & \rho^\varepsilon(t, x) &= \int_{\mathbb{R}^3} f^\varepsilon dv, \\ j^\varepsilon(t, x) &= \int_{\mathbb{R}^3} \frac{v}{\sqrt{1 + \varepsilon^2 |v|^2}} f^\varepsilon dv. \end{aligned} \quad (1.11)$$

We therefore observe that in this *long time* and *spatially high frequency* regime, the quasineutral limit comes down to the study of the classical limit of Vlasov-Maxwell to Vlasov-Poisson system, that is (1.1) with the speed of light $c = \frac{1}{\varepsilon}$. We shall thus mainly focus on the study of the classical limit.

Consequently, it appears natural to study the effect of instability of equilibria of the Vlasov-Maxwell system (1.10) to deduce instability for (1.3). There have many works devoted to the stability problem over the past few years; see, for instance, [16, 17, 19, 20, 28, 27, 29] and the references therein. It turns out that in the limit $\varepsilon \rightarrow 0$, instability for the Vlasov-Poisson system (1.2) is sufficient to get an instability in the classical limit, thanks to a careful use of the iterative scheme developed by Grenier in his study of instability of boundary layers [15]. In other words, we show that having an *approximate growing mode* of the linearized Vlasov-Maxwell equations is enough. Loosely speaking, we shall prove that if we deal with sequences of initial data converging polynomially (in ε) fast to an unstable equilibrium, then an instability is developed for the full Vlasov-Maxwell dynamics, in times of order $|\log \varepsilon|$. We shall finally deduce the invalidity of the derivation of (1.4) in the quasineutral limit of (1.3).

2 Main results

Our results rely on the existence of a growing mode for the linearized Vlasov-Maxwell system (1.3), around unstable *homogeneous* equilibria $(f, E, B) \equiv (\mu(v), 0, 0)$. To ensure the non-negativity of the distribution functions we shall construct, we introduce the following condition:

Definition 2.1 (δ -condition). *We say that a profile $\mu(v)$ satisfies the δ -condition if μ is positive and satisfies*

$$\sup_{v \in \mathbb{R}^3} \frac{|\nabla \mu(v)|}{(1 + |v|)\mu(v)} < +\infty. \quad (2.1)$$

The terminology is borrowed from [24]; in that reference, a further relaxed δ' -condition is also introduced, allowing non-negative equilibria. Our results here should apply to this δ' -condition as well. For the sake of simplicity, we restrict ourselves to the above δ -condition on μ throughout the paper.

In what follows, the statement that $\mu(v)$ is spectrally unstable means that there is a growing mode of the form $e^{\lambda t}g(y, v)$, $g \in L^2$, with $\Re\lambda > 0$, of the linearized Vlasov-Poisson problem around the homogeneous equilibrium $\mu(v)$, namely: (λ, g) is a solution of

$$\lambda g + v \cdot \nabla_y g + \nabla_v \mu(v) \cdot \nabla_y \Delta_y^{-1} \left(\int_{\mathbb{R}^3} g \, dv \right) = 0, \quad \iint g(y, v) \, dy dv = 0. \quad (2.2)$$

Looking for a solution of the form $g = e^{ik \cdot y} \hat{g}(v)$ and writing $\lambda = -ik \cdot \omega$, for some complex vector ω , with $\Im\omega \cdot k > 0$, we are led to study

$$ik \cdot (v - \omega) \hat{g}(v) - \frac{1}{|k|^2} ik \cdot \nabla_v \mu \hat{\rho} = 0,$$

in which $\hat{\rho} = \int \hat{g}(v) \, dv$. The above identity has a nonzero solution if and only if (k, ω) is such that the so-called *Penrose instability criterion*

$$\frac{1}{|k|^2} \int \frac{k \cdot \nabla_v \mu}{k \cdot (v - \omega)} \, dv = 1, \quad \Im\omega \cdot k > 0, \quad (2.3)$$

is satisfied. We have the following classical lemma (see [34, 7]).

Lemma 2.2. *Let μ satisfy the Penrose instability criterion (2.3) at some point (k_0, ω_0) . Then, μ is spectrally unstable and there exists a growing mode of the form*

$$f(s, y, v) = e^{\lambda_0 s} e^{ik_0 \cdot y} \hat{f}(v), \quad (2.4)$$

with $\lambda_0 = -ik_0 \cdot \omega_0$ and

$$\hat{f}(v) = \frac{1}{|k_0|^2} \frac{\nabla_v \mu \cdot k_0}{k_0 \cdot (v - \omega_0)}. \quad (2.5)$$

Reciprocally, if λ is an eigenvalue for the linearized operator around μ then all eigenfunctions are linear combinations of functions of the form (2.4) - (2.5).

In particular, it is known, see for instance [18], that if there is a vector $e \in \mathbb{S}^2$ such that the function

$$\mu_e(r) = \int_{r e + e^\perp} \mu(w) \, dw$$

admits a local strict minimum at a point $\bar{r} \in \mathbb{R}$ and is symmetric around \bar{r} , and that the following inequality holds:

$$\int_{\mathbb{R}} \frac{\mu_e(r) - \mu_e(\bar{r})}{|r - \bar{r}|^2} \, dr > 4\pi^2, \quad (2.6)$$

then the Penrose instability criterion is satisfied.

In this paper, we shall only consider equilibria that are

- *radial*, that is to say $\mu \equiv \mu(|v|^2)$;
- *smooth* (i.e. C^k , with $k \gg 1$) and *decaying sufficiently fast at infinity*;
- *normalized* in the sense that $\int_{\mathbb{R}^3} \mu(v) \, dv = 1$. Note also, since μ being radial, we have $\int_{\mathbb{R}^3} \mu(v) \hat{v} \, dv = 0$, for all $c > 0$.

As a result, any such equilibrium gives a stationary solution for all the systems we study in this paper, namely the relativistic Vlasov-Maxwell system (1.3), the Vlasov-Poisson system (1.2) and the kinetic eMHD system (1.4).

We are now in position to state our first result.

Theorem 2.3 (Instability in the classical limit). *Let $\mu(v)$ be a radial, smooth, normalized equilibrium that satisfies the δ -condition and the Penrose instability criterion. Then, for any $m, S, S', p > 0$, there exist a family of smooth solutions $(f^\varepsilon, E^\varepsilon, B^\varepsilon)_{\varepsilon>0}$ of (1.10), with $f^\varepsilon \geq 0$, and a sequence of times $s_\varepsilon = \mathcal{O}(|\log \varepsilon|)$ such that*

$$\|(1 + |v|^2)^{\frac{m}{2}} (f^\varepsilon|_{s=0} - \mu)\|_{H^S(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \varepsilon^p, \quad (2.7)$$

but

$$\liminf_{\varepsilon \rightarrow 0} \|f^\varepsilon(s_\varepsilon) - \mu\|_{H^{-S'}(\mathbb{T}^3 \times \mathbb{R}^3)} > 0, \quad (2.8)$$

$$\liminf_{\varepsilon \rightarrow 0} \|\rho^\varepsilon(s_\varepsilon) - 1\|_{H^{-S'}(\mathbb{T}^3)} > 0, \quad \liminf_{\varepsilon \rightarrow 0} \|j^\varepsilon(s_\varepsilon)\|_{H^{-S'}(\mathbb{T}^3)} > 0, \quad (2.9)$$

$$\liminf_{\varepsilon \rightarrow 0} \|E^\varepsilon(s_\varepsilon)\|_{L^2(\mathbb{T}^3)} > 0 \quad (2.10)$$

in which $\rho^\varepsilon, j^\varepsilon$ are defined as in (1.11).

Remark 2.4. *Note that a smooth double-bump equilibrium (see [18]) satisfies all the required assumptions of the theorem.*

Remark 2.5. *This nonlinear instability result for the full 3D Vlasov-Maxwell dynamics is concerned with the classical limit $c = 1/\varepsilon \rightarrow +\infty$; it is stronger in terms of admissible norms than those previously known for fixed values of the speed of light c , see e.g. [20].*

Remark 2.6. *From a view of our analysis, one can also extract a nonlinear instability result for the limiting Vlasov-Poisson equation. This result yields a stronger instability in much weaker Sobolev norms than the one obtained by Guo and Strauss in [18], however with the additional assumption that the equilibrium μ is sufficiently smooth. It is the exact analogue in higher dimension of the 1D result proved by the first author and Hauray [24, Theorem 3.1].*

Recalling the hyperbolic change of variables (1.9), Theorem 2.3 will allow us to prove short time instability of the Vlasov-Maxwell system (1.3). In this context, we can first introduce the following *sharp* Penrose instability condition, ensuring instability of equilibria in the quasineutral limit.

Definition 2.7 (Sharp Penrose instability condition). *We say that a profile $\mu(v)$ satisfies the sharp Penrose instability condition if there is a vector $e \in \mathbb{S}^2$ such that the function $\mu_e(r) = \int_{re+e^\perp} \mu(w) dw$ admits a local minimum at the point \bar{r} and the following inequality holds*

$$\int_{\mathbb{R}} \frac{\mu_e(r) - \mu_e(\bar{r})}{|r - \bar{r}|^2} dr > 0. \quad (2.11)$$

If the local minimum is flat, i.e. is reached on an interval $[\bar{r}_1, \bar{r}_2]$, then (2.11) has to be satisfied for all $\bar{r} \in [\bar{r}_1, \bar{r}_2]$.

The sharp Penrose instability condition does not directly yield a growing mode for the Vlasov-Poisson equations set on $\mathbb{T}^3 \times \mathbb{R}^3$; however the instability appears in the regime of small ε .

Our second main result reads as follows.

Theorem 2.8 (Invalidity of the quasineutral limit). *Let $\mu(v)$ be a radial, smooth, normalized equilibrium satisfying the δ -condition, decaying sufficiently fast at infinity, and satisfying the sharp Penrose instability condition. Then, for any $m, S, p > 0$, there exist a family of smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ of (1.3), with $f_\varepsilon \geq 0$ and a sequence of times $t_\varepsilon = \mathcal{O}(\varepsilon |\log \varepsilon|) \rightarrow 0$, such that*

$$\|(1 + |v|^2)^{\frac{m}{2}} (f_\varepsilon|_{t=0} - \mu)\|_{H^S(\mathbb{T}^3 \times \mathbb{R}^3)} \leq \varepsilon^p, \quad (2.12)$$

but

$$\liminf_{\varepsilon \rightarrow 0} \|f_\varepsilon(t_\varepsilon) - \mu\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} > 0, \quad (2.13)$$

$$\liminf_{\varepsilon \rightarrow 0} \|\rho_\varepsilon(t_\varepsilon) - 1\|_{L^2(\mathbb{T}^3)} > 0, \quad \liminf_{\varepsilon \rightarrow 0} \|j_\varepsilon(t_\varepsilon)\|_{L^2(\mathbb{T}^3)} > 0, \quad (2.14)$$

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \|E_\varepsilon(t_\varepsilon)\|_{L^2(\mathbb{T}^3)} > 0, \quad (2.15)$$

in which $\rho_\varepsilon, j_\varepsilon$ are defined as in (1.11).

The following of this paper is dedicated to the proofs of Theorem 2.3 and 2.8.

3 Instability in the classical limit

We study the instability of Vlasov-Maxwell systems in the classical limit $\varepsilon \rightarrow 0$:

$$\begin{cases} \partial_s f^\varepsilon + \hat{v} \cdot \nabla_y f^\varepsilon + (E^\varepsilon + \varepsilon \hat{v} \times B^\varepsilon) \cdot \nabla_v f^\varepsilon = 0, \\ \varepsilon \partial_s B^\varepsilon + \nabla_y \times E^\varepsilon = 0, & \nabla_y \cdot E^\varepsilon = \rho^\varepsilon - 1, \\ -\varepsilon \partial_s E^\varepsilon + \nabla_y \times B^\varepsilon = \varepsilon j, & \nabla_y \cdot B^\varepsilon = 0, \end{cases} \quad (3.1)$$

where

$$\hat{v} = \frac{v}{\sqrt{1 + \varepsilon^2 |v|^2}}, \quad \rho^\varepsilon(t, x) = \int_{\mathbb{R}^3} f^\varepsilon dv, \\ j^\varepsilon(t, x) = \int_{\mathbb{R}^3} \frac{v}{\sqrt{1 + \varepsilon^2 |v|^2}} f^\varepsilon dv.$$

Let $\mu(v)$ be a radial, smooth, normalized equilibrium of (3.1). We set $f^\varepsilon = \mu + f$, $E^\varepsilon = E$ and $B^\varepsilon = B$. The perturbation (f, E, B) solves

$$\partial_s f + \hat{v} \cdot \nabla_y f + (E + \varepsilon \hat{v} \times B) \cdot \nabla_v (\mu + f) = 0. \quad (3.2)$$

The fields E and B are constructed through the electromagnetic potentials:

$$E = -\nabla \phi - \varepsilon \partial_s A, \quad B = \nabla \times A, \quad (3.3)$$

with A satisfying the *Coulomb gauge*

$$\nabla \cdot A = 0.$$

The scalar and vector potentials ϕ, A solve

$$-\Delta \phi = \rho(f), \quad \varepsilon^2 \partial_s^2 A - \Delta A = \varepsilon j(f) - \varepsilon \partial_s \nabla \phi, \quad (3.4)$$

where we set

$$\rho(f) := \int f(s, y, v) dv, \quad j(f) := \int \hat{v} f(s, y, v) dv.$$

As discussed in the introduction, the instability comes from that of the underlying Vlasov-Poisson system. In other words, we rely on a growing mode of the linearization of Vlasov-Poisson around μ in order to build an approximate growing solution to the nonlinear perturbation systems (3.2)-(3.4).

Let us start by introducing the linearized Vlasov-Poisson operator, acting on functions f with zero mean:

$$Pf := \partial_s f - L_0 f, \quad L_0 f := -v \cdot \nabla_y f - \nabla_v \mu \cdot \nabla_y \Delta_y^{-1} \rho(f). \quad (3.5)$$

Note that $\rho(f) = \int f(s, y, v) dv$ is a function of (s, y) with zero mean in y .

We denote by H^k the usual Sobolev space of functions in y over \mathbb{T}^3 (or in v over \mathbb{R}^3) with all partial derivatives up to order k having finite L^2 norms, and denote by H_m^n the function space consisting of functions in y and v so that the norm

$$\|f\|_{H_m^n} := \sum_{|\alpha|+|\beta| \leq n} \|\langle v \rangle^m \partial_y^\alpha \partial_v^\beta f\|_{L^2}$$

is finite, with $\langle v \rangle := \sqrt{1 + |v|^2}$ and $m, n \geq 0$.

Defining the domain of L_0 as

$$D(L_0) = \left\{ f \in H_m^n, L_0 f \in H_m^n, \iint f dv dy = 0 \right\},$$

where n, m are large enough, we recall (see for instance [7]) that the unstable spectrum is made only of point spectrum. Moreover, there is at least one unstable eigenvalue associated to the largest positive real part $\Re \lambda$ among all elements of the spectrum; we pick one, denote it by λ_0 , and refer to it as the maximal unstable eigenvalue.

3.1 Grenier's iterative scheme

We shall construct an approximate solution f_{app} to the nonlinear problem (3.2), following the methodology introduced by Grenier [15] for the study of instability of boundary layers in the inviscid limit of the Navier-Stokes equations. In view of the nonlinear equation (3.2) and of the linearized Vlasov-Poisson operator (3.5), we introduce

$$\begin{aligned} S(f) &:= -[\partial_s A(f) - \hat{v} \times (\nabla \times A(f))] \cdot \nabla_v \mu, \\ T(f) &:= -\frac{|v|^2}{\sqrt{1 + \varepsilon^2 |v|^2} (1 + \sqrt{1 + \varepsilon^2 |v|^2})} v \cdot \nabla_y f, \\ Q(f, g) &:= -\nabla \phi(f) \cdot \nabla_v g - \varepsilon[\partial_s A(f) - \hat{v} \times (\nabla \times A(f))] \cdot \nabla_v g, \end{aligned}$$

in which the potentials $(\phi(f), A(f))$ solve the elliptic and wave problem (3.4) with the source associated to f . In what follows, the wave equation is solved with zero initial data:

$$A|_{s=0} = \partial_s A|_{s=0} = 0.$$

Finding a solution to the nonlinear problem (3.2) is equivalent to solving the following symbolic equation:

$$R(f) := Pf + \varepsilon S(f) + \varepsilon^2 T(f) + Q(f, f) = 0. \quad (3.6)$$

We first point out that, as it will become clear from estimate (3.25), the term $\varepsilon \partial_s A$ appearing in $\varepsilon S(f)$ is not small as compared to f , and will not be treated as a perturbation. It turns out that we can extract the leading part of $\varepsilon \partial_s A$ into the Vlasov-Poisson operator (see [32, Section 2], for a similar use of this idea). Indeed, using the definition of $P(f)$, we write

$$\begin{aligned} S(f) &= -[\partial_s A - \hat{v} \times (\nabla \times A)] \cdot \nabla_v \mu \\ &= -P(A \cdot \nabla_v \mu) + \hat{v} \cdot \nabla(A \cdot \nabla_v \mu) + \nabla_v \mu \cdot \nabla \Delta^{-1}(\rho(A \cdot \nabla_v \mu)) + [\hat{v} \times (\nabla \times A)] \cdot \nabla_v \mu \\ &= -P(A \cdot \nabla_v \mu) + \tilde{S}(f), \end{aligned}$$

in which we have set

$$\tilde{S}(f) := (\hat{v} \cdot \nabla) A \cdot \nabla_v \mu + [\hat{v} \times (\nabla \times A)] \cdot \nabla_v \mu. \quad (3.7)$$

Thus, the problem (3.6) is equivalent to

$$R(f) = P[f - \varepsilon A \cdot \nabla_v \mu] + \varepsilon \tilde{S}(f) + \varepsilon^2 T(f) + Q(f, f) = 0. \quad (3.8)$$

It is now straightforward to (formally) construct an approximate solution f_{app} so that the error $R(f_{\text{app}})$ is arbitrarily small. We start the construction with

$$(g_1, \phi_1) = e^{\lambda_0 s} (\hat{g}_1, \hat{\phi}_1)(y, v) \quad (3.9)$$

to be a growing mode associated to the maximal unstable eigenvalue λ_0 , constructed as in Lemma 2.2; we choose $(\hat{g}_1, \hat{\phi}_1)$ so that

$$\hat{g}_1(y, v) = r_1 e^{ik_0 \cdot y} \frac{1}{|k_0|^2} \frac{\nabla_v \mu \cdot k_0}{k_0 \cdot (v - \omega_0)} \quad (3.10)$$

with $r_1 > 0$ large enough in order to ensure

$$\|\hat{g}_1\|_{H^{-s'}} \geq 2\theta_0, \quad (3.11)$$

for $S' > 0$ as in Theorem 2.3, and some $\theta_0 > 2$. By construction, (g_1, ϕ_1) solves the linearized Vlasov-Poisson system:

$$P(g_1) = 0, \quad \phi_1 := -\Delta^{-1} \rho(g_1).$$

We assume for the moment that (g_1, ϕ_1) is real and shall explain later how to deal with the general case.

We then solve for (f_1, A_1) satisfying

$$f_1 - \varepsilon A_1 \cdot \nabla_v \mu = g_1, \quad \varepsilon^2 \partial_s^2 A_1 - \Delta A_1 = \varepsilon(j(f_1) - \partial_s \nabla \phi_1), \quad A_1|_{s=0} = \partial_s A_1|_{s=0} = 0. \quad (3.12)$$

We shall justify later (see Lemma 3.2) why this system indeed has a solution.

Let $p \in \mathbb{N} \setminus \{0, 1\}$. Observe that $\varepsilon^p f_1$ approximately solves the nonlinear equation (3.8), leaving an error

$$R(\varepsilon^p f_1) := \varepsilon^{p+1} \tilde{S}(f_1) + \varepsilon^{2p} Q(f_1, f_1) + \varepsilon^{p+2} T(f_1),$$

which formally is of order $\mathcal{O}(\varepsilon^{p+1})$. To obtain an error with higher order, we introduce (f_2, ϕ_2, A_2) so that (g_2, ϕ_2) , with $g_2 = f_2 - \varepsilon A_2 \cdot \nabla_v \mu$, solve

$$\begin{aligned} P(g_2) &= -\tilde{S}(f_1), & \phi_2 &= -\Delta^{-1} \rho(g_2), & g_2|_{s=0} &= 0, \\ \varepsilon^2 \partial_s^2 A_2 - \Delta A_2 &= \varepsilon(j(f_2) - \partial_s \nabla \phi_2), & A_2|_{s=0} &= \partial_s A_2|_{s=0} = 0. \end{aligned} \quad (3.13)$$

It follows directly that $\varepsilon^p f_1 + \varepsilon^{p+1} f_2$ approximately solves the nonlinear equation (3.8), with a better error:

$$\begin{aligned} R(\varepsilon^p f_1 + \varepsilon^{p+1} f_2) &:= \varepsilon^{p+2} \left[S(\tilde{f}_2) + T(f_1) \right] + \varepsilon^{p+3} T(f_2) \\ &\quad + \varepsilon^{2p} Q(f_1, f_1) + \varepsilon^{2p+1} \left[Q(f_1, f_2) + Q(f_2, f_1) \right] + \varepsilon^{2p+2} Q(f_2, f_2). \end{aligned}$$

Inductively, we construct

$$f_{\text{app}} = \sum_{k=1}^N \varepsilon^{p+k-1} f_k, \quad (3.14)$$

in which (f_k, ϕ_k, A_k) , $k \geq 3$, are defined as the unique solution to the linear problems:

$$\begin{cases} P(g_k) = -\tilde{S}(f_{k-1}) - T(f_{k-2}) - \sum_{\ell=1}^{k-1} Q(f_\ell, f_{k+1-p-\ell}), & \phi_k = -\Delta^{-1} \rho(g_k), & g_k|_{s=0} = 0 \\ f_k - \varepsilon A_k \cdot \nabla_v \mu = g_k, & \varepsilon^2 \partial_s^2 A_k - \Delta A_k = \varepsilon(j(f_k) - \partial_s \nabla \phi_k), & A_k|_{s=0} = \partial_s A_k|_{s=0} = 0, \end{cases} \quad (3.15)$$

with the convention that $f_{-j} = 0$ for $j \in \mathbb{N}$. The error of this approximation can be computed as

$$R(f_{\text{app}}) = -\varepsilon^{N+p} (\tilde{S}(f_N) + T(f_{N-1})) - \varepsilon^{N+p+1} T(f_N) - \sum_{k+\ell > N+1-p; 1 \leq k, \ell \leq N-1} \varepsilon^{2p+k+\ell-2} Q(f_k, f_\ell),$$

which is of order $\mathcal{O}(\varepsilon^{N+p})$ or higher.

In the following subsections, we shall derive relevant estimates on each f_k in the approximate solution f_{app} and deduce appropriate bounds on the approximation. In the proof, C (with various subscripts) shall always refer to a positive constant which can change from line to line but does not depend on ε .

3.2 Linear estimates

In this section, we obtain bounds on the profile solutions f_k , solving (3.12) and (3.15). We start by studying the linear semigroup $e^{L_0 s}$. We have the following sharp semigroup bounds with losses of derivatives and integrability in v . Here, sharp refers to the growth in time, in the sense that the optimal growth one could hope for would be in $e^{\Re \lambda_0 t}$ and we reach $e^{(\Re \lambda_0 + \beta)t}$ for all $\beta > 0$.

Proposition 3.1 (Sharp bounds on the solution operator). *Let $\mu(v)$ be a smooth unstable equilibrium of Vlasov-Poisson system which decays sufficiently fast as $v \rightarrow \infty$, and let λ_0 be the maximal unstable eigenvalue. Let $n, r \geq 0, m \geq 2$, and h in H_{m+2}^{n+2} . Then, $f = e^{L_0 s} h$ is well-defined as the solution of the linearized Vlasov-Poisson problem $P(f) = (\partial_s - L_0) f = 0$ with the initial data h . Furthermore, there holds*

$$\|e^{L_0 s} h\|_{H_m^n} \leq C_\beta e^{(\Re \lambda_0 + \beta)s} \|h\|_{H_{m+2}^{n+2}}, \quad \forall s \geq 0, \quad \forall \beta > 0, \quad (3.16)$$

for some constant C_β depending on μ and β .

Proof. Let $n \geq 0, m \geq 2$. For each f in H_2^n with $\iint f \, dy \, dv = 0$, we denote $\phi = -\Delta^{-1}\rho(f)$. The standard elliptic theory yields

$$\|\nabla\phi\|_{H^{n+1}} \leq C_0\|\rho(f)\|_{H^n} \leq C_0\|\langle v \rangle^2 f\|_{H_y^n L_v^2}, \quad (3.17)$$

in which C_0 is some universal constant. We consider the resolvent equation:

$$(\lambda - L_0)f = h, \quad \Re\lambda \geq 0.$$

Standard L^2 energy estimates yield at once

$$\Re\lambda\|\langle v \rangle^m f\|_{L^2} \leq \|\langle v \rangle^m \nabla_v \mu\|_{L^2(\mathbb{R}^3)}\|\nabla\phi\|_{L^2} + \|\langle v \rangle^m h\|_{L^2}.$$

Therefore we deduce the following weighted L^2 resolvent bound, for $\Re\lambda > \gamma_{0,m} := C_0\|\langle v \rangle^m \nabla_v \mu\|_{L^2(\mathbb{R}^3)}$,

$$\|\langle v \rangle^m (\lambda - L_0)^{-1} h\|_{L^2} \leq \frac{1}{\Re\lambda - \gamma_{0,m}} \|\langle v \rangle^m h\|_{L^2}$$

Similarly, higher derivatives estimates are obtained in the similar fashion, since we observe that we have for $\alpha, \beta \in \mathbb{N}^3$,

$$\lambda \partial_v^\beta \partial_y^\alpha f + v \cdot \nabla_y \partial_v^\beta \partial_y^\alpha f - \nabla_v \partial_v^\beta \mu \cdot \nabla \partial_y^\alpha \phi(f) + [\partial_v^\beta, v \cdot \nabla_y] \partial_y^\alpha f = \partial_v^\beta \partial_y^\alpha h$$

in which $[\partial_v^\beta, v \cdot \nabla_y] = \partial_v^\beta (v \cdot \nabla_y) - v \cdot \nabla_y \partial_v^\beta$. This identity, together with the elliptic estimate (3.17), first yields

$$\begin{aligned} \Re\lambda\|\langle v \rangle^m \partial_y^\alpha f\|_{L^2} &\leq \|\langle v \rangle^m \nabla_v \mu \cdot \nabla \partial_y^\alpha \phi(f)\|_{L^2} + \|\langle v \rangle^m \partial_y^\alpha h\|_{L^2} \\ &\leq C_0\|\langle v \rangle^m \nabla_v \mu\|_{L^2(\mathbb{R}^3)}\|\langle v \rangle^2 f\|_{H_y^{|\alpha|-1} L_v^2} + \|\langle v \rangle^m \partial_y^\alpha h\|_{L^2}. \end{aligned}$$

By a straightforward induction, we find

$$\Re\lambda\|\langle v \rangle^m f\|_{H_y^{|\alpha|} L_v^2} \leq C'_0\|\langle v \rangle^2 f\|_{L^2} + C'_0\|\langle v \rangle^m h\|_{H_y^{|\alpha|} L_v^2}. \quad (3.18)$$

Similarly, we have

$$\begin{aligned} \Re\lambda\|\langle v \rangle^m \partial_v^\beta \partial_y^\alpha f\|_{L^2} &\leq \|\langle v \rangle^m \nabla_v \partial_v^\beta \mu \cdot \nabla \partial_y^\alpha \phi(f)\|_{L^2} + \|\langle v \rangle^m [\partial_v^\beta, v \cdot \nabla_y] \partial_y^\alpha f\|_{L^2} + \|\langle v \rangle^m \partial_v^\beta \partial_y^\alpha h\|_{L^2} \\ &\leq C_0\|\langle v \rangle^m \nabla_v \partial_v^\beta \mu\|_{L^2(\mathbb{R}^3)}\|\langle v \rangle^2 f\|_{H_y^{|\alpha|-1} L_v^2} \\ &\quad + C_0\|\langle v \rangle^m f\|_{H_y^{|\alpha|+1} H_v^{|\beta|-1}} + \|\langle v \rangle^m \partial_v^\beta \partial_y^\alpha h\|_{L^2}. \end{aligned}$$

Again, by induction, this proves

$$\Re\lambda\|\langle v \rangle^m \partial_v^\beta \partial_y^\alpha f\|_{L^2} \leq C'_0\|\langle v \rangle^2 f\|_{H_y^{|\alpha|+|\beta|-1} L_v^2} + C'_0\|\langle v \rangle^m f\|_{H_y^{|\alpha|+|\beta|} L_v^2} + C'_0\|h\|_{H_m^n},$$

which together with the above bound (3.18) on $\partial_y^{|\alpha|+|\beta|} f$ gives, considering all multi-indices α, β such that $|\alpha| + |\beta| \leq n$, that there exists $\gamma_{n,m}, C_{n,m} > 0$ such that

$$\Re\lambda\|f\|_{H_m^n} \leq \gamma_{n,m}\|\langle v \rangle^2 f\|_{L^2} + C_{n,m}\|h\|_{H_m^n}, \quad (3.19)$$

for $m \geq 2$. In particular, this proves that

$$\|(\lambda - L_0)^{-1} h\|_{H_m^n} \leq \frac{C_{n,m}}{\Re\lambda - \gamma_{n,m}} \|h\|_{H_m^n},$$

for some positive constant $C_{\gamma_{n,m}}$, and for all $\lambda \in \mathbb{C}$ so that $\Re\lambda > \gamma_{n,m}$. The classical Hille-Yosida theorem then asserts that L_0 generates a continuous semigroup $e^{L_0 s}$ on the Banach space L_m^2 (and hence, on H_m^n); see, for instance, [33] or [39, Appendix A]. In addition, there holds the following representation for the semigroup:

$$e^{L_0 s} h = \text{P.V.} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda s} (\lambda - L_0)^{-1} h \, d\lambda \quad (3.20)$$

for any $\gamma > \gamma_{n,m}$, where P.V. denotes the Cauchy principal value.

Next, by assumption, λ_0 is an unstable eigenvalue with maximal real part, and the resolvent operator $(\lambda - L_0)^{-1}$ is in fact a well-defined and bounded operator on H_m^n for all λ so that $\Re\lambda > \Re\lambda_0$. Let $\beta > 0$ and take $\gamma = \Re\lambda_0 + \beta$. Using the boundedness of the resolvent operator, we obtain at once

$$\left\| \frac{1}{2\pi i} \int_{\gamma-iM}^{\gamma+iM} e^{\lambda s} (\lambda - L_0)^{-1} h \, d\lambda \right\|_{H_m^n} \leq C_{\beta,M} e^{(\Re\lambda_0 + \beta)s} \|h\|_{H_m^n}, \quad (3.21)$$

for any large but fixed constant M . To treat the integral for large $\Im\lambda$, we observe that directly from the equation $\lambda f = L_0 f + h$, one has

$$|\lambda| \|f\|_{H_m^n} \leq \|L_0 f + h\|_{H_m^n} \leq C(\|\nabla_y f\|_{H_{m+1}^n} + \|\nabla\phi\|_{H^n}) + \|h\|_{H_m^n}.$$

Using (3.19), there holds, for some $C'_{n+1,m+1} > 0$,

$$|\lambda| \|(\lambda - L_0)^{-1} h\|_{H_m^n} \leq C'_{n+1,m+1} \|\langle v \rangle^2 f\|_{L^2} + C'_{n+1,m+1} \|h\|_{H_{m+1}^n}.$$

We take $\Re\lambda = \gamma$, and consider

$$|\Im\lambda| > \frac{2}{3} C'_{n+1,m+1}.$$

We deduce

$$\|(\lambda - L_0)^{-1} h\|_{H_m^n} \leq \frac{C_\beta}{|\Im\lambda|} \|h\|_{H_{m+1}^{n+1}}. \quad (3.22)$$

Now, to estimate the integral for large $|\Im\lambda|$, we may write

$$(\lambda - L_0)^{-1} h = \frac{1}{\lambda} (\lambda - L_0)^{-1} L_0 h + \frac{h}{\lambda}.$$

Thus, with $\gamma = \Re\lambda_0 + \beta$, we get

$$\begin{aligned} & \text{P.V.} \frac{1}{2\pi i} \int_{\{|\Im\lambda| \geq M\}} e^{\lambda s} (\lambda - L_0)^{-1} h \, d\lambda \\ &= \text{P.V.} \frac{1}{2\pi i} \int_{\{|\Im\lambda| \geq M\}} e^{\lambda s} (\lambda - L_0)^{-1} \frac{L_0 h}{\lambda} \, d\lambda + \text{P.V.} \frac{1}{2\pi i} \left[\int_{\gamma-i\infty}^{\gamma+i\infty} - \int_{\{|\Im\lambda| < M\}} \right] e^{\lambda s} \frac{h}{\lambda} \, d\lambda \end{aligned}$$

in which the second integral on the right-hand side is equal to h , whereas the last integral is bounded by $C_0 e^{\gamma s} h$. We take $M \geq \frac{2}{3} C'_{n+1,m+1}$ so that the bound (3.22) holds. This yields

$$\begin{aligned} \left\| \int_{\{|\Im\lambda| \geq M\}} e^{\lambda s} (\lambda - L_0)^{-1} \frac{L_0 h}{\lambda} \, d\lambda \right\|_{H_m^n} &\leq C_{\beta,M} e^{\gamma s} \|L_0 h\|_{H_{m+1}^{n+1}} \int_{\{|\Im\lambda| \geq M\}} |\Im\lambda|^{-2} \, d\Im\lambda \\ &\leq C_{\beta,M} e^{\gamma s} \|h\|_{H_{m+2}^{n+2}}. \end{aligned}$$

Putting these together and combining with (3.21), we get

$$\|e^{L_0 s} h\|_{H_m^n} \leq C_\beta e^{(\Re\lambda_0 + \beta)s} \|h\|_{H_{m+2}^{n+2}}, \quad (3.23)$$

for any $\beta > 0$. The lemma is proved. \square

Next, we derive a few estimates on the electromagnetic field. Recall that the standard elliptic theory yields the elliptic estimate (3.17) for $\phi = -\Delta^{-1} \rho(f)$. In addition, together with a use of the Vlasov equation $(\partial_s - L)f = 0$, the function $\partial_s \phi$ satisfies $-\Delta \partial_s \phi = \partial_s \rho(f) = -\nabla \cdot j(f)$, which then yields

$$\|\partial_s \nabla \phi\|_{H^n} \leq C_0 \|j(f)\|_{H^n} \leq C_0 \|f\|_{H_3^n}.$$

Similarly, the standard H^n theory for the wave equation (3.4) for $A = A(f)$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \left(\|\varepsilon \partial_s A\|_{H^n}^2 + \|\nabla A\|_{H^n}^2 \right) &\leq C_0 \|\varepsilon \partial_s A\|_{H^n} \left(\|j(f)\|_{H^n} + \|\partial_s \nabla \phi\|_{H^n} \right) \\ &\leq C_0 \|\varepsilon \partial_s A\|_{H^n} \|f\|_{H_3^n}. \end{aligned} \quad (3.24)$$

Applying the Gronwall inequality to the above, we obtain

$$\|\varepsilon \partial_s A\|_{H^n} + \|\nabla A\|_{H^n} \leq C_0 \int_0^s \|f(\tau)\|_{H_3^n} d\tau. \quad (3.25)$$

In particular, by a view of the definition of the fields E, B in term of the electromagnetic potentials, we get

$$\|(E, B)\|_{H^n} \leq C_0 \int_0^s \|f(\tau)\|_{H_3^n} d\tau. \quad (3.26)$$

The following gives a link between the Vlasov-Poisson solution g_k and the Vlasov-Maxwell solution f_k as defined in (3.15).

Lemma 3.2. *Let g be in H_m^n , for $n \geq 0$ and $m \geq 3$, with $\int g dv dx = 0$ and let T be a positive number so that $\varepsilon T \ll 1$. There exists a solution f in $L^\infty(0, T; H_m^n)$ solving the linear problem:*

$$f - \varepsilon A \cdot \nabla_v \mu = g, \quad \varepsilon^2 \partial_s^2 A - \Delta A = \varepsilon(j(f) + \partial_s \nabla \Delta^{-1} \rho(g)), \quad A|_{s=0} = \partial_s A|_{s=0} = 0. \quad (3.27)$$

In addition, there holds

$$\sup_{\tau \in [0, s]} \|f(\tau)\|_{H_m^n} \leq C_0 \sup_{\tau \in [0, s]} \|g(\tau)\|_{H_m^n} + C_0 \int_0^s \|\langle v \rangle^3 g(\tau)\|_{L^2} d\tau, \quad \forall s \in [0, T],$$

for some constant C_0 that is independent of ε .

Furthermore, in the case where $\int \hat{v} g dv dx = 0$, we have $\int_{\mathbb{T}^3} A dx = 0$ and the following upper and lower bound on f :

$$c_0 \sup_{\tau \in [0, s]} \|g(\tau)\|_{H_m^n} \leq \sup_{\tau \in [0, s]} \|f(\tau)\|_{H_m^n} \leq C_0 \sup_{\tau \in [0, s]} \|g(\tau)\|_{H_m^n}, \quad \forall s \in [0, T], \quad (3.28)$$

for some constants $c_0 > 0, C_0 > 0$ that are independent of ε .

Proof. We start by establishing a priori estimates. We first note that there holds the Poincaré inequality:

$$\|A - \langle A \rangle\|_{H^n} \lesssim \|\nabla A\|_{H^n} \leq C_0 \int_0^s \|f(\tau)\|_{H_3^n} d\tau,$$

in which $\langle A \rangle$ denotes the average of A over \mathbb{T}^3 . This yields at once

$$\begin{aligned} \|f - \varepsilon \langle A \rangle \cdot \nabla_v \mu\|_{H_m^n} &\leq \|f - \varepsilon A \cdot \nabla_v \mu\|_{H_m^n} + \|\varepsilon A \cdot \nabla_v \mu - \varepsilon \langle A \rangle \cdot \nabla_v \mu\|_{H_m^n} \\ &\leq \|g\|_{H_m^n} + C_0 \varepsilon \int_0^s \|f(\tau)\|_{H_3^n} d\tau. \end{aligned} \quad (3.29)$$

Let us bound the average of A . Directly from the wave equation for A and the equation for f in terms of g , we get

$$\begin{aligned} \varepsilon \frac{d^2}{ds^2} \langle A \rangle &= \langle j \rangle = \left\langle \int \hat{v} g dv \right\rangle + \varepsilon \left\langle \int \hat{v} (A \cdot \nabla_v \mu) dv \right\rangle \\ &= \left\langle \int \hat{v} g dv \right\rangle - r_0 \varepsilon \langle A \rangle, \end{aligned}$$

in which we have used integration by parts in v , the fact that μ is radial, and set

$$r_0 := \int_{\mathbb{R}^3} \frac{1 + \frac{2}{3}\varepsilon^2|v|^2}{(1 + \varepsilon^2|v|^2)^{3/2}} \mu \, dv > 0.$$

First, we consider the case when the average of $\int \hat{v}g(s) \, dv$ is equal to zero. In this case, we clearly have $\langle A(s) \rangle = 0$ and thus the bound (3.29) reads

$$\|f(s)\|_{H_m^n} \leq \|g\|_{H_m^n} + C_0\varepsilon \int_0^s \|f(\tau)\|_{H_3^n} \, d\tau.$$

By a standard fixed point argument, we obtain the existence of f in $L^\infty(0, T; H_m^n)$ and satisfying (3.27), as long as $\varepsilon T \ll 1$ and $m \geq 3$. We straightaway deduce the upper bound of (3.28), while the lower bound is obtained as follows:

$$\begin{aligned} \sup_{\tau \in [0, s]} \|g(\tau)\|_{H_m^n} &\leq \sup_{\tau \in [0, s]} \|f(\tau)\|_{H_m^n} + C_0\varepsilon \sup_{\tau \in [0, s]} \|A(\tau)\|_{H^n} \\ &\leq \sup_{\tau \in [0, s]} \|f(\tau)\|_{H_m^n} + C_0\varepsilon \int_0^s \|f(\tau)\|_{H_3^n} \, d\tau \\ &\leq \frac{1}{C_0} \sup_{\tau \in [0, s]} \|f(\tau)\|_{H_m^n}. \end{aligned}$$

In the general case when the average of $\int \hat{v}g(s) \, dv$ is not equal to zero, we define $\langle \varepsilon A(s) \rangle$ as the solution of the ordinary differential equation

$$\frac{d^2}{ds^2} \langle \varepsilon A(s) \rangle + r_0 \langle \varepsilon A(s) \rangle = \left\langle \int \hat{v}g(s) \, dv \right\rangle.$$

Since the fundamental solutions to the homogeneous equation $y'' + r_0 y = 0$ are bounded, the above yields at once

$$\varepsilon |\langle A(s) \rangle| \leq \int_0^s \left| \left\langle \int \hat{v}g(\tau) \, dv \right\rangle \right| \, d\tau \leq C_0 \int_0^s \|\langle v \rangle^3 g(\tau)\|_{L^2} \, d\tau.$$

We now establish a bound for f satisfying

$$f - \varepsilon(A - \langle A \rangle) \cdot \nabla_v \mu = g + \varepsilon \langle A \rangle \cdot \nabla_v \mu.$$

The second term on the left-hand side is again a small perturbation in terms of f , yielding

$$\|f(s)\|_{H_m^n} \leq \|g\|_{H_m^n} + C_0 \int_0^s \|\langle v \rangle^3 g(\tau)\|_{L^2} \, d\tau + C_0\varepsilon \int_0^s \|f(\tau)\|_{H_3^n} \, d\tau.$$

Thus, as long as $\varepsilon T \ll 1$, we can use as well a fixed point argument, yielding the existence of f as well as the claimed bound. \square

3.3 Error estimates

Let us now give estimates on the approximate solution f_{app} and the error of the approximation $R(f_{\text{app}})$. Let N be a fixed number of the iteration in f_{app} and let

$$m \geq 5N + 3 \text{ and } n > 3N + 5/2.$$

By a view of the wave estimate (3.25) and the fact that μ decays rapidly at infinity, we have for all $n, m \geq 0$,

$$\begin{aligned} \|\tilde{S}(f)(s)\|_{H_m^n} &\leq C_0 \|\nabla A\|_{H^n} \leq C_0 \int_0^s \|f(\tau)\|_{H_3^n} \, d\tau \\ \|T(f)(s)\|_{H_{m-3}^{n-1}} &\leq C_0 \|f\|_{H_m^n}, \\ \|Q(f, g)(s)\|_{H_{m-2}^{n-1}} &\leq C_0 \|(E, B)(s)\|_{H^n} \|g(s)\|_{H_m^n} \leq C_0 \|g(s)\|_{H_m^n} \int_0^s \|f(\tau)\|_{H_3^n} \, d\tau. \end{aligned} \tag{3.30}$$

We apply Lemma 3.2 to the linear problem (3.12) for f_1 , which yields

$$\|f_1\|_{H_m^n} \leq C_0 e^{\Re\lambda_0 s}. \quad (3.31)$$

By induction, we shall prove

$$\|f_k(s)\|_{H_{m-5k+5}^{n-3k+3}} + \|g_k(s)\|_{H_{m-5k+5}^{n-3k+3}} \leq C_k e^{(1+\frac{k-1}{p})\Re\lambda_0 s}, \quad 1 \leq k \leq N, \quad (3.32)$$

for all $s \in [0, T]$, with $\varepsilon T \ll 1$. The case $k = 1$ is clear. Assuming the bound holds for all $j \in \{1, \dots, k\}$, with $k \geq 1$, we now prove the bound for $j = k+1 \geq 2$. Writing a Duhamel formula for the nonhomogeneous equation (3.15) on g_{k+1} , we find

$$g_{k+1} = - \int_0^s e^{L_0(s-\tau)} \left[\tilde{S}(f_k) + T(f_{k-1}) + \sum_{\ell=1}^k Q(f_\ell, f_{k+2-p-\ell}) \right] (\tau) d\tau.$$

Using Proposition 3.1, the bounds in (3.30) and the induction assumption, we can estimate

$$\begin{aligned} & \|g_{k+1}\|_{H_{m-5k}^{n-3k}} \\ & \leq C_\beta \int_0^s e^{(\Re\lambda_0 + \beta)(s-\tau)} \left[\|\tilde{S}(f_k)\|_{H_{m-5k+2}^{n-3k+2}} + \|T(f_{k-1})\|_{H_{m-5k+2}^{n-3k+2}} + \sum_{\ell=1}^k \|Q(f_\ell, f_{k+2-p-\ell})\|_{H_{m-5k+2}^{n-3k+2}} \right] (\tau) d\tau \\ & \leq C_\beta \int_0^s e^{(\Re\lambda_0 + \beta)(s-\tau)} \left[C_k e^{(1+\frac{k-1}{p})\Re\lambda_0 \tau} + \sum_{\ell=1}^k C_{k,\ell} e^{(1+\frac{\ell-1}{p})\Re\lambda_0 \tau} e^{(1+\frac{k+1-p-\ell}{p})\Re\lambda_0 \tau} \right] d\tau \\ & \leq C_\beta \int_0^s e^{(\Re\lambda_0 + \beta)(s-\tau)} e^{(1+\frac{k}{p})\Re\lambda_0 \tau} d\tau \\ & \leq C_k e^{(1+\frac{k}{p})\Re\lambda_0 s}, \end{aligned}$$

in which we have chosen $\beta = \Re\lambda_0/p$. Next, we apply Lemma 3.2, yielding

$$\sup_{\tau \in [0, s]} \|f_{k+1}(\tau)\|_{H_{m-5k}^{n-3k}} \leq C_0 \sup_{\tau \in [0, s]} \|g_{k+1}(\tau)\|_{H_{m-5k}^{n-3k}} + C_0 \int_0^s \|\langle v \rangle^3 g_{k+1}(\tau)\|_{L^2} d\tau \leq C_k e^{(1+\frac{k}{p})\Re\lambda_0 s},$$

which finishes the proof of the inductive bound (3.32), for all $k \geq 1$.

Using these bounds on f_k as well as (3.30), we can estimate the error of the approximation:

$$\begin{aligned} \|R(f_{\text{app}})(s)\|_{H_{m-5N}^{n-3N}} & \leq \varepsilon^{N+p} (\|\tilde{S}(f_N)\|_{H_{m-5N}^{n-3N}} + \|T(f_{N-1})\|_{H_{m-5N}^{n-3N}}) + \varepsilon^{N+p+1} \|T(f_N)\|_{H_{m-5N}^{n-3N}} \\ & \quad + \sum_{k+\ell > N+1-p; 1 \leq k, \ell \leq N-1} \varepsilon^{2p+k+\ell-2} \|Q(f_k, f_\ell)\|_{H_{m-5N}^{n-3N}} \\ & \leq C_0 \varepsilon^{N+p} e^{(1+\frac{N-1}{p})\Re\lambda_0 s} + \sum_{k+\ell > N+1-p; 1 \leq k, \ell \leq N-1} \varepsilon^{2p+k+\ell-2} e^{(2+\frac{k+\ell-2}{p})\Re\lambda_0 s} \quad (3.33) \\ & \leq C_0 \varepsilon^{N+p} e^{(1+\frac{N-1}{p})\Re\lambda_0 s} + C_0 \varepsilon^{N+p} e^{(1+\frac{N}{p})\Re\lambda_0 s} \\ & \leq C_0 \left(\varepsilon^p e^{\Re\lambda_0 s} \right)^{\left(1+\frac{N}{p}\right)}, \end{aligned}$$

for all $s \geq 0$, as long as $\varepsilon^p e^{\Re\lambda_0 s}$ remains bounded.

3.4 Nonlinear instability

We are ready to conclude the proof of Theorem 2.3. The instability result now follows from a standard energy estimate. Indeed, let (f, ϕ, A) be the exact perturbative solution to the Vlasov-Maxwell system:

$$\partial_s f + \hat{v} \cdot \nabla_y f + (E + \varepsilon \hat{v} \times B) \cdot \nabla_v (\mu + f) = 0.$$

with the electromagnetic field solving the Maxwell equations. Consider $(f_{\text{app}}, \phi_{\text{app}}, A_{\text{app}})$ the approximate solution constructed and studied in the previous sections. Let the difference be

$$(h, \phi_h, A_h) := (f - f_{\text{app}}, \phi - \phi_{\text{app}}, A - A_{\text{app}}),$$

which solves

$$\partial_s h + \hat{v} \cdot \nabla_y h + (E_h + \varepsilon \hat{v} \times B_h) \cdot \nabla_v (\mu + f_{\text{app}}) + (E_{\text{app}} + E_h + \varepsilon \hat{v} \times (B_{\text{app}} + B_h)) \cdot \nabla_v h = R(f_{\text{app}}),$$

with $h|_{t=0} = 0$. Standard weighted energy estimates yield, for $k > 7/2$,

$$\frac{1}{2} \frac{d}{ds} \|h\|_{H_3^k}^2 \leq C_0 \|h\|_{H_3^k} \left[\|(E_h, B_h)\|_{H^k} + \|R(f_{\text{app}})\|_{H_3^k} \right] + \|h\|_{H_3^k}^2 \left[1 + \|(E_h, B_h)\|_{H^k} \right].$$

Combining with the estimates (3.26) on E_h, B_h and with (3.33) yields at once

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \left(\|h(s)\|_{H_3^k}^2 + \|(E_h, B_h)(s)\|_{H^k}^2 \right) &\leq \left[C + \|h(s)\|_{H_3^k} \right] \left(\|h(s)\|_{H_3^k}^2 + \|(E_h, B_h)(s)\|_{H^k}^2 \right) \\ &\quad + C_0 \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^{2(1+\frac{N}{p})}. \end{aligned}$$

We now introduce

$$T^\varepsilon := \sup \left\{ s \geq 0 : \sup_{\tau \in [0, s]} \|h(\tau)\|_{H_3^k} \leq \frac{\theta_0}{2} \varepsilon^p e^{\Re \lambda_0 \tau} \right\},$$

where $\theta_0 > 2$ was fixed in (3.11). By the standard local existence theory, we know that $T^\varepsilon > 0$.

Now define

$$\tilde{T}^\varepsilon := \frac{1}{\Re \lambda_0} |\log(\varepsilon^p \frac{\theta_0}{2})|.$$

It follows that for all $s \in [0, \tilde{T}^\varepsilon]$,

$$\frac{\theta_0}{2} \varepsilon^p e^{\Re \lambda_0 s} \leq 1, \tag{3.34}$$

If $\tilde{T}^\varepsilon > T^\varepsilon$, then for any $s \in [0, T^\varepsilon]$, the above differential inequality yields

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \left(\|h(s)\|_{H_3^k}^2 + \|(E_h, B_h)(s)\|_{H^k}^2 \right) &\leq (1 + C) \left(\|h(s)\|_{H_3^k}^2 + \|(E_h, B_h)(s)\|_{H^k}^2 \right) \\ &\quad + C_0 \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^{2(1+\frac{N}{p})}. \end{aligned}$$

Using the Gronwall inequality and imposing N large enough so that

$$N \geq C_0, \quad 1 + C \leq \left(1 + \frac{N}{p}\right) \Re \lambda_0, \quad \frac{32C_0}{\theta_0^2} \leq (\theta_0/2)^{\frac{2N}{p}},$$

there holds

$$\begin{aligned} \|h(s)\|_{H_3^k}^2 + \|(E_h, B_h)(s)\|_{H^k}^2 &\leq 2C_0 \int_0^s e^{2(1+C)(s-\tau)} \left(\varepsilon^p e^{\Re \lambda_0 \tau} \right)^{2(1+\frac{N}{p})} d\tau \\ &\leq 2C_0 \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^{2(1+\frac{N}{p})} \\ &\leq \frac{2C_0}{(\theta_0/2)^{\frac{2N}{p}}} \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^2 \\ &\leq \frac{1}{16} \theta_0^2 \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^2 = \left(\frac{\theta_0}{4} \varepsilon^p e^{\Re \lambda_0 s} \right)^2 \\ &< \frac{\theta_0}{4} \varepsilon^p e^{\Re \lambda_0 s}. \end{aligned}$$

This contradicts the definition of T^ε and proves that we necessarily have $T^\varepsilon \geq \tilde{T}^\varepsilon$.

Finally, recall that $f = f_{\text{app}} + h$. Thus, by the triangle inequality, as long as $s \in [0, \tilde{T}^\varepsilon]$, we get

$$\begin{aligned} \|f\|_{L^2} &\geq \|f_{\text{app}}\|_{L^2} - \|h\|_{L^2} \\ &\geq \|f_{\text{app}}\|_{L^2} - \frac{\theta_0}{2} \varepsilon^p e^{\Re \lambda_0 s}. \end{aligned} \quad (3.35)$$

We therefore need to get a lower bound on the L^2 norm f_{app} . First, we have

$$\|f_1\|_{L^2} \geq \|g_1\|_{L^2} - \varepsilon \|A_1 \cdot \nabla_v \mu\|_{L^2}.$$

It follows from the construction that the average of $\int v g_1 dv$ is equal to zero. Indeed, by definition, we have

$$\int v \hat{g}_1 dv dy = \int e^{ik_0 \cdot y} dy \int \frac{1}{|k_0|^2} \frac{\nabla_v \mu \cdot k_0}{k_0 \cdot (v - \omega_0)} v dv = 0.$$

From Lemma 3.2, we know that $\langle A_1 \rangle = 0$, so that

$$\varepsilon \|A_1 \cdot \nabla_v \mu\|_{L^2} \leq \varepsilon C_0 \int_0^s \|f(\tau)\|_{H_3^n} d\tau.$$

Recalling (3.31), we end up with

$$\varepsilon \|A_1 \cdot \nabla_v \mu\|_{L^2} \leq \varepsilon C_0 e^{\Re \lambda_0 s}.$$

By (3.11), we deduce (at least for $\varepsilon > 0$ small enough),

$$\|f_1\|_{L^2} \geq \theta_0 e^{\Re \lambda_0 s}.$$

Finally using (3.32) to bound the contribution of the terms f_k , $k \geq 2$, we obtain

$$\|f_{\text{app}}\|_{L^2} \geq \theta_0 \varepsilon^p e^{\Re \lambda_0 s} - C_N \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^{1+\frac{1}{p}}$$

and thus for $s \in [0, \tilde{T}^\varepsilon]$,

$$\begin{aligned} \|f(s)\|_{L^2} &\geq \theta_0 \varepsilon^p e^{\Re \lambda_0 s} - C_N \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^{1+\frac{1}{p}} - \frac{\theta_0}{2} \varepsilon^p e^{\Re \lambda_0 s} \\ &\geq \frac{\theta_0}{2} \varepsilon^p e^{\Re \lambda_0 s} \left(1 - \frac{2C_N}{\theta_0} \left(\varepsilon^p e^{\Re \lambda_0 s} \right)^{\frac{1}{p}} \right). \end{aligned} \quad (3.36)$$

Define finally $\bar{T}^\varepsilon := \frac{1}{\Re \lambda_0} \left| \log(\varepsilon^p \frac{4C_N}{\theta_0}) \right|$. For $s_\varepsilon := \min(\tilde{T}^\varepsilon, \bar{T}^\varepsilon)$ we end up with the lower bound

$$\|f(s_\varepsilon)\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \geq \delta_0, \quad (3.37)$$

with $\delta_0 = \min\left(\frac{1}{2}, \frac{\theta_0^2}{16C_N}\right)$.

Similarly, for what concerns $H^{-S'}$ instability, we can get as well

$$\begin{aligned} \|f(s_\varepsilon)\|_{H^{-S'}} &\geq \varepsilon^p \|f_1\|_{H^{-S'}} - \|f_{\text{app}} - \varepsilon^p f_1\|_{H^{-S'}} - \|h\|_{H^{-S'}} \\ &\geq \varepsilon^p \|f_1\|_{H^{-S'}} - \|f_{\text{app}} - \varepsilon^p f_1\|_{L^2} - \|h\|_{L^2} \\ &\geq \theta_0 \varepsilon^p e^{\Re \lambda_0 s_\varepsilon} - C_N \left(\varepsilon^p e^{\Re \lambda_0 s_\varepsilon} \right)^{1+\frac{1}{p}} - \frac{\theta_0}{2} \varepsilon^p e^{\Re \lambda_0 s_\varepsilon} \\ &\geq \delta_0, \end{aligned} \quad (3.38)$$

with δ_0 as defined in (3.37). Recalling that $f^\varepsilon - \mu = f$, this proves the first instability result of (2.8).

The instability on ρ^ε and j^ε is proved with similar estimates, using the *weighted* L^2 error estimates. The only thing to notice is that, recalling $g_1 = e^{\lambda_0 s} \hat{g}_1$, and by a view of (3.10) and of the Penrose condition (2.3), we have $\rho(\hat{g}_1) \neq 0$ and $j(\hat{g}_1) \neq 0$. With the same arguments, we end up with

$$\|\rho^\varepsilon(s) - 1\|_{H^{-s'}} \geq \delta'_0, \quad \|j^\varepsilon(s)\|_{H^{-s'}} \geq \delta'_0, \quad (3.39)$$

for some $\delta'_0 > 0$.

The instability on E^ε then follows by a view of (3.3). We write

$$\|E^\varepsilon\|_{L^2}^2 = \|\nabla\phi^\varepsilon\|_{L^2}^2 + \|\varepsilon\partial_s A^\varepsilon\|_{L^2}^2 + \varepsilon \int \nabla\phi^\varepsilon \cdot \partial_s A^\varepsilon dx,$$

and note that since A^ε satisfies the Coulomb gauge $\nabla \cdot A^\varepsilon = 0$,

$$\int \nabla\phi^\varepsilon \cdot \partial_s A^\varepsilon dx = \int \phi^\varepsilon \partial_s \nabla \cdot A^\varepsilon dx = 0.$$

We thus use (3.39) to get $\|\rho^\varepsilon - 1\|_{H^{-1}} \geq \delta'_0$ and obtain, using finally the Poisson equation (3.4),

$$\|E^\varepsilon(s)\|_{L^2} \geq \|\nabla\phi^\varepsilon(s)\|_{L^2} \geq C_0 \delta'_0.$$

Let us finally complete the proof of Theorem 2.3 by briefly explaining how to deal with complex eigenvalues and eigenfunctions, as well as getting non-negative distribution functions.

We assume here that $\Im\lambda_0 \neq 0$. Writing $\hat{g}_1 = \Re g_1 + i\Im g_1$, and assuming without loss of generality that $\Re g_1 \neq 0$, we set $g_1 = \Re(e^{\lambda_0 s} \hat{g}_1)$ instead of the definition (3.9). Then one can perform exactly the same construction and analysis, except that the lower bound for f_{app} in (3.36) is achieved for all s of the form $s = \frac{2\pi k}{\Im\lambda_0}$. This is sufficient to get the instability as in (3.37).

For what concerns non-negativity, we just need to notice that the δ -condition and the form of the eigenfunctions (recall (3.10)) ensure that $\varepsilon^p |g_1|_{t=0} \leq \mu$, so that the initial condition satisfies $f|_{t=0} \geq 0$.

4 Invalidity of the quasineutral limit

Let $\mu(v)$ be a radial, smooth, normalized profile satisfying the δ -condition and the sharp Penrose instability condition.

For any $M > 0$, we shall denote

$$\mathbb{T}_M^3 := \mathbb{R}^3 / (M\mathbb{Z} \times M\mathbb{Z} \times M\mathbb{Z}).$$

We recall that for a given length M , the sharp Penrose instability condition does not necessarily ensure the existence of a growing mode for the linearized equations. However, the latter is true for large enough values of M . This is the content of the following Proposition, taken from [24, Proposition 3.2].

Proposition 4.1. *Assume that μ is a smooth homogeneous profile satisfying the sharp Penrose instability condition. There exists $M_0 > 0$ such that if $M \geq M_0$, then the Penrose instability condition (2.3) is satisfied for the equations posed on $\mathbb{T}_M^3 \times \mathbb{R}^3$.*

As a matter of fact, the framework of [24] is one-dimensional but this particular result straightforwardly extends to higher dimensions. Using this Proposition, we fix some large enough parameter $M > 0$ such that the linearized Vlasov-Poisson operator L_0 on $\mathbb{T}_M^3 \times \mathbb{R}^3$ has an eigenvalue with positive real part. From now on, we consider the sequence $\varepsilon_k = \frac{1}{kM}$, for $k \in \mathbb{N}^*$, but we forget about the k subscript for readability.

As already explained in the introduction, in the high spatial frequency regime, the study of the quasineutral limit comes down to that of the classical limit. More precisely, we shall first consider εM -periodic (in all spatial directions) solutions to (1.3), that is we look for solutions to the system

$$\begin{cases} \partial_t \tilde{f}_\varepsilon + \hat{v} \cdot \nabla_x \tilde{f}_\varepsilon + (\tilde{E}_\varepsilon + \hat{v} \times \tilde{B}_\varepsilon) \cdot \nabla_v \tilde{f}_\varepsilon = 0, \\ \partial_t \tilde{B}_\varepsilon + \nabla_x \times \tilde{E}_\varepsilon = 0, & \varepsilon^2 \nabla_x \cdot \tilde{E}_\varepsilon = \tilde{\rho}_\varepsilon - 1, \\ -\varepsilon^2 \partial_t \tilde{E}_\varepsilon + \nabla_x \times \tilde{B}_\varepsilon = \tilde{j}_\varepsilon, & \nabla_x \cdot \tilde{B}_\varepsilon = 0, \end{cases} \quad (4.1)$$

for $t \geq 0$, $x \in \mathbb{T}_{\varepsilon M}^3$, $v \in \mathbb{R}^3$ and where

$$\hat{v} = \frac{v}{\sqrt{1 + \varepsilon^2 |v|^2}},$$

$$\tilde{\rho}_\varepsilon(t, x) = \int_{\mathbb{R}^3} \tilde{f}_\varepsilon \, dv, \quad \tilde{j}_\varepsilon(t, x) = \int_{\mathbb{R}^3} \hat{v} \tilde{f}_\varepsilon \, dv.$$

We can then obtain a solution $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ to (1.3) by patching $(\varepsilon M)^{-3}$ copies of $(\tilde{f}_\varepsilon, \tilde{E}_\varepsilon, \tilde{B}_\varepsilon)$. This means, identifying \mathbb{T}_M^3 to $[0, M]^3$, writing

$$f_\varepsilon(t, x, v) = \tilde{f}_\varepsilon(t, x_1 - j_1 \varepsilon M, x_2 - j_2 \varepsilon M, x_3 - j_3 \varepsilon M, v),$$

for all $x = (x_1, x_2, x_3)$ in

$$\prod_{i=1}^3 [j_i \varepsilon M, (j_i + 1) \varepsilon M), \quad j_1, j_2, j_3 = 0, \dots, k-1.$$

Similar formulas are given for $(E_\varepsilon, B_\varepsilon)$.

We can now perform the hyperbolic change of variables $(t, x, v) \rightarrow (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v)$, i.e. we consider $(g_\varepsilon, \mathbb{E}_\varepsilon, \mathbb{B}_\varepsilon)$ such that:

$$\tilde{f}_\varepsilon(t, x, v) = g_\varepsilon \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v \right), \quad \tilde{E}_\varepsilon(t, x, v) = \frac{1}{\varepsilon} \mathbb{E}_\varepsilon \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right), \quad \tilde{B}_\varepsilon(t, x, v) = \mathbb{B}_\varepsilon \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right). \quad (4.2)$$

This leads to the study of the classical limit, for $s \geq 0$, $y \in \mathbb{T}_M$, $v \in \mathbb{R}$

$$\begin{cases} \partial_s g_\varepsilon + \hat{v} \cdot \nabla_y g_\varepsilon + (\mathbb{E}_\varepsilon + \varepsilon \hat{v} \times \mathbb{B}_\varepsilon) \cdot \nabla_v g_\varepsilon = 0, \\ \varepsilon \partial_s \mathbb{B}_\varepsilon + \nabla_y \times \mathbb{E}_\varepsilon = 0, & \nabla_y \cdot \mathbb{E}_\varepsilon = \int g_\varepsilon \, dv - 1 \\ -\varepsilon \partial_s \mathbb{E}_\varepsilon + \nabla_y \times \mathbb{B}_\varepsilon = \varepsilon \int \hat{v} g_\varepsilon \, dv, & \nabla_y \cdot \mathbb{B}_\varepsilon = 0. \end{cases} \quad (4.3)$$

We apply Theorem 2.3 (M being considered as a fixed transparent parameter). Let $S, N \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$, such that $p > S + N$. We take $k = 0$. By Theorem 2.3, we find for all $\varepsilon \in (0, 1]$ a solution $(g_\varepsilon, \mathbb{E}_\varepsilon, \mathbb{B}_\varepsilon)$ to (4.3) with $g_\varepsilon \geq 0$, such that

$$\|(1 + |v|^2)^{\frac{m}{2}} (g_{\varepsilon|_{s=0}} - \mu)\|_{H^s(\mathbb{T}_M^3 \times \mathbb{R}^3)} \leq \varepsilon^p, \quad (4.4)$$

but there is a sequence of times $s_\varepsilon = \mathcal{O}(|\log \varepsilon|)$ such that

$$\liminf_{\varepsilon \rightarrow 0} \|g_\varepsilon(s_\varepsilon) - \mu\|_{L^2(\mathbb{T}_M^3 \times \mathbb{R}^3)} > 0, \quad \liminf_{\varepsilon \rightarrow 0} \|\mathbb{E}_\varepsilon(s_\varepsilon)\|_{L^2(\mathbb{T}_M^3)} > 0, \quad (4.5)$$

$$\liminf_{\varepsilon \rightarrow 0} \|\rho_{g_\varepsilon}(s_\varepsilon) - 1\|_{L^2(\mathbb{T}_M^3)} > 0, \quad \liminf_{\varepsilon \rightarrow 0} \|j_{g_\varepsilon}(s_\varepsilon)\|_{L^2(\mathbb{T}_M^3)} > 0, \quad (4.6)$$

with the notation $\rho_{g_\varepsilon}(t) = \int_{\mathbb{R}^3} g_\varepsilon(t, x, v) \, dv$ and $j_{g_\varepsilon}(t) = \int_{\mathbb{R}^3} \hat{v} g_\varepsilon(t, x, v) \, dv$.

Next, a consequence of the change of variable (4.2) and of the εM -periodicity of f_ε is that:

$$\begin{aligned} \|(1 + |v|^2)^{\frac{m}{2}} (f_{\varepsilon|t=0} - \mu)\|_{H^S(\mathbb{T}^3 \times \mathbb{R}^3)} &\leq C \frac{\varepsilon^{-S}}{M^3} \|g_{\varepsilon|s=0} - \mu\|_{H^S(\mathbb{T}_M^3 \times \mathbb{R}^3)} \\ \|f_\varepsilon(t) - 1\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} &= \frac{1}{M^3} \|g_\varepsilon(t/\varepsilon) - 1\|_{L^2(\mathbb{T}_M^3 \times \mathbb{R}^3)}, \quad \varepsilon \|E_\varepsilon(t)\|_{L^2(\mathbb{T}^3)} = \frac{1}{M^3} \|\mathbb{E}_\varepsilon(t/\varepsilon)\|_{L^2(\mathbb{T}_M^3)}, \\ \|\rho_\varepsilon(t) - 1\|_{L^2(\mathbb{T}^3)} &= \frac{1}{M^3} \|\rho_{g_\varepsilon}(t/\varepsilon) - 1\|_{L^2(\mathbb{T}_M^3)}, \quad \|j_\varepsilon(t) - 1\|_{L^2(\mathbb{T}^3)} = \frac{1}{M^3} \|j_{g_\varepsilon}(t/\varepsilon) - 1\|_{L^2(\mathbb{T}_M^3)}. \end{aligned} \quad (4.7)$$

We set $t_\varepsilon := \varepsilon s_\varepsilon = \mathcal{O}(\varepsilon |\log \varepsilon|)$ and deduce, at least for ε small enough,

$$\begin{aligned} \|(1 + |v|^2)^{\frac{m}{2}} (f_{\varepsilon|t=0} - \mu)\|_{H^S(\mathbb{T}^3 \times \mathbb{R}^3)} &\leq \frac{1}{M^3} \varepsilon^{p-S} \leq \varepsilon^N, \\ \liminf_{\varepsilon \rightarrow 0} \|f_\varepsilon(t_\varepsilon) - \mu\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} &> 0, \quad \liminf_{\varepsilon \rightarrow 0} \varepsilon \|E_\varepsilon(t_\varepsilon)\|_{L^2(\mathbb{T}^3)} > 0, \\ \liminf_{\varepsilon \rightarrow 0} \|\rho_\varepsilon(t_\varepsilon) - 1\|_{L^2(\mathbb{T}^3)} &> 0, \quad \liminf_{\varepsilon \rightarrow 0} \|j_\varepsilon(t_\varepsilon)\|_{L^2(\mathbb{T}^3)} > 0, \end{aligned}$$

which proves Theorem 2.8.

Acknowledgements. We are grateful to Pennsylvania State University and École polytechnique for hospitality during the preparation of this work. TN was partly supported by the NSF under grant DMS-1405728 and by a two-month visiting position at École polytechnique during Spring 2015.

We thank the referee for several suggestions that helped to improve the presentation of the paper and especially for correcting an earlier incorrect interpretation of Theorem 2.3.

References

- [1] K. Asano. On local solutions of the initial value problem for the Vlasov-Maxwell equation. *Comm. Math. Phys.*, 106(4):551–568, 1986.
- [2] K. Asano and S. Ukai. On the Vlasov-Poisson limit of the Vlasov-Maxwell equation. In *Patterns and waves*, volume 18 of *Stud. Math. Appl.*, pages 369–383. North-Holland, Amsterdam, 1986.
- [3] F. Bouchut, F. Golse, and C. Pallard. Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 170(1):1–15, 2003.
- [4] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4):737–754, 2000.
- [5] Yann Brenier, Norbert Mauser, and Marjolaine Puel. Incompressible Euler and e-MHD as scaling limits of the Vlasov-Maxwell system. *Commun. Math. Sci.*, 1(3):437–447, 2003.
- [6] P. Degond. Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equations for infinite light velocity. *Math. Methods Appl. Sci.*, 8(4):533–558, 1986.
- [7] P. Degond. Spectral theory of the linearized Vlasov-Poisson equation. *Trans. Amer. Math. Soc.*, 294(2):435–453, 1986.
- [8] R. J. DiPerna and P.-L. Lions. Global weak solutions of Vlasov-Maxwell systems. *Comm. Pure Appl. Math.*, 42(6):729–757, 1989.
- [9] R. Glassey and J. Schaeffer. The “two and one-half-dimensional” relativistic Vlasov Maxwell system. *Comm. Math. Phys.*, 185(2):257–284, 1997.
- [10] R. Glassey and J. Schaeffer. The relativistic Vlasov-Maxwell system in two space dimensions. I, II. *Arch. Rational Mech. Anal.*, 141(4):331–354, 355–374, 1998.

- [11] R. Glassey and W. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rational Mech. Anal.*, 92(1):59–90, 1986.
- [12] E. Grenier. Defect measures of the Vlasov-Poisson system in the quasineutral regime. *Comm. Partial Differential Equations*, 20(7-8):1189–1215, 1995.
- [13] E. Grenier. Oscillations in quasineutral plasmas. *Comm. Partial Differential Equations*, 21(3-4):363–394, 1996.
- [14] E. Grenier. Limite quasineutre en dimension 1. In *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1999)*, pages Exp. No. II, 8. Univ. Nantes, Nantes, 1999.
- [15] Emmanuel Grenier. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.*, 53(9):1067–1091, 2000.
- [16] Yan Guo. Stable magnetic equilibria in collisionless plasmas. *Comm. Pure Appl. Math.*, 50(9):891–933, 1997.
- [17] Yan Guo. Stable magnetic equilibria in a symmetric collisionless plasma. *Comm. Math. Phys.*, 200(1):211–247, 1999.
- [18] Yan Guo and Walter A. Strauss. Nonlinear instability of double-humped equilibria. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 12(3):339–352, 1995.
- [19] Yan Guo and Walter A. Strauss. Unstable oscillatory-tail waves in collisionless plasmas. *SIAM J. Math. Anal.*, 30(5):1076–1114 (electronic), 1999.
- [20] Yan Guo and Walter A. Strauss. Magnetically created instability in a collisionless plasma. *J. Math. Pures Appl. (9)*, 79(10):975–1009, 2000.
- [21] D. Han-Kwan and M. Iacobelli. The quasineutral limit of the Vlasov-Poisson equation in Wasserstein metric. *Comm. Math. Sci.*, to appear, 2014.
- [22] D. Han-Kwan and M. Iacobelli. Quasineutral limit for Vlasov-Poisson via Wasserstein stability estimates in higher dimension. *Submitted*, 2015.
- [23] D. Han-Kwan and F. Rousset. Quasineutral limit for Vlasov-Poisson with Penrose stable data. *Ann. Sci. École Norm. Sup.*, to appear, 2015.
- [24] Daniel Han-Kwan and Maxime Hauray. Stability Issues in the Quasineutral Limit of the One-Dimensional Vlasov–Poisson Equation. *Comm. Math. Phys.*, 334(2):1101–1152, 2015.
- [25] A. Kingsep, K. Chukbar, and V. Ian’kov. Electron magnetohydrodynamics. *Voprosy Teorii Plazmy*, 16:209–250, 1987.
- [26] S. Klainerman and G. Staffilani. A new approach to study the Vlasov-Maxwell system. *Commun. Pure Appl. Anal.*, 1(1):103–125, 2002.
- [27] Zhiwu Lin and Walter Strauss. Nonlinear stability and instability of relativistic Vlasov-Maxwell systems. *Comm. Pure Appl. Math.*, 60(6):789–837, 2007.
- [28] Zhiwu Lin and Walter A. Strauss. Linear stability and instability of relativistic Vlasov-Maxwell systems. *Comm. Pure Appl. Math.*, 60(5):724–787, 2007.
- [29] Zhiwu Lin and Walter A. Strauss. A sharp stability criterion for the Vlasov-Maxwell system. *Invent. Math.*, 173(3):497–546, 2008.
- [30] Jonathan Luk and Robert M. Strain. A new continuation criterion for the relativistic Vlasov-Maxwell system. *Comm. Math. Phys.*, 331(3):1005–1027, 2014.

- [31] N. Masmoudi. From Vlasov-Poisson system to the incompressible Euler system. *Comm. Partial Differential Equations*, 26(9-10):1913–1928, 2001.
- [32] Toan T. Nguyen and Walter A. Strauss. Linear stability analysis of a hot plasma in a solid torus. *Arch. Ration. Mech. Anal.*, 211(2):619–672, 2014.
- [33] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [34] O. Penrose. Electrostatic instability of a uniform non-Maxwellian plasma. *Phys. Fluids*, 3:258–265, 1960.
- [35] Marjolaine Puel and Laure Saint-Raymond. Quasineutral limit for the relativistic Vlasov-Maxwell system. *Asymptot. Anal.*, 40(3-4):303–352, 2004.
- [36] J. Schaeffer. The classical limit of the relativistic Vlasov-Maxwell system. *Comm. Math. Phys.*, 104(3):403–421, 1986.
- [37] S. Wollman. An existence and uniqueness theorem for the Vlasov-Maxwell system. *Comm. Pure Appl. Math.*, 37(4):457–462, 1984.
- [38] S. Wollman. Local existence and uniqueness theory of the Vlasov-Maxwell system. *J. Math. Anal. Appl.*, 127(1):103–121, 1987.
- [39] Kevin Zumbrun. Planar stability criteria for viscous shock waves of systems with real viscosity. In *Hyperbolic systems of balance laws*, volume 1911 of *Lecture Notes in Math.*, pages 229–326. Springer, Berlin, 2007.