On nonlinear instability of Prandtl's boundary layers: the case of Rayleigh's stable shear flows

Emmanuel Grenier[*] Toan T. Nguyen[†]

June 4, 2017

Abstract

In this paper, we study Prandtl's boundary layer asymptotic expansion for incompressible fluids on the half-space in the inviscid limit. In [7], E. Grenier proved that Prandtl's Ansatz is false for data with Sobolev regularity near Rayleigh's unstable shear flows. In this paper, we show that this Ansatz is also false for Rayleigh's stable shear flows. Namely we construct unstable solutions near arbitrary stable monotonic boundary layer profiles. Such shear flows are stable for Euler equations, but not for Navier-Stokes equations: adding a small viscosity destabilizes the flow.

Contents

[*]Equipe Projet Inria NUMED, INRIA Rhône Alpes, Unité de Mathématiques Pures et Appliquées., UMR 5669, CNRS et École Normale Supérieure de Lyon, 46, allée d'Italie, 69364 Lyon Cedex 07, France. Email: Emmanuel.Grenier@ens-lyon.fr

[†]Department of Mathematics, Penn State University, State College, PA 16803. Email: nguyen@math.psu.edu.

1 Introduction

In this paper, we are interested in the inviscid limit $\nu \to 0$ of the Navier-Stokes equations for incompressible fluids, possibly subject to some external forcing f^ν, namely

$$\partial_t u^\nu + (u^\nu \cdot \nabla)u^\nu + \nabla p^\nu = \nu \Delta u^\nu + f^\nu, \tag{1.1}$$

$$\nabla \cdot u^\nu = 0, \tag{1.2}$$

on the half plane $\Omega = \{(x, y) \in \mathbb{T} \times \mathbb{R}^+\}$, with the no-slip boundary condition

$$u^\nu = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

As ν goes to 0, one would expect the solutions u^ν to converge to solutions of Euler equations for incompressible fluids

$$\partial_t u^0 + (u^0 \cdot \nabla)u^0 + \nabla p^0 = f^0, \tag{1.4}$$

$$\nabla \cdot u^0 = 0, \tag{1.5}$$

with the boundary condition

$$u^0 \cdot n = 0 \quad \text{on} \quad \partial\Omega, \tag{1.6}$$

where n is the unit normal to $\partial\Omega$.

At the beginning of the twentieth century, Prandtl introduced his well known boundary layers in order to describe the transition from Navier-Stokes to Euler equations as the viscosity tends to zero. Formally, we expect that

$$u^\nu(t, x, y) \approx u^0(t, x, y) + u_P\Big(t, x, \frac{y}{\sqrt{\nu}}\Big) + \mathcal{O}(\sqrt{\nu}) \tag{1.7}$$

where u^0 solves the Euler equations (1.4)-(1.6), and u_P is the Prandtl boundary layer correction, which is of order one in term of small viscosity. The size of Prandtl's boundary layer is of order $\sqrt{\nu}$. Formally it is even possible to write an asymptotic expansion for u^ν in terms of powers of $\sqrt{\nu}$. The aim of this paper is to investigate whether (1.7) holds true. The Prandtl's boundary layer equations read

$$\begin{aligned} \partial_t u_{P,1} + u_P \cdot \nabla u_{P,1} + \partial_x p^0 = \partial_z^2 u_{P,1} + f^P, \\ \nabla \cdot u_P = 0, \end{aligned} \tag{1.8}$$

together with appropriate boundary conditions to correct the no-slip boundary conditions of Navier-Stokes solutions. In the above, $\partial_x p^0$ denotes the pressure gradient of the Euler flow on the boundary, and $u_{P,1}$ is the horizontal component of the velocity.

Prandtl boundary layers have been intensively studied in the mathematical literature. Notably, solutions to the Prandtl equations have been constructed for monotonic data [21, 22, 1, 20] or data with Gevrey or analytic regularity [23, 5, 16]. In the case of non-monotonic data with Sobolev regularity, the Prandtl equations are ill-posed [3, 6, 15].

The validity of Prandtl's Ansatz (1.7) has been established in [23, 24] for initial data with analytic regularity, leaving a remainder of order $\sqrt{\nu}$. A similar result is also obtained in [19]. The Ansatz (1.7), with a specific boundary layer profile, has been recently justified for data with Gevrey regularity [4]. When only data with Sobolev regularity are assumed, E. Grenier proved in [7] that such an asymptotic expansion is false, up to a remainder of order $\nu^{1/4}$ in L^∞ norm. The invalidity of the expansion is proved near boundary layers with an inflection point or more precisely near those that are spectrally unstable for the Rayleigh equations.

In this paper, we shall prove the nonlinear instability of the Ansatz (1.7) near boundary layer profiles which are stable for Rayleigh's equations, for instance near monotonic profiles. Roughly speaking, given an arbitrary stable boundary layer, the two main results in this paper are

- in the case of time-dependent boundary layers, we construct Navier-Stokes solutions, with arbitrarily small forcing, of order $\mathcal{O}(\nu^P)$, with P as large as we want, so that the Ansatz (1.7) is false near the boundary layer, up to a remainder of order $\nu^{1/4+\epsilon}$ in L^∞ norm, ϵ being arbitrarily small.

- in the case of stationary boundary layers, we construct Navier-Stokes solutions, without forcing term, so that the Ansatz (1.7) is false, up to a remainder of order $\nu^{5/8}$ in L^∞ norm.

These results prove that there exist no asymptotic expansion of Prandtl's type. The proof does not run as smoothly as in [7], and is based on the construction of an approximate solution. However the second term of this approximate solution grows faster than expected, and as a consequence, we are not able to construct an approximate solution which reaches to an amplitude of order one $\mathcal{O}(1)$, even allowing a non zero forcing term. This point will be detailed in a forthcoming paper.

In the next sections, we shall introduce the precise notion of Rayleigh's stable boundary layers and present our main results. After a brief recall of the linear instability results [9, 11] in Section 2, we give the proof of the main results in Sections 3 and 4, respectively.

1.1 Stable boundary layer profiles

Throughout this paper, by a boundary layer profile, we mean a shear flow of the form

$$U_{\mathrm{bl}} := \begin{pmatrix} U_{\mathrm{bl}}(t, \frac{y}{\sqrt{\nu}}) \\ 0 \end{pmatrix} \tag{1.9}$$

that solves the Prandtl's boundary layer problem (1.8), with initial data $U_{\mathrm{bl}}(0,z) = U(z)$. Without forcing, U_{bl} is the solution of heat equation

$$\partial_t U_{\mathrm{bl}} - \partial_{YY} U_{\mathrm{bl}} = 0.$$

Boundary layer profiles can also be generated by adding a forcing term f^P, in which case we shall focus precisely on the corresponding stationary boundary layers $U_{\mathrm{bl}} = U(z)$, with $-U''(z) = f^P$. We will consider these two different cases, namely time dependent boundary layers (without forcing) and time independent boundary layers (with given, time independent, forcing).

As mentioned, the Ansatz (1.7) is proven to be false for initial boundary layer profiles $U(z)$ that are spectrally unstable to the Euler equations [7]. *In this paper, we shall thus focus on stable profiles,* those that are spectrally stable to the Euler equations. This includes, for instance, boundary layer profiles without an inflection point by view of the classical Rayleigh's inflection point theorem. In this paper we assume in addition that $U(z)$ is strictly monotonic, real analytic, that $U(0) = 0$ and that $U(z)$ converges exponentially fast at infinity to a finite constant U_+. By a slight abuse of language, such profiles will be referred to as stable profiles in this paper.

In order to study the instability of such boundary layers, we first analyze the spectrum of the corresponding linearized problem around initial profiles $U(z)$. We introduce the isotropic boundary layer variables

$(t, x, z) = (t, x, y)/\sqrt{\nu}$, and will use the Fourier transform in the x variable only, denoting by α the corresponding wavenumber. The linearization of Navier-Stokes equations, written in the vorticity formulation for each wave number α, reads

$$(\partial_t - L_\alpha)\omega_\alpha = 0, \qquad L_\alpha \omega_\alpha := \sqrt{\nu}\Delta_\alpha \omega_\alpha - i\alpha U \omega_\alpha + i\alpha \phi_\alpha U'', \tag{1.10}$$

with vorticity

$$\omega_\alpha = \Delta_\alpha \phi_\alpha,$$

together with the zero boundary conditions $\phi_\alpha = \phi'_\alpha = 0$ on $z = 0$. Here,

$$\Delta_\alpha = \partial_z^2 - \alpha^2.$$

Together with Y. Guo, we proved in [8, 9] that, even for profiles U which are stable as $\nu = 0$, there are unstable eigenvalues to the Navier-Stokes problem (1.10) for sufficiently small viscosity ν and for a range of wavenumber $\alpha \in [\alpha_1, \alpha_2]$, with $\alpha_1 \sim \nu^{1/8}$ and $\alpha_2 \sim \nu^{1/12}$. The unstable eigenvalues λ_* of L_α, found in [9], satisfy

$$\Re \lambda_* \sim \nu^{1/4}. \tag{1.11}$$

Such an instability was first observed by Heisenberg [13, 14], then Tollmien and C. C. Lin [17, 18]; see also Drazin and Reid [2, 25] for a complete account of the physical literature on the subject. See also Theorem 2.1 below for precise details. In coherence with the physical literature [2], we believe that, α being fixed, this eigenvalue is the most unstable one. However, this point is an open question from the mathematical point of view.

Next, we observe that L_α is a compact perturbation of the Laplacian $\sqrt{\nu}\Delta_\alpha$, and hence its unstable spectrum in the usual L^2 space is discrete. Thus, for each α, ν, we can define the maximal unstable eigenvalue $\lambda_{\alpha,\nu}$ so that $\Re\lambda_{\alpha,\nu}$ is maximum. We set $\lambda_{\alpha,\nu} = 0$, if no unstable eigenvalues exist.

In this paper, we assume that the unstable eigenvalues found in the spectral instability result, Theorem 2.1, are maximal eigenvalues. Precisely, we introduce

$$\gamma_0 := \lim_{\nu \to 0} \sup_{\alpha \in \mathbb{R}} \nu^{-1/4} \Re \lambda_{\alpha,\nu}. \tag{1.12}$$

The existence of unstable eigenvalues in Theorem 2.1 implies that γ_0 is positive. Our spectral assumption is that γ_0 is finite (that is, the eigenvalues in Theorem 2.1 are maximal).

1.2 Main results

We are ready to state two main results of this paper.

1.2.1 Approximate solutions

Theorem 1.1. *Let $U_{\rm bl}(t,z)$ be a time-dependent stable boundary layer profile as described in Section 1.1. Then, for arbitrarily large s, N and arbitrarily small positive ϵ, there exists a sequence of functions u^ν that solves the Navier-Stokes equations* (1.1)-(1.3)*, with some forcing f^ν, so that*

$$\|u^\nu(0) - U_{\rm bl}(0)\|_{H^s} + \sup_{t\in[0,T^\nu]} \|f^\nu(t)\|_{H^s} \le \nu^N,$$

but

$$\|u^\nu(T^\nu) - U_{\rm bl}(T^\nu)\|_{L^\infty} \ge \nu^{\frac14+\epsilon},$$

$$\|\omega^\nu(T^\nu) - \omega_{\rm bl}(T^\nu)\|_{L^\infty} \to \infty,$$

for time sequences $T^\nu \to 0$, as $\nu \to 0$. Here, $\omega^\nu = \nabla \times u^\nu$ denotes the vorticity of fluids.

This Theorem proves that the Ansatz (1.7) is false, even near stable boundary layers, for data with Sobolev regularity. As the maximal unstable mode grows slowly in time of order $e^{\nu^{1/4}t}$ (in the scaled variable), the instability occurs in a very large time of order $\nu^{-1/4}\log\nu^{-1}$. Hence, an integration over this long period of times causes a loss of order $\nu^{-1/4}$, which prevents us to reach instability of order one in the above theorem.

1.2.2 Nonlinear instability

Theorem 1.2 (Instability result for stable profiles)**.** *Let $U_{\rm bl} = U(z)$ be a stable stationary boundary layer profile as described in Section 1.1. Then, for any s, N arbitrarily large, there exists a sequence of solutions u^ν to the Navier-Stokes equations, with forcing $f^\nu = f^P$ (boundary layer forcing), so that u^ν satisfy*

$$\|u^\nu(0) - U_{\rm bl}\|_{H^s} \le \nu^N,$$

but

$$\|u^\nu(T^\nu) - U_{\rm bl}\|_{L^\infty} \gtrsim \nu^{5/8},$$

$$\|\omega^\nu(T^\nu) - \omega_{\rm bl}\|_{L^\infty} \gtrsim 1,$$

for some time sequences $T^\nu \to 0$, as $\nu \to 0$.

The spectral instability for stable profiles gives rise to sublayers (or critical layers) whose thickness is of order $\nu^{5/8}$. The velocity gradient in this sublayer grows like $\nu^{-5/8}e^{t/\nu^{1/4}}$, and becomes larger when t is of order T^ν. As a consequence, they may in turn become unstable after the instability time T^ν obtained in the above theorem. Thus, in order to improve the $\nu^{5/8}$ instability, one needs to further examine the stability property of the sublayer itself (see [12]).

1.3 Boundary layer norms

We end the introduction by introducing the boundary layer norms to be used throughout the paper. These norms were introduced in [10] to capture the large, but localized, behavior of vorticity near the boundary. Precisely, for each vorticity function $\omega_\alpha = \omega_\alpha(z)$, we introduce the following boundary layer norms

$$\|\omega_\alpha\|_{\beta,\gamma,p} := \sup_{z\ge 0}\Big[\Big(1+\sum_{q=1}^{p}\delta^{-q}\phi_{P-1+q}(\delta^{-1}z)\Big)^{-1}e^{\beta z}|\omega_\alpha(z)|\Big], \qquad (1.13)$$

where P is a large, fixed integer, $p \ge 0$, $\beta > 0$,

$$\phi_p(z) = \frac{1}{1+z^p},$$

and with the boundary layer thickness

$$\delta = \gamma\nu^{1/8}$$

for some $\gamma > 0$. In the case when $p = 0$, $\|\omega_\alpha\|_{\beta,\gamma,p}$ reduces to the usual exponentially weighted L^∞ norm $\|\omega_\alpha\|_{L^\infty_\beta}$. We introduce the boundary layer space $\mathcal{B}^{\beta,\gamma,p}$ to consist of functions whose $\|\cdot\|_{\beta,\gamma,p}$ norm is finite, and write $L^\infty_\beta = \mathcal{B}^{\beta,\gamma,0}$. Clearly,

$$L^\infty_\beta \subset \mathcal{B}^{\beta,\gamma,q} \subset \mathcal{B}^{\beta,\gamma,p}$$

for $0 \le q \le p$. In addition, it is straightforward to check that

$$\|fg\|_{\beta,\gamma,p+q} \le \|f\|_{\beta,\gamma,p}\|g\|_{\beta,\gamma,q}, \qquad (1.14)$$

for all $p, q \ge 0$. Finally, for functions $\omega(x, z)$, we introduce

$$\|\omega\|_{\sigma,\beta,\gamma,p} := \sup_{\alpha\in\mathbb{R}}\|\omega_\alpha\|_{\beta,\gamma,p},$$

in which ω_α is the Fourier transform of ω in the tangential variable x.

2 Linear instability

In this section, we shall recall the spectral instability of stable boundary layer profiles [9] and the semigroup estimates on the corresponding linearized Navier-Stokes equation [10, 11]. Precisely, we consider the linearized problem for vorticity $\omega = \partial_z v_1 - \partial_x v_2$, which reads

$$(\partial_t - L)\omega = 0, \qquad L\omega := \sqrt{\nu}\Delta\omega - U\partial_x\omega - v_2 U'', \tag{2.1}$$

together with $v = \nabla^\perp \phi$ and $\Delta\phi = \omega$, satisfying the no-slip boundary conditions $\phi = \partial_z\phi = 0$ on $\{z = 0\}$. The linearized problem (2.1) will be studied in the Fourier space with respect to x variable; namely, for each horizontal wavenumber α, we study the following problem

$$(\partial_t - L_\alpha)\omega = 0, \qquad L_\alpha\omega := \sqrt{\nu}\Delta_\alpha\omega - i\alpha U\omega - i\alpha\phi U'', \tag{2.2}$$

together with $\Delta_\alpha\phi = \omega$, satisfying the no-slip boundary conditions $\phi = \partial_z\phi = 0$ on $\{z = 0\}$.

2.1 Spectral instability

The following theorem, proved in [9], provides an unstable eigenvalue of L for generic shear flows.

Theorem 2.1 (Spectral instability; [9]). *Let $U(z)$ be an arbitrary shear profile with $U(0) = 0$ and $U'(0) > 0$ and satisfy*

$$\sup_{z\ge 0} |\partial_z^k (U(z) - U_+)e^{\eta_0 z}| < +\infty, \qquad k = 0, \cdots, 4,$$

for some constants U_+ and $\eta_0 > 0$. Let $R = \nu^{-1/2}$ be the Reynolds number, and set $\alpha_{\mathrm{low}}(R) \sim R^{-1/4}$ and $\alpha_{\mathrm{up}}(R) \sim R^{-1/6}$ be the lower and upper stability branches.

Then, there is a critical Reynolds number R_c so that for all $R \ge R_c$ and all $\alpha \in (\alpha_{\mathrm{low}}(R), \alpha_{\mathrm{up}}(R))$, there exist a nontrivial triple $c(R), \hat v(z; R), \hat p(z; R)$, with $\mathrm{Im}\, c(R) > 0$, such that $v_R := e^{i\alpha(x-ct)}\hat v(z; R)$ and $p_R := e^{i\alpha(x-ct)}\hat p(z; R)$ solve the linearized Navier-Stokes problem (2.1). Moreover there holds the following estimate for the growth rate of the unstable solutions:

$$\alpha \Im c(R) \quad \approx \quad R^{-1/2}$$

as $R \to \infty$.

The proof of the previous Theorem, which can be found in [9] gives a detailed description of the unstable mode. The vorticity is of the form

$$\omega_0 = e^{\lambda_\nu t}\Delta(e^{i\alpha_\nu x}\phi_0(z)) \quad + \quad \text{complex conjugate} \tag{2.3}$$

The stream function ϕ_0 is constructed through asymptotic expansions, and is of the form

$$\phi_0 := \phi_{in,0}(z) + \delta_{\rm bl}\phi_{bl,0}(\delta_{\rm bl}^{-1}z) + \delta_{cr}\phi_{cr,0}(\delta_{cr}^{-1}\eta(z)),$$

for some boundary layer function $\phi_{bl,0}$ and some critical layer function $\phi_{cr,0}$. The critical layer depends on the so called Langer's variable $\eta(z)$. Let us focus on the lower branch of instability. In this case the critical layer and the boundary layer merge in one single layer. Namely we have

$$\alpha_\nu \approx R^{-1/4} = \nu^{1/8}, \qquad \Re\lambda_\nu \approx R^{-1/2} = \nu^{1/4},$$

and the critical layer thickness is of order

$$\delta_{cr} = (\alpha_\nu R)^{-1/3} \approx \nu^{1/8}.$$

As $\lambda_\nu = i\alpha_\nu c_\nu \approx R^{-1/2}$, the boundary sublayer thickness is of order

$$\delta_{\rm bl} = \Big(\frac{1}{\alpha_\nu(U_0 - c_\nu)R}\Big)^{1/2} \approx R^{-1/4} = \nu^{1/8}.$$

Next, by construction, we recall that derivatives of $\phi_{\rm bl,0}$ satisfy

$$|\partial_z^k\phi_{\rm bl,0}(\delta_{\rm bl}^{-1}z)| \le C_k\delta_{\rm bl}^{-k}e^{-\eta_0 z/\delta_{\rm bl}}.$$

Meanwhile, due to the structure of the Airy functions, there holds

$$|\partial_z^k\phi_{cr,0}(\delta_{cr}^{-1}\eta(z))| \le C_k\delta_{cr}^{-k}\langle Z\rangle^{k/2}e^{-\eta_0|Z|^{3/2}} \le C_k\delta_{cr}^{-k}e^{-\eta_1|Z|^{3/2}}.$$

Here, the Langer's variable satisfies $Z \approx \delta_{cr}^{-1}z^{2/3}$ for large z. Thus, since $\delta_{cr} \ll 1$, the exponential term is bounded by

$$e^{-\eta_1|Z|^{3/2}} \le Ce^{-\eta_1 z/\delta_{cr}^{3/2}} \le Ce^{-\eta_2 z/\delta_{cr}}.$$

This proves that the k^{th} derivatives of both critical layers and boundary sublayers are both bounded by $\nu^{-k/8}$ multiplied by an exponentially localized term whose thickness is of order $\nu^{1/8}$.

To summarize, the maximal growing mode is of the form

$$\phi_0 = \phi_{in,0}(z) + \nu^{1/8}\phi_{\rm bl,0}(\nu^{-1/8}z)$$

in which $\phi_{\rm bl,0}(\cdot)$ decays rapidly fast in its argument. In addition, it is clear that each x-derivative of ω_0 gains a small factor of $\alpha_\nu \approx \nu^{1/8}$. We therefore have an accurate description of the linear unstable mode.

2.2 Linear estimates

The corresponding semigroup e^{Lt} of the linear problem (2.1) is constructed through the path integral

$$e^{Lt}\omega = \int_{\mathbb{R}} e^{i\alpha x} e^{L_\alpha t}\omega_\alpha \, d\alpha \tag{2.4}$$

in which ω_α is the Fourier transform of ω in tangential variables and L_α, defined as in (2.2), is the Fourier transform of L. One of the main results proved in [11] is the following theorem.

Theorem 2.2. *[11] Let $\omega_\alpha \in \mathcal{B}^{\beta,\gamma,1}$ for some positive β, γ_0 be defined as in* (1.12)*, and $C_{\nu,\alpha}$ be the constant defined by*

$$C_{\nu,\alpha} := 1 + \alpha^2 \nu^{-1/4} \chi_{\{\alpha \ll 1\}} \tag{2.5}$$

for $\chi_{\{\cdot\}}$ being the characteristic function. Assume that γ_0 is finite. Then, for any $\gamma_1 > \gamma_0$, there is some positive constant C_γ so that

$$\|e^{L_\alpha t}\omega_\alpha\|_{\beta,\gamma,1} \le C_\gamma C_{\nu,\alpha} e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4}\alpha^2\sqrt{\nu}t} \|\omega_\alpha\|_{\beta,\gamma,1},$$
$$\|\partial_z e^{L_\alpha t}\omega_\alpha\|_{\beta,\gamma,1} \le C_\gamma C_{\nu,\alpha} \Big(\nu^{-1/8} + (\sqrt{\nu}t)^{-1/2}\Big) e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4}\alpha^2\sqrt{\nu}t} \|\omega_\alpha\|_{\beta,\gamma,1}.$$

In particular, when $\alpha \lesssim \nu^{1/8}$, the constant $C_{\nu,\alpha}$ in the above estimate is uniformly bounded in α, ν. This remark leads to the choice of $\alpha \sim \nu^{1/8}$, namely to focus on the lower branch of instability.

3 Approximate solutions

Let us now construct an approximate solution u_{app}, which solves Navier-Stokes equations, up to very small error terms. First, we introduce the rescaled isotropic space time variables

$$\tilde{t} = \frac{t}{\sqrt{\nu}}, \quad \tilde{x} = \frac{x}{\sqrt{\nu}}, \quad \tilde{z} = \frac{z}{\sqrt{\nu}}.$$

Without any confusion, we drop the tildes. The Navier-Stokes equations in these scaled variables read

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla p &= \sqrt{\nu}\Delta u, \\ \nabla\cdot u &= 0, \end{aligned} \tag{3.1}$$

with the no-slip boundary conditions on $z = 0$. Theorem 1.1 follows at once from the following theorem.

Theorem 3.1. *Let $U(z)$ be a stable boundary layer profile, and let $U_{\mathrm{bl}}(\sqrt{\nu}t, z)$ be the corresponding Prandtl's boundary layer. Then, there exist an approximate solution $\widetilde{u}_{\mathrm{app}}$ that approximately solves (3.1) in the following sense: for arbitrarily large numbers p, M and for any $\epsilon > 0$, the functions $\widetilde{u}_{\mathrm{app}}$ solve*

$$\begin{aligned} \partial_t \widetilde{u}_{\mathrm{app}} + (\widetilde{u}_{\mathrm{app}} \cdot \nabla)\widetilde{u}_{\mathrm{app}} + \nabla \widetilde{p}_{\mathrm{app}} &= \sqrt{\nu}\Delta \widetilde{u}_{\mathrm{app}} + \mathcal{E}_{\mathrm{app}}, \\ \nabla \cdot \widetilde{u}_{\mathrm{app}} &= 0, \end{aligned} \tag{3.2}$$

for some remainder $\mathcal{E}_{\mathrm{app}}$ and for time $t \le T_\nu$, with T_ν being defined through

$$\nu^p e^{\Re\lambda_0 T_\nu} = \nu^{\frac{1}{4}+\epsilon}.$$

In addition, for all $t \in [0, T_\nu]$, there hold

$$\begin{aligned} \|\mathrm{curl}(\widetilde{u}_{\mathrm{app}} - U_{\mathrm{bl}}(\sqrt{\nu}t, z))\|_{\beta,\gamma,1} &\lesssim \nu^{\frac{1}{4}+\epsilon}, \\ \|\mathrm{curl}\mathcal{E}_{\mathrm{app}}(t)\|_{\beta,\gamma,1} &\lesssim \nu^M. \end{aligned}$$

Furthermore, there are positive constants $\theta_0, \theta_1, \theta_2$ independent of ν so that there holds

$$\theta_1 \nu^p e^{\Re\lambda_0 t} \le \|(\widetilde{u}_{\mathrm{app}} - U_{\mathrm{bl}})(t)\|_{L^\infty} \le \theta_2 \nu^p e^{\Re\lambda_0 t}$$

for all $t \in [0, T_\nu]$. In particular,

$$\|(\widetilde{u}_{\mathrm{app}} - U_{\mathrm{bl}})(T_\nu)\|_{L^\infty} \gtrsim \nu^{\frac{1}{4}+\epsilon}.$$

3.1 Formal construction

The construction is classical, following [7]. Indeed, the approximate solutions are constructed in the following form

$$\widetilde{u}_{\mathrm{app}}(t,x,z) = U_{\mathrm{bl}}(\sqrt{\nu}t, z) + \nu^p \sum_{j=0}^{M} \nu^{j/8} u_j(t,x,z). \tag{3.3}$$

For convenience, let us set $v = u - U_{\mathrm{bl}}$, where u denotes the genuine solution to the Navier-Stokes equations (3.1). Then, the vorticity $\omega = \nabla \times v$ solves

$$\partial_t \omega + (U_{\mathrm{bl}}(\sqrt{\nu}t, y) + v) \cdot \nabla \omega + v_2 \partial_y^2 U_s(\sqrt{\nu}t, y) - \sqrt{\nu}\Delta\omega = 0$$

in which $v = \nabla^\perp \Delta^{-1}\omega$ and v_2 denotes the vertical component of velocity. Here and in what follows, Δ^{-1} is computed with the zero Dirichlet boundary condition. As $U_{\rm bl}$ depends slowly on time, we can rewrite the vorticity equation as follows:

$$(\partial_t - L)\omega + \nu^{1/8} S\omega + Q(\omega,\omega) = 0. \tag{3.4}$$

In (3.4), L denotes the linearized Navier-Stokes operator around the stationary boundary layer $U = U_s(0,z)$:

$$L\omega := \sqrt{\nu}\Delta\omega - U\partial_x\omega - u_2 U'',$$

$Q(\omega,\tilde\omega)$ denotes the quadratic nonlinear term $u\cdot\nabla\tilde\omega$, with $v = \nabla^\perp\Delta^{-1}\omega$, and S denotes the perturbed operator defined by

$$S\omega : = \nu^{-1/8}[U_s(\sqrt{\nu}t,z) - U(z)]\partial_x\omega + \nu^{-1/8}u_2[\partial_y^2 U_s(\sqrt{\nu}t,z) - U''(z)].$$

Recalling that U_s solves the heat equation with initial data $U(z)$, we have

$$|U_s(\sqrt{\nu}t,z) - U(z)| \le C\|U''\|_{L^\infty}\sqrt{\nu}t$$

and

$$|\partial_z^2 U_s(\sqrt{\nu}t,z) - U''(z)| \le C\|U''\|_{W^{2,\infty}}\sqrt{\nu}t.$$

Hence,

$$S\omega = \nu^{-1/8}\mathcal{O}(\sqrt{\nu}t)\Big[|\partial_x\omega| + |\partial_x\Delta^{-1}\omega|\Big] \tag{3.5}$$

in which $\Delta^{-1}\omega$ satisfies the zero boundary condition on $z=0$. The approximate solutions are then constructed via the asymptotic expansion:

$$\omega_{\rm app} = \nu^p\sum_{j=0}^M \nu^{j/8}\omega_j, \tag{3.6}$$

in which p is an arbitrarily large integer. Plugging this Ansatz into (3.4) and matching order in ν, we are led to solve

- for $j=0$:

  $$(\partial_t - L)\omega_0 = 0$$

  with zero boundary conditions on $v_0 = \nabla^\perp(\Delta)^{-1}\omega_0$ on $z=0$;

- for $0 < j \le M$:

  $$(\partial_t - L)\omega_j = R_j, \qquad \omega_{j|_{t=0}} = 0, \tag{3.7}$$

  with zero boundary condition on $v_j = \nabla^\perp(\Delta)^{-1}\omega_j$ on $z=0$. Here, the remainders R_j are defined by

  $$R_j = S\omega_{j-1} + \sum_{k+\ell+8p=j} Q(\omega_k,\omega_\ell).$$

As a consequence, the approximate vorticity $\omega_{\rm app}$ solves (3.4), leaving the error $R_{\rm app}$ defined by

$$R_{\rm app} = \nu^{p+\frac{M+1}{8}} S\omega_M + \sum_{k+\ell>M+1-8p; 1\le k,\ell\le M} \nu^{2p+\frac{k+\ell}{8}} Q(\omega_k,\omega_\ell) \tag{3.8}$$

which formally is of order $\nu^{p+\frac{M+1}{8}}$, for arbitrary p and M.

3.2 Estimates

We start the construction with ω_0 being the maximal growing mode, constructed in Section 2.1. We recall

$$\omega_0 = e^{\lambda_\nu t} e^{i\alpha_\nu x} \Delta_{\alpha_\nu}\Big(\phi_{in,0}(z) + \nu^{1/8}\phi_{\rm bl,0}(\nu^{-1/8}z)\Big) \quad + \quad \text{c.c.} \tag{3.9}$$

with $\alpha_\nu \sim \nu^{1/8}$ and $\Re\lambda_\nu \sim \nu^{1/4}$. In what follows, α_ν and λ_ν are fixed. We obtain the following lemma.

Lemma 3.2. *Let ω_0 be the maximal growing mode* (3.9)*, and let ω_j be inductively constructed by* (3.7)*. Then, there hold the following uniform bounds:*

$$\|\partial_x^a \partial_z^b \omega_j\|_{\sigma,\beta,\gamma,1} \le C_0 \nu^{a/8} \nu^{-b/8} \nu^{-\frac14[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}t} \tag{3.10}$$

for all $a, j \ge 0$ and for $b = 0,1$. In addition, the approximate solution $\omega_{\rm app}$ defined as in (3.6) *satisfies*

$$\|\partial_x^a \partial_z^b \omega_{\rm app}\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{a/8}\nu^{-b/8} \sum_{j=0}^{M} \nu^{-\frac14[\frac{j}{8p}]} \Big(\nu^p e^{\gamma_0 \nu^{1/4} t}\Big)^{1+\frac{j}{8p}}, \tag{3.11}$$

for $a \ge 0$ and $b = 0,1$. Here, $[k]$ denotes the largest integer so that $[k] \le k$.

Proof. For $j \ge 1$, we construct ω_j having the form

$$\omega_j = \sum_{n\in\mathbb{Z}} e^{in\alpha_\nu x}\omega_{j,n}$$

It follows that $\omega_{j,n}$ solves

$$(\partial_t - L_{\alpha_n})\omega_{j,n} = R_{j,n}, \qquad \omega_{j,n}{}_{|_{t=0}} = 0$$

with $\alpha_n = n\alpha_\nu$ and $R_{j,n}$ the Fourier transform of R_j evaluated at the Fourier frequency α_n. Precisely, we have

$$R_{j,n} = S_{\alpha_n}\omega_{j-1,n} + \sum_{k+\ell+8p=j} \sum_{n_1+n_2=n} Q_{\alpha_n}(\omega_{k,n_1}, \omega_{\ell,n_2}),$$

in which S_{α_n} and Q_{α_n} denote the corresponding operator S and Q in the Fourier space. The Duhamel's integral reads

$$\omega_{j,n}(t) = \int_0^t e^{L_{\alpha_n}(t-s)} R_{j,n}(s)\, ds \tag{3.12}$$

for all $j \geq 1$ and $n \in \mathbb{Z}$.

It follows directly from an inductive argument and the quadratic nonlinearity of $Q(\cdot,\cdot)$ that for all $0 \leq j \leq M$, $\omega_{j,n} = 0$ for all $|n| \geq 2^{j+1}$. This proves that $|\alpha_n| \leq 2^{M+1}\alpha_\nu \lesssim \nu^{1/8}$, for all $|n| \leq 2^{M+1}$. Since $\alpha_n \lesssim \nu^{1/8}$, the semigroup bounds from Theorem 2.2 read

$$\begin{aligned} \|e^{L_\alpha t}\omega_\alpha\|_{\beta,\gamma,1} &\lesssim e^{\gamma_1 \nu^{1/4} t} e^{-\frac{1}{4}\alpha^2\sqrt{\nu}t} \|\omega_\alpha\|_{\beta,\gamma,1}, \\ \|\partial_z e^{L_\alpha t}\omega_\alpha\|_{\beta,\gamma,1} &\lesssim \Big(\nu^{-1/8} + (\sqrt{\nu}t)^{-1/2}\Big) e^{\gamma_1\nu^{1/4}t} e^{-\frac{1}{4}\alpha^2\sqrt{\nu}t}\|\omega_\alpha\|_{\beta,\gamma,1}. \end{aligned} \tag{3.13}$$

In addition, since $\alpha_n \lesssim \nu^{1/8}$, from (3.5), we compute

$$S_{\alpha_n}\omega_{j-1,n} = \mathcal{O}(\sqrt{\nu}t)\Big[|\omega_{j-1,n}| + |\Delta_{\alpha_n}^{-1}\omega_{j-1,n}|\Big]$$

and hence by induction we obtain

$$\begin{aligned} \|S_{\alpha_n}\omega_{j-1,n}\|_{\beta,\gamma,1} &\lesssim \sqrt{\nu}t\Big[\|\omega_{j-1,n}\|_{\beta,\gamma,1} + \|\Delta_{\alpha_n}^{-1}\omega_{j-1,n}\|_{\beta,\gamma,1}\Big] \\ &\lesssim \sqrt{\nu}t\nu^{-\frac{1}{4}[\frac{j-1}{8p}]} e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4}t}. \end{aligned} \tag{3.14}$$

Let us first consider the case when $1 \leq j \leq 8p-1$, for which $R_{j,n} = S_{\alpha_n}\omega_{j-1,n}$. That is, there is no nonlinearity in the remainder. Using the above estimate on S_{α_n} and the semigroup estimate (3.13) into (3.12), we obtain, for $1 \leq j \leq 8p-1$,

$$\begin{aligned} \|\omega_{j,n}(t)\|_{\beta,\gamma,1} &\leq \int_0^t \|e^{L_{\alpha_n}(t-s)} S_{\alpha_n}\omega_{j-1,n}(s)\|_{\beta,\gamma,1}\, ds \\ &\leq C\int_0^t e^{\gamma_1\nu^{1/4}(t-s)} \|S_{\alpha_n}\omega_{j-1,n}(s)\|_{\beta,\gamma,1}\, ds \\ &\leq C\int_0^t e^{\gamma_1\nu^{1/4}(t-s)}\sqrt{\nu}s e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4}s}\, ds. \end{aligned}$$

We choose

$$\gamma_1 = \gamma_0\Big(1 + \frac{j-1}{8p} + \frac{1}{16p}\Big)$$

in (3.13) and use the inequality

$$\nu^{1/4} t \le C e^{\frac{\gamma_0}{16p}\nu^{1/4} t}.$$

and obtain

$$\begin{aligned}
\|\omega_{j,n}(t)\|_{\beta,\gamma,1} &\le C \int_0^t e^{\gamma_1 \nu^{1/4}(t-s)} \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4} s}\, ds \\
&\le C\nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4} t} \int_0^t ds \\
&\le C\nu^{1/4} t e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4} t} \\
&\le C e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4} t}.
\end{aligned} \tag{3.15}$$

Similarly, as for derivatives, we obtain

$$\begin{aligned}
&\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \\
&\le \int_0^t \|e^{L_{\alpha_n}(t-s)} S_{\alpha_n} \omega_{j-1,n}(s)\|_{\beta,\gamma,1}\, ds \\
&\le C \int_0^t \Big(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2}\Big) e^{\gamma_1 \nu^{1/4}(t-s)} \|S_{\alpha_n} \omega_{j-1,n}(s)\|_{\beta,\gamma,1}\, ds \\
&\le C \int_0^t \Big(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2}\Big) e^{\gamma_1 \nu^{1/4}(t-s)} \sqrt{\nu} s e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4} s}\, ds,
\end{aligned}$$

in which the integral involving $\nu^{-1/8}$ is already treated in (3.15) and bounded by $C\nu^{-1/8} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4} t}$. As for the second integral, we estimate

$$\begin{aligned}
&\int_0^t (\sqrt{\nu}(t-s))^{-1/2} e^{\gamma_1 \nu^{1/4}(t-s)} \sqrt{\nu} s e^{\gamma_0(1+\frac{j-1}{8p})\nu^{1/4} s}\, ds \\
&\quad \le \int_0^t (\sqrt{\nu}(t-s))^{-1/2} e^{\gamma_1 \nu^{1/4}(t-s)} \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4} s}\, ds \\
&\quad \le \nu^{1/4} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4} t} \int_0^t (\sqrt{\nu}(t-s))^{-1/2}\, ds \\
&\quad \le C\sqrt{t} e^{\gamma_0(1+\frac{j-1}{8p}+\frac{1}{16p})\nu^{1/4} t} \\
&\quad \le C\nu^{-1/8} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4} t}.
\end{aligned} \tag{3.16}$$

Thus,

$$\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \le C\nu^{-1/8} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4} t}.$$

This and (3.15) prove the inductive bound (3.10) for $j \le 8p-1$.

For $j \geq 8p$, the quadratic nonlinearity starts to play a role. For $k+\ell = j-8p$, we compute

$$Q_{\alpha_n}(\omega_{k,n_1}, \omega_{\ell,n_2}) = i\alpha_\nu \Big(n_2 \partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1} \omega_{\ell,n_2} - n_1 \Delta_{\alpha_n}^{-1} \omega_{k,n_1} \partial_z \omega_{\ell,n_2} \Big). \quad (3.17)$$

Using the algebra structure of the boundary layer norm (see (1.14)), we have

$$\begin{aligned} \alpha_\nu \|\partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1} \omega_{\ell,n_2}\|_{\beta,\gamma,1} &\lesssim \nu^{1/8} \|\partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1}\|_{\beta,\gamma,0} \|\omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{1/8} \|\omega_{k,n_1}\|_{\beta,\gamma,1} \|\omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{1/8} \nu^{-\frac14 [\frac{k}{8p}]} \nu^{-\frac14 [\frac{\ell}{8p}]} e^{\gamma_0 (2 + \frac{k+\ell}{8p}) \nu^{1/4} t} \end{aligned}$$

where we used

$$\|\partial_z \Delta_{\alpha_n}^{-1} \omega_{k,n_1}\|_{\beta,\gamma,0} \leq C \|\omega_{k,n_1}\|_{\beta,\gamma,1},$$

an inequality which is proven in the Appendix. Moreover,

$$\begin{aligned} \alpha_\nu \|\Delta_\alpha^{-1} \omega_{k,n_1} \partial_z \omega_{\ell,n_2}\|_{\beta,\gamma,1} &\lesssim \nu^{1/8} \|\Delta_\alpha^{-1} \omega_{k,n_1}\|_{\beta,\gamma,0} \|\partial_z \omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{1/8} \|\omega_{k,n_1}\|_{\beta,\gamma,1} \|\partial_z \omega_{\ell,n_2}\|_{\beta,\gamma,1} \\ &\lesssim \nu^{-\frac14 [\frac{k}{8p}]} \nu^{-\frac14 [\frac{\ell}{8p}]} e^{\gamma_0 (2 + \frac{k+\ell}{8p}) \nu^{1/4} t}, \end{aligned}$$

in which the derivative estimate (3.10) was used. We note that

$$[\frac{k}{8p}] + [\frac{\ell}{8p}] \leq [\frac{k+\ell}{8p}] = [\frac{j}{8p}] - 1.$$

This proves

$$\|Q_{\alpha_n}(\omega_{k,n_1}, \omega_{\ell,n_2})\|_{\beta,\gamma,1} \lesssim \nu^{1/4} \nu^{-\frac14 [\frac{j}{8p}]} e^{\gamma_0 (1 + \frac{j}{8p}) \nu^{1/4} t}$$

for all $k+\ell = j-8p$. This, together with the estimate (3.14) on S_{α_n}, yields

$$\begin{aligned} \|R_{j,n}(t)\|_{\beta,\gamma,1} &\lesssim \sqrt{\nu} t \nu^{-\frac14 [\frac{j-1}{8p}]} e^{\gamma_0 (1 + \frac{j-1}{8p}) \nu^{1/4} t} + \nu^{1/4} \nu^{-\frac14 [\frac{j}{8p}]} e^{\gamma_0 (1 + \frac{j}{8p}) \nu^{1/4} t} \\ &\lesssim \nu^{1/4} \nu^{-\frac14 [\frac{j}{8p}]} e^{\gamma_0 (1 + \frac{j}{8p}) \nu^{1/4} t}, \end{aligned}$$

for all $j \geq 8p$ and $n \in \mathbb{Z}$, in which we used $\nu^{1/4} t \leq e^{\gamma_0 t / 8p}$.

Putting these estimates into the Duhamel's integral formula (3.12), we obtain, for $j \geq 8p$,

$$\begin{aligned} \|\omega_{j,n}(t)\|_{\beta,\gamma,1} &\leq C \int_0^t e^{\gamma_1 \nu^{1/4} (t-s)} \|R_{j,n}(s)\|_{\beta,\gamma,1} \, ds \\ &\leq C \int_0^t e^{\gamma_1 \nu^{1/4} (t-s)} \nu^{1/4} \nu^{-\frac14 [\frac{j}{8p}]} e^{\gamma_0 (1 + \frac{j}{8p}) \nu^{1/4} s} \, ds \\ &\lesssim \nu^{-\frac14 [\frac{j}{8p}]} e^{\gamma_0 (1 + \frac{j}{8p}) \nu^{1/4} s} \end{aligned}$$

and

$$\begin{aligned}
&\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \\
&\le C\int_0^t \Big(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2}\Big) e^{\gamma_1\nu^{1/4}(t-s)} \|R_{j,n}(s)\|_{\beta,\gamma,1}\, ds \\
&\le C\int_0^t \Big(\nu^{-1/8} + (\sqrt{\nu}(t-s))^{-1/2}\Big) e^{\gamma_1\nu^{1/4}(t-s)} \nu^{1/4}\nu^{-\frac14[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}s}\, ds.
\end{aligned}$$

Using (3.16), we obtain

$$\|\partial_z \omega_{j,n}(t)\|_{\beta,\gamma,1} \lesssim \nu^{-1/8}\nu^{-\frac14[\frac{j}{8p}]} e^{\gamma_0(1+\frac{j}{8p})\nu^{1/4}s},$$

which completes the proof of (3.10). The lemma follows. □

3.3 The remainder

We recall that the approximate vorticity $\omega_{\rm app}$, constructed as in (3.6), approximately solves (3.4), leaving the error $R_{\rm app}$ defined by

$$R_{\rm app} = \nu^{p+\frac{M+1}{8}} S\omega_M + \sum_{k+\ell > M+1-8p; 1\le k,\ell\le M} \nu^{2p+\frac{k+\ell}{8}} Q(\omega_k,\omega_\ell).$$

Using the estimates in Lemma 3.2, we obtain

$$\begin{aligned}
\|S\omega_M\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4}\nu^{-\frac14[\frac{M+1}{8p}]} e^{\gamma_0(1+\frac{M+1}{8p})\nu^{1/4}t} \\
\|Q(\omega_k,\omega_\ell)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4}\nu^{-\frac14[\frac{k+\ell}{8p}]} e^{\gamma_0(2+\frac{k+\ell}{8p})\nu^{1/4}t}.
\end{aligned}$$

This yields

$$\| R_{\rm app}\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{1/4}\sum_{j=M+1}^{2M} \nu^{-\frac14[\frac{j}{8p}]}\Big(\nu^p e^{\gamma_0\nu^{1/4}t}\Big)^{1+\frac{j}{8p}}. \tag{3.18}$$

3.4 Proof of Theorem 3.1

The proof of the Theorem now straightforwardly follows from the estimates from Lemma 3.2 and the estimate (3.18) on the remainder. Indeed, we choose the time T_* so that

$$\nu^p e^{\gamma_0\nu^{1/4}T_*} = \nu^\tau \tag{3.19}$$

for some fixed $\tau > \frac14$. It then follows that for all $t \le T_*$ and $j \ge 0$, there holds

$$\nu^{-\frac14[\frac{j}{8p}]}\Big(\nu^p e^{\gamma_0\nu^{1/4}t}\Big)^{1+\frac{j}{8p}} \lesssim \nu^\tau \nu^{(\tau-\frac14)\frac{j}{8p}}.$$

Using this into the estimates (3.11) and (3.18), respectively, we obtain

$$
\begin{aligned}
\|\partial_z^b \omega_{\mathrm{app}}(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{p-b/8} e^{\gamma_0 \nu^{1/4} t} \lesssim \nu^{\tau - b/8}, \\
\|R_{\mathrm{app}}(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4} \nu^{-\frac{1}{4}[\frac{M}{8p}]} \Big(\nu^p e^{\gamma_0 \nu^{1/4} t} \Big)^{1+\frac{M}{8p}} \\
&\lesssim \nu^{\tau+1/4} \nu^{(\tau - \frac{1}{4})\frac{M}{8p}},
\end{aligned} \tag{3.20}
$$

for all $t \le T_*$. Since $\tau > \frac{1}{4}$ and M is arbitrarily large (and fixed), the remainder is of order ν^P for arbitrarily large number P. The theorem is proved.

4 Nonlinear instability

We are now ready to give the proof of Theorem 1.2. Let $\widetilde{u}_{\mathrm{app}}$ be the approximate solution constructed in Theorem 3.1 and let

$$
v = u - \widetilde{u}_{\mathrm{app}},
$$

with u being the genuine solution to the nonlinear Navier-Stokes equations. The corresponding vorticity $\omega = \nabla \times v$ solves

$$
\partial_t \omega + (\widetilde{u}_{\mathrm{app}} + v) \cdot \nabla \omega + v \cdot \nabla \widetilde{\omega}_{\mathrm{app}} = \sqrt{\nu} \Delta \omega + R_{\mathrm{app}}
$$

for the remainder $R_{\mathrm{app}} = \operatorname{curl} \mathcal{E}_{\mathrm{app}}$ satisfying the estimate (3.18). Let us write

$$
u_{\mathrm{app}} = \widetilde{u}_{\mathrm{app}} - U_{\mathrm{bl}}.
$$

To make use of the semigroup bound for the linearized operator $\partial_t - L$, we rewrite the vorticity equation as

$$
(\partial_t - L)\omega + (u_{\mathrm{app}} + v) \cdot \nabla \omega + v \cdot \nabla \omega_{\mathrm{app}} = R_{\mathrm{app}}
$$

with $\omega_{|_{t=0}} = 0$. We note that since the boundary layer profile is stationary, the perturbative operator S defined as in (3.5) is in fact zero. The Duhamel's principle then yields

$$
\omega(t) = \int_0^t e^{L(t-s)} \Big(R_{\mathrm{app}} - (u_{\mathrm{app}} + v) \cdot \nabla \omega - v \cdot \nabla \omega_{\mathrm{app}} \Big)\, ds. \tag{4.1}
$$

Using the representation (4.1), we shall prove the existence and give estimates on ω. We shall work with the following norm

$$
|||\omega(t)||| := \|\omega(t)\|_{\sigma,\beta,\gamma,1} + \nu^{1/8} \|\partial_x \omega(t)\|_{\sigma,\beta,\gamma,1} + \nu^{1/8} \|\partial_z \omega(t)\|_{\sigma,\beta,\gamma,1} \tag{4.2}
$$

in which the factor $\nu^{1/8}$ added in the norm is to overcome the loss of $\nu^{-1/8}$ for derivatives (see (4.4) for more details).

Let p be an arbitrary large number. We introduce the maximal time T_ν of existence, defined by

$$T_\nu := \max\Big\{t \in [0, T_*] \ : \ \sup_{0\le s\le t} |||\omega(s)||| \le \nu^p e^{\gamma_0 \nu^{1/4} t}\Big\} \tag{4.3}$$

in which T_* is defined as in (3.19). By the short time existence theory, with zero initial data, T_ν exists and is positive. It remains to give a lower bound estimate on T_ν. First, we obtain the following lemmas.

Lemma 4.1. *For $t \in [0, T_*]$, there hold*

$$\begin{aligned} \|\partial_x^a \partial_z^b \omega_{\mathrm{app}}(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{a/8-b/8}\Big(\nu^p e^{\gamma_0 \nu^{1/4} t}\Big) \\ \|\partial_x^a \partial_z^b R_{\mathrm{app}}(t)\|_{\sigma,\beta,\gamma,1} &\lesssim \nu^{1/4+a/8-b/8}\nu^{-\frac14[\frac{M}{8p}]}\Big(\nu^p e^{\gamma_0 \nu^{1/4} t}\Big)^{1+\frac{M}{8p}}. \end{aligned}$$

Proof. This follows directly from Lemma 3.2 and the estimate (3.18) on the remainder R_{app}, upon noting the fact that for $t \in [0, T_*]$, $\nu^p e^{\gamma_0 \nu^{1/4} t}$ remains sufficiently small. □

Lemma 4.2. *There holds*

$$\Big\|(u_{\mathrm{app}} + v)\cdot\nabla\omega + v\cdot\nabla\omega_{\mathrm{app}}\Big\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{-\frac18}\Big(\nu^p e^{\gamma_0 \nu^{1/4} t}\Big)^2.$$

for $t \in [0, T_\nu]$.

Proof. We first recall the elliptic estimate

$$\|u\|_{\sigma,\beta,\gamma,0} \lesssim \|\omega\|_{\sigma,\beta,\gamma,1}$$

which is proven in the Appendix A), and the following uniform bounds (see (1.14))

$$\begin{aligned} \|u\cdot\nabla\tilde\omega\|_{\sigma,\beta,\gamma,1} &\le \|u\|_{\sigma,\beta,\gamma,0}\|\nabla\tilde\omega\|_{\sigma,\beta,\gamma,1} \\ &\le \|\omega\|_{\sigma,\beta,\gamma,1}\|\nabla\tilde\omega\|_{\sigma,\beta,\gamma,1}. \end{aligned}$$

Using this and the bounds on ω_{app}, we obtain

$$\|v\cdot\nabla\omega_{\mathrm{app}}\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{-1/8}\Big(\nu^p e^{\gamma_0 \nu^{1/4} t}\Big)\|\omega\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{-\frac18}\Big(\nu^p e^{\gamma_0 \nu^{1/4} t}\Big)^2$$

and

$$
\begin{aligned}
\|(u_{\mathrm{app}}+v)\cdot\nabla\omega\|_{\sigma,\beta,\gamma,1} &\lesssim \Big(\nu^p e^{\gamma_0\nu^{1/4}t}+\|\omega\|_{\sigma,\beta,\gamma,1}\Big)\|\nabla\omega\|_{\sigma,\beta,\gamma,1}\\
&\lesssim \nu^{-\frac{1}{8}}\Big(\nu^p e^{\gamma_0\nu^{1/4}t}\Big)^2.
\end{aligned}
$$

This proves the lemma. □

Next, using Theorem 2.2 and noting that $\alpha e^{-\alpha^2\nu t}\lesssim 1+(\nu t)^{-1/2}$, we obtain the following uniform semigroup bounds:

$$
\begin{aligned}
\|e^{Lt}\omega\|_{\sigma,\beta,\gamma,1} &\le C_0\nu^{-1/4}e^{\gamma_1\nu^{1/4}t}\|\omega\|_{\sigma,\beta,\gamma,1}\\
\|\partial_x e^{Lt}\omega\|_{\sigma,\beta,\gamma,1} &\le C_0\nu^{-1/4}\Big(1+(\sqrt{\nu}t)^{-1/2}\Big)e^{\gamma_1\nu^{1/4}t}\|\omega\|_{\sigma,\beta,\gamma,1}\\
\|\partial_z e^{Lt}\omega\|_{\sigma,\beta,\gamma,1} &\le C_0\nu^{-1/4}\Big(\nu^{-1/8}+(\sqrt{\nu}t)^{-1/2}\Big)e^{\gamma_1\nu^{1/4}t}\|\omega\|_{\sigma,\beta,\gamma,1}.
\end{aligned}
\tag{4.4}
$$

We are now ready to apply the above estimates into the Duhamel's integral formula (4.1). We obtain

$$
\begin{aligned}
\|\omega(t)\|_{\sigma,\beta,\gamma,1} \lesssim{}& \nu^{-1/4}\int_0^t e^{\gamma_1\nu^{1/4}(t-s)}\nu^{-\frac{1}{8}}\Big(\nu^p e^{\gamma_0\nu^{1/4}s}\Big)^2\,ds\\
&+\nu^{-1/4}\int_0^t e^{\gamma_1\nu^{1/4}(t-s)}\nu^{1/4}\nu^{-\frac{1}{4}[\frac{M}{8p}]}\Big(\nu^p e^{\gamma_0\nu^{1/4}s}\Big)^{1+\frac{M}{8p}}\,ds\\
\lesssim{}& \nu^{-5/8}\Big(\nu^p e^{\gamma_0\nu^{1/4}t}\Big)^2+\nu^P\Big(\nu^p e^{\gamma_0\nu^{1/4}t}\Big),
\end{aligned}
$$

upon taking γ_1 sufficiently close to γ_0. Set T_1 so that

$$
\nu^p e^{\gamma_0\nu^{1/4}T_1}=\theta_0\nu^{\frac{5}{8}},
\tag{4.5}
$$

for some sufficiently small and positive constant θ_0. Then, for all $t\le T_1$, there holds

$$
\|\omega(t)\|_{\sigma,\beta,\gamma,1}\lesssim \nu^p e^{\gamma_0\nu^{1/4}t}\Big[\theta_0+\nu^P\Big]
$$

Similarly, we estimate the derivatives of ω. The Duhamel integral and the semigroup bounds yield

$$
\begin{aligned}
\|\nabla\omega(t)\|_{\sigma,\beta,\gamma,1} \lesssim{}& \nu^{-1/4}\int_0^t e^{\gamma_1\nu^{1/4}(t-s)}\Big(\nu^{-1/8}+(\sqrt{\nu}(t-s))^{-1/2}\Big)\\
&\times\Big[\nu^{-\frac{1}{8}}\Big(\nu^p e^{\gamma_0\nu^{1/4}s}\Big)^2+\nu^{-\frac{1}{4}[\frac{M}{8p}]}\Big(\nu^p e^{\gamma_0\nu^{1/4}s}\Big)^{1+\frac{M}{8p}}\Big]\,ds\\
\lesssim{}& \nu^{-5/8}\Big[\nu^{-\frac{1}{8}}\Big(\nu^p e^{\gamma_0\nu^{1/4}s}\Big)^2+\nu^{-\frac{1}{4}[\frac{M}{8p}]}\Big(\nu^p e^{\gamma_0\nu^{1/4}s}\Big)^{1+\frac{M}{8p}}\Big]
\end{aligned}
$$

By view of (4.5) and the estimate (3.16), the above yields

$$\|\nabla\omega(t)\|_{\sigma,\beta,\gamma,1} \lesssim \nu^{p-\frac{1}{8}} e^{\gamma_0\nu^{1/4}t}\Big[\theta_0 + \nu^P\Big].$$

To summarize, for $t \le \min\{T_*, T_1, T_\nu\}$, with the times defined as in (3.19), (4.3), and (4.5), we obtain

$$|||w(t)||| \lesssim \nu^p e^{\gamma_0\nu^{1/4}t}\Big[\theta_0 + \nu^P\Big].$$

Taking θ_0 sufficiently small, we obtain

$$|||w(t)||| \ll \nu^p e^{\gamma_0\nu^{1/4}t}$$

for all time $t \le \min\{T_*, T_1, T_\nu\}$. In particular, this proves that the maximal time of existence T_ν is greater than T_1, defined as in (4.5). This proves that at the time $t = T_*$, the approximate solution grows to order of $\nu^{5/8}$ in the L^∞ norm. Theorem 1.2 is proved.

A Elliptic estimates

In this section, for sake of completeness, we recall the elliptic estimates with respect to the boundary layer norms. These estimates are proven in [10, Section 3].

First, we consider the classical one-dimensional Laplace equation

$$\Delta_\alpha \phi = \partial_z^2\phi - \alpha^2\phi = f \tag{A.1}$$

on the half line $z \ge 0$, with the Dirichlet boundary condition $\phi(0) = 0$. We start with bounds in the space $\mathcal{A}^\beta := L^\infty_\beta$, namely in the space of bounded functions with the norm $\sup_{z\ge 0} |\phi(z)|e^{\beta z}$. We will prove

Proposition A.1. *If $f \in \mathcal{A}^\beta$, then $\phi \in \mathcal{A}^\beta$ provided $\beta < 1/2$. In addition, there holds*

$$\alpha^2\|\phi\|_\beta + |\alpha|\,\|\partial_z\phi\|_\beta + \|\partial_z^2\phi\|_\beta \le C\|f\|_\beta, \tag{A.2}$$

where the constant C is independent of the integer $\alpha \ne 0$.

Proof. The solution ϕ of (A.1) is explicitly given by

$$\phi(z) = -\frac{1}{2\alpha}\int_0^\infty \Big(e^{-\alpha|x-z|} - e^{-\alpha|x+z|}\Big) f(x)dx. \tag{A.3}$$

A direct bound leads to

$$\|\phi\|_\beta \le \frac{C}{\alpha^2}\|f\|_\beta$$

in which the extra factor of α^{-1} is due to the x-integration. Differentiating the integral formula, we get

$$\|\partial_z\phi\|_\beta \le \frac{C}{\alpha}\|f\|_\beta.$$

We then use the equation to bound $\partial_z^2\phi$, which ends the proof. □

We now establish a similar property for $\mathcal{B}^{\beta,\gamma}$ norms:

Proposition A.2. *If $f \in \mathcal{B}^{\beta,\gamma}$, then $\phi \in \mathcal{A}^\beta$ provided $\beta < 1/2$. In addition, there holds*

$$|\alpha|\,\|\phi\|_{\beta,0} + \|\partial_z\phi\|_{\beta,0} \le C\|f\|_{\beta,\gamma,1}, \tag{A.4}$$

where the constant C is independent of the integer α.

Proof. We will only consider the case $\alpha > 0$, the opposite case being similar. The Green function of $\partial_z^2 - \alpha^2$ is

$$G(x,z) = \frac{1}{\alpha}\Big(e^{-\alpha|z-x|} - e^{-\alpha|z+x|}\Big)$$

and is bounded by α^{-1}. Therefore, using (A.3),

$$\begin{aligned}|\phi(z)| &\le \alpha^{-1}\|f\|_{\beta,\gamma,1}\int_0^\infty e^{-\alpha|z-x|}e^{-\beta x}\Big(1+\delta^{-1}\phi_P(\delta^{-1}x)\Big)dx\\ &\le \alpha^{-1}\|f\|_{\beta,\gamma,1}\Big(\alpha^{-1}+\delta^{-1}\int_0^\infty \phi_P(\delta^{-1}x)dx\Big)\end{aligned}$$

which yields the claimed bound for ϕ since $P > 1$. A similar proof applies for $\partial_z\phi$. □

Note that the above proposition only gives bounds on first order derivatives of ϕ. As the source term f has a boundary layer behavior, we cannot get a good control on second order derivatives. To get bounds on second order derivatives we need to use an extra control on f. For instance, as a direct consequence of the previous proposition, we have, for nonzero integers α,

$$\alpha^2\|\phi\|_{\beta,0} + |\alpha|\,\|\partial_z\phi\|_{\beta,0} + \|\partial_z^2\phi\|_{\beta,\gamma,1} \le C\|\alpha f\|_{\beta,\gamma,1}, \tag{A.5}$$

in which the bound on $\partial_z^2\phi$ is recovered using directly $\partial_z^2\phi = \alpha^2\phi + f$.

Next, let us now turn to the two dimensional Laplace operator.

Proposition A.3. *Let ϕ be the solution of*

$$-\Delta\phi = \omega$$

with the zero Dirichlet boundary condition, and let

$$v = \nabla^\perp \phi.$$

If $\omega \in \mathcal{B}^{\sigma,\beta,\gamma,1}$, then $\phi \in \mathcal{B}^{\sigma,\beta,0}$ and $v = (v_1, v_2) \in \mathcal{B}^{\sigma,\beta,0}$. Moreover, there hold the following elliptic estimates

$$\|\phi\|_{\sigma,\beta,0} + \|v_1\|_{\sigma,\beta,0} + \|v_2\|_{\sigma,\beta,0} \le C\|\omega\|_{\sigma,\beta,\gamma,1}, \tag{A.6}$$

$$\begin{aligned} \|\partial_x v_1\|_{\sigma,\beta,0} + \|\partial_x v_2\|_{\sigma,\beta,0} + \|\partial_z v_1\|_{\sigma,\beta,1} + \|\partial_z v_2\|_{\sigma,\beta,0} \\ \le C\|\omega\|_{\sigma,\beta,\gamma,1} + C\|\partial_x \omega\|_{\sigma,\beta,\gamma,1}, \end{aligned} \tag{A.7}$$

and, with $\psi(z) = z/(1+z)$,

$$\|\psi^{-1} v_2\|_{\sigma,\beta,0} \le C\|\omega\|_{\sigma,\beta,\gamma,1} + C\|\partial_x \omega\|_{\sigma,\beta,\gamma,1}, \tag{A.8}$$

for some constant C.

Proof. The proof relies on the Fourier transform in the x variable, with dual integer Fourier component α. We then have

$$\partial_z^2 \phi_\alpha - \alpha^2 \phi_\alpha = -f_\alpha.$$

Bounds (A.6) is then a direct consequence of Proposition A.2. Bound (A.7) is a consequence of (A.5), and (A.8) comes from the integration of $\partial_z v_2$ together with (A.7). □

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