

Generator functions and their applications

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Abstract

In [13], we introduced so called *generators* functions to precisely follow the regularity of analytic solutions of Navier Stokes equations. In this short note, we give a short presentation of these generator functions and use them to give existence results of analytic solutions to some classical equations, namely to hyperbolic equations, to incompressible Euler equations, and to hydrostatic Euler and Vlasov models. The use of these generator functions appear to be an alternative way to the use of the classical abstract Cauchy-Kovalevskaya theorem [1, 10, 20, 21].

1 Introduction

The general abstract Cauchy-Kovalevskaya theorem has been intensively used to construct analytic solutions to various evolution partial differential equations, including first order hyperbolic and parabolic equations, or Euler and Navier-Stokes equations. We refer for instance to Asano [1], Caffisch [10], Nirenberg [20], Safonov [21], among others. In this short note, we use generators functions as introduced in [13] to obtain existence results for these equations. The results are not new, but show the versatility and simplicity of use of these generator functions. We believe that such approaches could be used on many other equations and provide easy ways to get existence results or to investigate instabilities.

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Let us first introduce generator functions in the particular case of a periodic function $f(t, x)$ on $t \geq 0$ and $x \in \mathbb{T}^d$, $d \geq 1$. For $z \in \mathbb{R}$, we introduce the *generator function* $Gen[f]$ defined by

$$Gen[f](t, z) = \sum_{\alpha \in \mathbb{Z}^d} e^{z|\alpha|} |f_\alpha(t)| \quad (1.1)$$

in which f_α denotes the Fourier transform of $f(t, x)$ with respect to $x \in \mathbb{T}^d$. If f is analytic in x , $Gen[f]$ is only defined for small enough $|z|$, up to the analyticity radius of $f(t, \cdot)$. The results in this note also apply to the case when $x \in \mathbb{R}^d$, with which the above summation is replaced by the integral over \mathbb{R}^d . In applications, we may also introduce generator functions depending on multi-variables $z = (z_1, \dots, z_d)$ that correspond to the analyticity radius of $f(x)$ in (x_1, \dots, x_d) , respectively; see, for instance, [13] for the case of boundary layers on the half space $\mathbb{T} \times \mathbb{R}_+$.

First note that generator functions are non negative, and that all their derivatives are non negative and non decreasing in z . Moreover generator functions have very nice properties with respect to algebraic operations and differentiation. Namely they "commute" with the product, the sum and the differentiation, making their use very versatile.

Lemma 1.1. *For any f, g , there hold the following properties*

$$Gen[f + g] \leq Gen[f] + Gen[g] \quad (1.2)$$

$$Gen[fg] \leq Gen[f]Gen[g] \quad (1.3)$$

$$Gen[\nabla_x f] = \partial_z Gen[f] \quad (1.4)$$

$$\partial_t Gen[f] \leq Gen[\partial_t f] \quad (1.5)$$

for all $z \geq 0$.

Proof. Let f_α and g_α be the Fourier transform of f and g , respectively. It follows that

$$(fg)_\alpha = f_\alpha \star_\alpha g_\alpha$$

for $\alpha \in \mathbb{Z}^d$. For $z \geq 0$, we compute

$$\begin{aligned} Gen[fg](z) &= \sum_{\alpha \in \mathbb{Z}^d} e^{z|\alpha|} |f_\alpha \star_\alpha g_\alpha| \leq \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \in \mathbb{Z}^d} e^{z|\beta|} e^{z|\alpha-\beta|} |f_\beta| |g_{\alpha-\beta}| \\ &\leq Gen[f]Gen[g] \end{aligned}$$

which is the second inequality. The first and third identities follow directly from the definition. The last inequality is a direct consequence of

$$\partial_t |f_\alpha| \leq |\partial_t f_\alpha|.$$

□

The use of generator functions are not limited to polynomial operations. Namely, we have

Lemma 1.2. *Let F be an analytic function*

$$F(z) = \sum_{n \geq 0} a_n z^n$$

with some convergence radius ρ . Let us introduce

$$\tilde{F}(z) = \sum_{n \geq 0} |a_n| z^n.$$

Then, for any function f , provided $|f| < \rho$, there holds

$$\text{Gen}[F(f)] \leq \tilde{F}(\text{Gen}[f]). \quad (1.6)$$

Proof. First, using (1.3), we have

$$\text{Gen}[a_n f^n] \leq |a_n| \text{Gen}[f]^n.$$

The lemma follows from multiplying the inequality by z^n and summing over n . □

Moreover generator functions easily combine with truncation. Let P_N be the projection of the first N Fourier modes; namely

$$P_N(f)(x) = \sum_{\alpha \in \mathbb{Z}^d, |\alpha| \leq N} e^{i\alpha x} f_\alpha. \quad (1.7)$$

Then it follows that

$$\text{Gen}[P_N(f)] \leq \text{Gen}[f]. \quad (1.8)$$

The main point is that if $f(t, x)$ satisfies a non linear partial differential equation then $\text{Gen}[f]$ satisfies a similar differential inequality. This inequality describes in an acute way how the radius of analyticity shrinks as time goes on, and allows to get analytic bounds on f , and in particular to bound all its derivatives at the same time. We now illustrate this remark in the particular case of hyperbolic equations.

2 Hyperbolic equations

We consider the following system of evolution equations

$$\partial_t u = A(u, \nabla_x u) \quad (2.1)$$

for a vector valued function $u = u(x, t)$, with $x \in \mathbb{T}^d$, $d \geq 1$, and $t \geq 0$. We assume that the function $A(\cdot, \cdot)$ satisfies

$$\text{Gen}[A(u, \nabla_x u)] \leq C_0 F(\text{Gen}[u]) \left(1 + \partial_z \text{Gen}[u]\right) \quad (2.2)$$

for some constant C_0 and some analytic functions F . For instance, A may be a quadratic polynomial in u and ∇u , in which case $F = Id$, or more generally, A may be of the form $G(u) \cdot \nabla_x u$, for which $F = \tilde{G}$ (see Lemma 1.2).

We shall construct solutions in the function space X_ρ defined by

$$X_\rho := \left\{ u : \text{Gen}[u](\rho) < \infty \right\}. \quad (2.3)$$

Theorem 2.1. *Let $\rho > 0$ and u_0 be in X_ρ . Then the Cauchy problem (2.4) with initial data u_0 has a unique solution $u(t)$ in $X_{\rho(t)}$ for positive times t as long as $\rho(t) = \rho - C_1 t$ remains positive, C_1 being some large positive constant depending on u_0 .*

Proof. We shall construct solutions via the standard approximation. First, recall the projection P_N defined as in (1.7). Let $u_N(t)$ be a solution to the following regularized equations

$$\partial_t u_N = P_N A(P_N u_N, \nabla_x P_N u_N) \quad (2.4)$$

with initial data $u_N(0, x) = P_N u_0(x)$ for all N . As the right hand side consists of only a finite number of Fourier modes, this equation is in fact an ordinary differential equation. Hence there exists a unique solution $u_N(t)$, defined for t small enough. It suffices to prove that $u_N(t)$ is a Cauchy sequence in $X_{\rho(t)}$, as long as $\rho(t)$ remains positive.

The Fourier coefficients of the solution to (2.4) satisfy

$$\partial_t u_{N,\alpha} = P_{N,\alpha} A(P_N u_N, \nabla_x P_N u_N)$$

where $P_{N,\alpha} f$ denotes the Fourier coefficient f_α if $|\alpha| \leq N$, and zero, if otherwise. We therefore get

$$\begin{aligned} \sum_{|\alpha| \leq N} \partial_t |u_{N,\alpha}| e^{z|\alpha|} &\leq \sum_{|\alpha| \leq N} |\partial_t u_{N,\alpha}| e^{z|\alpha|} \\ &\leq \sum_{|\alpha| \leq N} |P_{N,\alpha} A(P_N u_N, \nabla_x P_N u_N)| e^{z|\alpha|}. \end{aligned}$$

Using (1.8) and the assumption (2.2), we obtain the following Hopf-type differential inequality

$$\partial_t \text{Gen}[u_N] \leq C_0 F(\text{Gen}[u_N]) \left(1 + \partial_z \text{Gen}[u_N]\right), \quad (2.5)$$

in which C_0 is independent of N . For convenience, we set

$$G_N(t, z) = \text{Gen}[u_N(t)](z).$$

The previous inequality yields

$$\partial_t G_N \leq C_0 F(G_N)(1 + \partial_z G_N), \quad (2.6)$$

which is a differential inequality that we will now exploit in order to bound u_N and all its derivatives. Note that G_N is a finite sum and is therefore defined for all z . As N goes to $+\infty$, $G_N(0, z)$ converges to $\text{Gen}[u_0](z)$, which is defined for $|z| \leq \rho$.

As usual with analytic solutions, the domain of analyticity shrinks with time, hence we introduce

$$F_N(t, z) = G_N(t, \theta(t)z)$$

for $t, z \geq 0$, where $\theta(\cdot)$ will be determined later. It follows that F_N satisfies

$$\partial_t F_N \leq C_0 F(F_N) + C_0 (F(F_N) + \theta'(t)z) \partial_z F_N. \quad (2.7)$$

Note that F_N is defined for any z . In the limit $N \rightarrow +\infty$ we focus on $0 \leq z \leq \rho$.

We will choose $\theta(t)$ is such a way that the characteristics of (2.7) are outgoing on $[0, \rho]$, namely such that at $z = 0$, $F(F_N) > 0$ (which is always satisfied) and such that at $z = \rho$, $F(F_N) + \theta'(t)\rho < 0$.

At $t = 0$, we choose $\theta(0) = 1$ and thus

$$F_N(0, z) = G_N(0, z) = \text{Gen}[P_N u_0](z) \leq \text{Gen}[u_0](z),$$

which is well-defined on $[0, \rho]$. Set

$$M_0 = \sup_{0 \leq z \leq \rho} \text{Gen}[u_0](z).$$

We will focus on times $0 \leq t \leq T_N$ such that $F_N(t, z) \leq 2M_0$ for $0 \leq t \leq T_N$ and $0 \leq z \leq \rho$. We choose

$$\theta(t) = 1 - F(3M_0)\rho^{-1}t.$$

Observe that on $0 \leq t \leq T_N$, as long as $\theta(t) > 0$, $F(F_N) + \theta'(t)\rho < 0$. On such a time interval, (2.7) is a nonlinear transport equation with a source term and with outgoing characteristics at 0 and ρ . As a consequence, we have

$$\partial_t \sup_{0 \leq z \leq 1} F_N(t, z) \leq C_0 \sup_{0 \leq z \leq 1} F(F_N(t, z)) \leq C_0 F(2M_0),$$

hence

$$F_N(t, z) \leq M_0 + C_0 t F(2M_0).$$

Classical arguments then leads to the fact that T_N is bounded away from 0, namely there exists some $T > 0$ such that $T_N \geq T$ for any N , and such that $\rho(t) > 0$ for any $t \leq T$. This implies that

$$F_N(t, z) \leq C(F_N(0, z)) \leq C(M_0)$$

for any $0 \leq z \leq \rho$ and any $0 \leq t \leq T$. Thus $u_N(t)$ is uniformly bounded in $X_{\rho(t)}$ for all N . As a consequence, u_N and all its derivatives of all orders are uniformly bounded in L^∞ . Up to the extraction of a subsequence, u_N and all its derivatives converge uniformly, towards some function u , solution of (2.4). Moreover, classical arguments show that $u(t) \in X_{\rho(t)}$ for any $0 \leq t \leq T$, which ends the proof of the theorem. \square

3 Euler equations

In this section, we apply the previous framework to construct analytic solutions to incompressible Euler equations on \mathbb{T}^d or \mathbb{R}^d , $d \geq 2$. Namely, we consider

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned} \tag{3.1}$$

on \mathbb{T}^d (the case \mathbb{R}^d is treated similarly). Again, the existence result is classical (see, for instance, [4, 5]). The function space X_ρ is defined in (2.3). We have

Theorem 3.1. *Let $\rho > 0$ and u_0 be a divergence-free vector field in X_ρ . Then the Cauchy problem (3.1) with initial data u_0 has a solution $u(t)$ in $X_{\rho(t)}$ for positive times t as long as $\rho(t) = \rho - C_1 t$ remains positive, C_1 being a constant depending on u_0 .*

Proof. Introduce the Leray projector \mathbb{P} , projection onto the divergence-free L^2 vector space. In Fourier coefficients, \mathbb{P}_α is an $d \times d$ matrix with entries

$$(\mathbb{P}_\alpha)_{jk} = \delta_{jk} - \frac{\alpha_j \alpha_k}{|\alpha|^2}.$$

In particular, $|\mathbb{P}_\alpha|$ is bounded, uniformly in $\alpha \in \mathbb{Z}^d$. Taking the Leray projection of (3.1), we obtain

$$\partial_t u = -\mathbb{P}(u \cdot \nabla u), \quad (3.2)$$

which falls into the previous abstract framework. It remains to check the assumption (2.2). Indeed, using (1.2) and the uniform boundedness of $|\mathbb{P}_\alpha|$ in α , we compute

$$\text{Gen}[\mathbb{P}(u \cdot \nabla u)] \leq \text{Gen}[u \cdot \nabla u] \leq \text{Gen}[u] \text{Gen}[\nabla u] \leq \text{Gen}[u] \partial_z \text{Gen}[u].$$

Theorem 3.1 follows from Theorem 2.1, upon noting that the divergence-free condition is invariant under (3.2). \square

4 Hydrostatic Euler equation and Vlasov models

In this section, we apply the abstract framework to two more examples that arise as a singular limit of Euler equations and Vlasov-Poisson systems. The first is the hydrostatic Euler equations which read

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad (4.1)$$

on $\mathbb{T} \times [-1, 1]$ for vorticity function

$$\omega = \partial_y^2 \varphi \quad (4.2)$$

and velocity field $u = \nabla^\perp \varphi$, with φ being the stream function, satisfying the boundary condition

$$\varphi|_{y=\pm 1} = 0. \quad (4.3)$$

The model (4.1)-(4.3) can be derived from the two-dimensional incompressible Euler equations in a narrow channel [19, 12, 9]. The Cauchy problem for data with Sobolev regularity is illposed [15], except for convex profiles [8], due to the loss of derivatives in x through $u_2 = -\partial_x \varphi = -\partial_x \partial_y^{-2} \omega$. However, the local solution can be constructed for data with analytic regularity [18].

We now show that the previous abstract framework also applies to the hydrostatic Euler model (4.1)-(4.3). For $z \geq 0$, we introduce the notion of generator functions as follows:

$$\text{Gen}[\omega](z) := \sum_{\alpha \in \mathbb{Z}} \sum_{\beta \geq 0} e^{z|\alpha|} \|\partial_y^\beta \omega_\alpha\|_{L^\infty[-1,1]} \frac{z^\beta}{\beta!}, \quad (4.4)$$

in which $\omega_\alpha(y)$ denotes the Fourier transform of $\omega(x, y)$ with respect to x . The analyticity radius for x and y needs not to be the same [13], however for simplicity, we have taken the generator of the form (4.4). We obtain the following.

Theorem 4.1. *Let $\rho > 0$ and ω_0 be in X_ρ . Then, the Cauchy problem (4.1)-(4.3) with initial data ω_0 has a solution $\omega(t)$ in $X_{\rho(t)}$ for positive times t as long as $\rho(t) = \rho - 3C_1t$ remains positive, C_1 being a constant depending on ω_0 .*

Proof. Let us first check the properties of the new generator function. For any f, g , we compute

$$\partial_y^\beta (fg)_\alpha = \sum_{\alpha' \in \mathbb{Z}} \sum_{0 \leq \beta' \leq \beta} \frac{\beta!}{\beta'!(\beta - \beta')!} \partial_y^{\beta'} f_{\alpha'} \partial_y^{\beta - \beta'} g_{\alpha - \alpha'},$$

for any $\alpha \in \mathbb{Z}$ and $\beta \geq 0$. Therefore, for $z \geq 0$, we have

$$\begin{aligned} & Gen(fg)(z) \\ &= \sum_{\alpha \in \mathbb{Z}} \sum_{\beta \geq 0} e^{z|\alpha|} \|\partial_y^\beta (fg)_\alpha\|_{L^\infty} \frac{z^\beta}{\beta!} \\ &\leq \sum_{\alpha \in \mathbb{Z}} \sum_{\beta \geq 0} \sum_{\alpha' \in \mathbb{Z}} \sum_{0 \leq \beta' \leq \beta} e^{z|\alpha|} \|\partial_y^{\beta'} f_{\alpha'}\|_{L^\infty} \|\partial_y^{\beta - \beta'} g_{\alpha - \alpha'}\|_{L^\infty} \frac{z^\beta}{\beta'!(\beta - \beta')!} \\ &\leq \sum_{\alpha, \alpha' \in \mathbb{Z}} \sum_{\beta \geq 0} \sum_{\beta \geq \beta'} e^{z|\alpha|} e^{z|\alpha - \alpha'|} \|\partial_y^{\beta'} f_{\alpha'}\|_{L^\infty} \|\partial_y^{\beta - \beta'} g_{\alpha - \alpha'}\|_{L^\infty} \frac{z^{\beta'} z^{\beta - \beta'}}{\beta'!(\beta - \beta')!} \\ &\leq Gen[f](z) Gen[g](z), \end{aligned}$$

which proves the second identity in Lemma 1.1. The other identities are straightforward.

Let us now verify assumption (2.2). In this case

$$A(\omega, \nabla \omega) = -u \cdot \nabla \omega,$$

where $u = \nabla^\perp \varphi$, with φ defined through (4.2)-(4.3). We first note that

$$Gen[u \cdot \nabla \omega] \leq Gen[u] \partial_z Gen[\omega].$$

Next, $\partial_y^2 \varphi = \omega$ (with the Dirichlet boundary conditions on $[-1, 1]$) can be explicitly solved, leading to

$$\|\varphi\|_{L^\infty} + \|\partial_y \varphi\|_{L^\infty} + \|\partial_y^2 \varphi\|_{L^\infty} \leq C_0 \|\omega\|_{L^\infty}.$$

This yields to

$$Gen[u] \leq C_0 \left[Gen[\omega] + \partial_z Gen[\omega] \right],$$

recalling the loss of derivatives in variable x through $u_2 = -\partial_x \varphi$. As a result, we have obtained

$$Gen[A(\omega, \nabla \omega)] \leq C_0 \left[Gen[\omega] + \partial_z Gen[\omega] \right] \partial_z Gen[\omega]. \quad (4.5)$$

To fit into the previous framework, we introduce the vector functions

$$U := [\omega, \nabla \omega], \quad \mathcal{A}(U, \nabla U) := [A(\omega, \nabla \omega), \nabla A(\omega, \nabla \omega)].$$

It follows that $\partial_t U = \mathcal{A}(U, \nabla U)$. The theorem follows from the previous abstract framework. \square

We conclude our paper with an application to kinetic models, known as the kinetic incompressible Euler and Vlasov-Dirac-Benney systems. These models are derived from the quasineutral limit or the vanishing Debye length limit of Vlasov-Poisson systems [7, 11, 14, 16, 2, 15]. The Vlasov equation reads

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = 0, \quad (4.6)$$

on $\mathbb{T}^d \times \mathbb{R}^d$, $d \geq 1$, where the potential function φ is defined by

$$\varphi = \int_{\mathbb{R}^d} f(t, x, v) dv - 1 \quad (4.7)$$

for the Vlasov-Dirac-Benney system [2, 14], or alternatively, defined through the elliptic equation

$$-\Delta \varphi = \nabla \cdot \left(\nabla \cdot \int f v \otimes v dv \right) \quad (4.8)$$

for the kinetic incompressible Euler model [7]. Again, the system (4.6)-(4.8) experiences a loss of derivatives in x . As a consequence, the Cauchy problem for data with Sobolev regularity is illposed [15], except for stable data [3, 16]. The local analytic solutions were constructed in [14, 17, 6].

We now show that the abstract framework also applies to the kinetic model (4.6)-(4.8). Indeed, we introduce the following modified notion of generator functions:

$$Gen[f](z) := \sum_{\alpha \in \mathbb{Z}^d} \sum_{\beta \geq 0} e^{z|\alpha|} \|\langle v \rangle^m \partial_v^\beta f_\alpha\|_{L^\infty(\mathbb{R}^d)} \frac{z^\beta}{\beta!}, \quad (4.9)$$

for some $m > d + 2$, in which $f_\alpha(v)$ denotes the Fourier transform of $f(x, v)$ with respect to x , and $\langle v \rangle = (1 + \|v\|^2)^{1/2}$. With exception of the weight in v , the generator function is identical to (4.4). Thus, Lemma 1.1 follows similarly. As a result, we have the following

Theorem 4.2. *Let $\rho > 0$ and f_0 be in X_ρ . Then, the Cauchy problem (4.6)-(4.8) with initial data f_0 has a solution $f(t)$ in $X_{\rho(t)}$ for positive times t as long as $\rho(t) = \rho - 3C_1 t$ remains positive, where C_1 is a constant depending on f_0 .*

Proof. It remains to verify assumption (2.2) for

$$A(f, \nabla_v f) = \nabla_x \varphi[f] \cdot \nabla_v f$$

for $\varphi[f]$ defined by the relation (4.7) or (4.8). In the first case, for $\alpha \neq 0$, we have

$$|\varphi_\alpha| \leq \int_{\mathbb{R}^d} |f_\alpha(t, v)| dv \leq C_0 \|\langle v \rangle^m f_\alpha\|_{L^\infty}$$

for some $m > d$. The second case is similar, with $m > d + 2$. This proves that

$$Gen[\nabla_x \varphi] \leq C_0 \partial_z Gen[f]$$

and hence

$$Gen[A(f, \nabla_v f)] \leq C_0 \partial_z Gen[f] \partial_z Gen[f].$$

Again, as in the previous case, to fit into the abstract framework, we introduce the vector functions

$$U := [f, \nabla_x f, \nabla_v f]$$

$$\mathcal{A}(U, \nabla U) := [A(f, \nabla_x f), \nabla_x A(f, \nabla_x f), \nabla_v A(f, \nabla_x f)],$$

which yields $\partial_t U = \mathcal{A}(U, \nabla U)$. The theorem follows from Theorem (2.1). \square

References

- [1] K. Asano. A note on the abstract Cauchy-Kowalewski theorem. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(4):102–105, 1988.
- [2] C. Bardos. About a Variant of the 1d Vlasov equation, dubbed “Vlasov-Dirac-Benney” Equation. *Séminaire Laurent Schwartz - EDP et applications*, 15:21 p., 2012-2013.

- [3] C. Bardos and N. Besse. The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi-classical limits. *Kinet. Relat. Models*, 6(4):893–917, 2013.
- [4] C. Bardos and S. Benachour. Domaine d’analyticité des solutions de l’équation d’Euler dans un ouvert de \mathbb{R}^n . (French) *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 4 (1977), no. 4, 647-687.
- [5] C. Bardos, S. Benachour, and M. Zerner. Analyticité des solutions périodiques de l’équation d’Euler en deux dimensions. (English summary) *C. R. Acad. Sci. Paris Sér. A-B* 282 (1976), no. 17, Aiii, A995-A998.
- [6] M. Bossy, J. Fontbona, P.-E. Jabin, and J.-F. Jabir. Local existence of analytical solutions to an incompressible Lagrangian stochastic model in a periodic domain. *Comm. Partial Differential Equations*, 38(7):1141–1182, 2013.
- [7] Y. Brenier. A Vlasov-Poisson type formulation of the Euler equations for perfect incompressible fluids. *Rapport de recherche INRIA*, 1989.
- [8] Y. Brenier. Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity*, 12(3):495–512, 1999.
- [9] Y. Brenier. Remarks on the derivation of the hydrostatic Euler equations. *Bull. Sci. Math.*, 127(7):585–595, 2003.
- [10] R. E. Caflisch. A simplified version of the abstract Cauchy-Kowalewski theorem with weak singularities. *Bull. Amer. Math. Soc. (N.S.)*, 23(2):495–500, 1990.
- [11] E. Grenier. Limite quasineutre en dimension 1. In *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1999)*, pages Exp. No. II, 8. Univ. Nantes, Nantes, 1999.
- [12] E. Grenier. On the derivation of homogeneous hydrostatic equations. *M2AN Math. Model. Numer. Anal.*, 33(5):965–970, 1999.
- [13] E. Grenier and T. T. Nguyen. L^∞ instability of Prandtl’s layers. *Annals of PDE*, 2019.
- [14] D. Han-Kwan. Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Comm. Partial Differential Equations*, 36(8):1385–1425, 2011.

- [15] D. Han-Kwan and T. T. Nguyen. Ill-posedness of the hydrostatic Euler and singular Vlasov equations. *Arch. Ration. Mech. Anal.*, 221(3):1317–1344, 2016.
- [16] D. Han-Kwan and F. Rousset. Quasineutral limit for Vlasov-Poisson with Penrose stable data. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(6):1445–1495, 2016.
- [17] P.-E. Jabin and A. Nouri. Analytic solutions to a strongly nonlinear Vlasov equation. *C. R. Math. Acad. Sci. Paris*, 349(9-10):541–546, 2011.
- [18] I. Kukavica, R. Temam, V. C. Vicol, and M. Ziane. Local existence and uniqueness for the hydrostatic Euler equations on a bounded domain. *J. Differential Equations*, 250(3):1719–1746, 2011.
- [19] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.
- [20] L. Nirenberg. An abstract form of the nonlinear Cauchy-Kowalewski theorem. *J. Differential Geometry*, 6:561–576, 1972. Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.
- [21] M. V. Safonov. The abstract Cauchy-Kovalevskaya theorem in a weighted Banach space. *Comm. Pure Appl. Math.*, 48(6):629–637, 1995.