Green function for linearized Navier-Stokes around a boundary layer profile: away from critical layers

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Abstract

In this paper, we construct the Green function for the classical Orr-Sommerfeld equations, which are the linearized Navier-Stokes equations around a boundary layer profile. As an immediate application, we derive uniform sharp bounds on the semigroup of the linearized Navier-Stokes problem around unstable profiles in the vanishing viscosity limit.

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1 Introduction

In this paper, we are interested in the study of linearized Navier Stokes around a given fixed profile $U_s = (U(z), 0)$ in the inviscid limit; namely, we consider the following set of equations

$$\partial_t v + U_s \cdot \nabla v + v \cdot \nabla U_s + \nabla p - \nu \Delta v = 0, \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

on the half plane $x \in \mathbb{T}$, $z \geq 0$, with boundary conditions

$$v = 0 \quad \text{on} \quad z = 0. \tag{1.3}$$

Throughout this paper, the background profile $U(z)$ is assumed to be sufficiently smooth, $U(0) = 0$, and satisfies

$$|\partial_z^k (U(z) - U_+)| \leq C_k e^{-\eta_0 z}, \qquad \forall\ z \geq 0, \quad k \geq 0, \tag{1.4}$$

for some finite U_+ and positive constants C_k and η_0.

We are interested in the problem (1.1)-(1.3) in the case when $\nu \to 0$. This very classical problem has led to a huge physical and mathematical literature, focussing in particular on the linear stability, on the dispersion relation, on the study of eigenvalues and eigenmodes, and on the onset of nonlinear instabilities and turbulence; see [2] for an introduction on these topics, and the classical achievements of Rayleigh, Orr, Sommerfeld, Heisenberg, Tollmien, C.C. Lin, and Schlichting.

Two cases arise. Either the profile U is linearly stable, or it is unstable for the linearized Euler equations, that is, when $\nu = 0$. In this paper we will focus on the second case, which turns out to be easier than the first one. We will therefore assume that U is linearly unstable for Euler equations (the precise assumption will be detailed below). In this case, it is well known that the profile U is linearly unstable for the linearized Navier Stokes equations provided ν is sufficiently small, or equivalently, the Reynolds number $R = \frac{1}{\nu}$ is sufficiently large.

Let λ_0 be the most unstable eigenvalue for the Euler equations, that is the eigenvalue with the largest real part. The aim of this paper is to bound the solution v of (1.1)-(1.3), uniformly as ν goes to 0. We will prove that, for any positive number τ, there exist positive constants C_τ, ν_τ such that solutions v, with initial data $v(0)$, satisfy

$$\|v(t)\|_\eta \le C_\tau e^{(\Re\lambda_0+\tau)t}\|v(0)\|_\eta \tag{1.5}$$

uniformly for any positive t and any $0 \le \nu \le \nu_\tau$. In this bound, $\eta > 0$ is an arbitrary small positive number, and $\|\cdot\|_\eta$ denotes the following L^∞ weighted norm

$$\|v\|_\eta = \sup_{z\ge 0} |v(z)|e^{\eta z}.$$

A precise statement of this result is detailed in the next section.

The bound (1.5) is a very natural result, but, up to the best of our knowledge, it has never been proven in the literature. Its proof relies on a very careful and detailed construction and analysis of the Green function of linear Navier-Stokes equations. This Green function is the Laplace transform of the Green function of the classical Orr-Sommerfeld equations, that we will construct in details.

In the next section we will recall the classical Orr-Sommerfeld equations, discuss its spectrum, and present our main results on OS, namely new bounds on Green functions for this equation. We then state our main result on the growth of solutions of linear Navier-Stokes equations. Finally, we turn to the construction of Green functions of Orr-Sommerfeld and of linear Navier Stokes equations.

1.1 Orr-Sommerfeld equations

To take advantage of the incompressibility relation (1.2), following the classical literature, we introduce the stream function ψ of v, defined by

$$v = \nabla^\perp \psi,$$

and take its Fourier transform in the x variable, with wave number α, and Laplace transform in time, with Laplace variable $\lambda = -i\alpha c$, following historical notations. We therefore focus on solutions v of (1.1)-(1.3) of the form

$$\lambda = -i\alpha c, \qquad v = \nabla^\perp \Big(e^{i\alpha(x-ct)} \phi(z) \Big),$$

This leads to the classical Orr-Sommerfeld (OS) ODE equations

$$(U-c)(\partial_z^2 - \alpha^2)\phi - U''\phi = \epsilon(\partial_z^2 - \alpha^2)^2\phi, \qquad \epsilon = \frac{1}{i\alpha R} \tag{1.6}$$

on the half line $z \ge 0$, together with boundary conditions:

$$\phi_{|z=0} = \phi'_{|z=0} = 0, \qquad \lim_{z\to\infty} \phi(z) = 0. \tag{1.7}$$

Here, $\alpha \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ denotes the wave number, $c \in \mathbb{C}$ is the complex phase velocity, and $R = \nu^{-1}$ is the Reynolds number.

Such a spectral formulation of the linearized Navier-Stokes equations about a boundary layer shear profile has been studied intensively in the literature. We in particular refer to [2, 15, 12, 13] for the major works of Heisenberg, Tollmien, C.C. Lin, and Schlichting on the subject. We also refer to [6, 7, 8] for the rigorous spectral analysis on the Orr-Sommerfeld equations.

In this paper, we shall derive pointwise bounds on the Green function of the Orr-Sommerfeld problem, and establish uniform bounds on the semigroup of the linearized Navier-Stokes equations. For convenience, let us denote

$$\Delta_\alpha := \partial_z^2 - \alpha^2$$

and

$$\mathrm{OS}(\phi) := (U-c)\Delta_\alpha \phi - U''\phi - \epsilon \Delta_\alpha^2 \phi. \tag{1.8}$$

For each fixed $\alpha \in \mathbb{N}^*$ and $c \in \mathbb{C}$, we let $G_{\alpha,c}(x,z)$ be the corresponding Green kernel of the OS problem. By definition, for each $x \in \mathbb{R}_+$, $G_{\alpha,c}(x,z)$ solves

$$\mathrm{OS}(G_{\alpha,c}(x,\cdot)) = \delta_x(\cdot)$$

on $z \geq 0$, together with the boundary conditions:

$$G_{\alpha,c}(x,0) = \partial_z G_{\alpha,c}(x,0) = 0, \qquad \lim_{z\to\infty} G_{\alpha,c}(x,z) = 0.$$

The Green kernel allows us to solve the inhomogenous OS problem

$$\mathrm{OS}(\phi) = f, \tag{1.9}$$

or equivalently, the resolvent equations to the linearized Navier-Stokes operator about $U(z)$, yielding the solution ϕ of the form

$$\phi(z) = \int_0^\infty G_{\alpha,c}(x,z) f(x)\, dx.$$

1.2 Spectrum of the Orr-Sommerfeld problem

To construct the Green function of the Orr-Sommerfeld problem, we first need to analyze its spectrum, or by definition, the complement of the range of c over which the Orr-Sommerfeld equation (1.9), together with the boundary conditions (1.7), is solvable for each function f. For this, we study the corresponding homogenous Orr-Sommerfeld problem

$$\mathrm{OS}_{\alpha,c}(\phi) = 0 \tag{1.10}$$

together with the boundary conditions $\phi = \phi' = 0$ on $z = 0$.

In what follows, we focus on the case $\alpha > 0$; the other case being similar. The unstable spectrum then corresponds to the case when $\Im c > 0$. By multiplying the Orr-Sommerfeld equations by ϕ^*, the complex conjugate of ϕ, it follows that the spectrum c, if exists, must satisfy $|\Re c| \leq C_0$ and $\Im c + \alpha\sqrt{\nu} \leq C_0$, for the universal positive constant C_0 (depending only on the profile U); see Section 6.1 for an equivalent estimate on the spectrum of the linearized Navier-Stokes problem.

Next, since $U(z)$ converges to a finite constant U_+ as $z \to \infty$ (see (1.4), below), solutions to the homogenous Orr-Sommerfeld equation (1.10) converge to solutions of the following constant-coefficient equation

$$\mathrm{OS}_+(\phi) = -\varepsilon\Delta_\alpha^2\phi_\alpha + (U_+ - c)\Delta_\alpha\phi_\alpha = 0. \tag{1.11}$$

This constant-coefficient equation has four independent solutions $e^{\pm\mu_s z}$ and $e^{\pm\mu_f^+ z}$, with

$$\mu_s = \alpha, \qquad \mu_f = \epsilon^{-1/2}\sqrt{U - c + \alpha^2\epsilon}, \qquad \mu_f^+ = \lim_{z\to\infty}\mu_f, \tag{1.12}$$

in which the square root takes the positive real part. Observe that as long as $|U(z) - c| \gg \epsilon$, or equivalently $|\lambda + i\alpha U(z)| \gg \sqrt{\nu}$, we have $\mu_s \ll \mu_f$.

As will be proved later, solutions to the Orr-Sommerfeld equation consist of "slow behavior" $e^{\pm\mu_s z}$ and "fast behavior" $e^{\pm\mu_f z}$, asymptotically near the infinity. By view of (1.10) and (1.11), the two slow modes are perturbations from the Rayleigh solutions $(U - c)\Delta_\alpha\phi - U''\phi = 0$ and the two fast modes are linked to so-called Airy-type solutions $(-\varepsilon\Delta_\alpha + U - c)\Delta_\alpha\phi = 0$. In this paper, we restrict to the case when c is away from the range of U.

Precisely, throughout the paper, letting ϵ_0 be an arbitrarily small, but fixed, positive constant, we shall consider the range of (α, c) in $\mathbb{R}_+ \times \mathbb{C}$ so that

$$d_c := |c - \text{Range}(U)| \ge \frac{\epsilon_0}{1+\alpha} \tag{1.13}$$

in which Range(U) denotes the range of U over $\mathbb{R}_+$.

To understand the Orr-Sommerfeld spectrum, we let $\phi^s_{\alpha,c}, \phi^f_{\alpha,c}$ be two independent, slow and fast decaying solutions of the Orr-Sommerfeld equations $\text{OS}_{\alpha,c}(\phi) = 0$, with their normalized amplitude $\|\phi^s_{\alpha,c}\|_{L^\infty} = \|\phi^f_{\alpha,c}\|_{L^\infty} = 1$. Set

$$D(\alpha, c) := \mu_f^{-1} \det \begin{pmatrix} \phi^s_{\alpha,c} & \phi^f_{\alpha,c} \\ \partial_z\phi^s_{\alpha,c} & \partial_z\phi^f_{\alpha,c} \end{pmatrix}_{|z=0}, \tag{1.14}$$

which is often referred to as the Evans function. Clearly, there are non trivial solutions (α, c, ϕ) to the Orr-Sommerfeld problem if and only if $D(\alpha, c) = 0$. In addition, the Orr-Sommerfeld solutions and hence the Evans function $D(\alpha, c)$ is analytic in c away from the range of $U(z)$. As a consequence, the unstable spectrum of the Orr-Sommerfeld problem is discrete; that is, there are at most finitely many zeros c on $\Im c > 0$. In addition, eigenfunctions corresponding to each unstable eigenvalue c (if exists) are of the form

$$\phi = \phi^s_{\alpha,c} - a(\alpha, c)\phi^f_{\alpha,c}, \qquad a(\alpha, c) = \frac{\phi^s_{\alpha,c}(0)}{\phi^f_{\alpha,c}(0)}. \tag{1.15}$$

The corresponding inviscid Evans function is defined in a similar way. Indeed, let $\phi_{\alpha,c}$ be a decaying solution of the Rayleigh equation (the Orr-Sommerfeld equations with $\epsilon = 0$), with the normalized amplitude $\|\phi_{\alpha,c}\|_{L^\infty} = 1$. Set

$$E(\alpha, c) := \phi_{\alpha,c}(0). \tag{1.16}$$

Then, the discrete spectrum of the Rayleigh problem is equivalent to the zeros of $E(\alpha, c)$, for each fixed wavenumber α.

We end this section with the following simple proposition whose proof is given in Section 3.6.

Proposition 1.1. *Let U be a spectrally unstable shear profile for the Rayleigh problem (linearized Euler equations). That is, $E(\alpha_0, c_0) = 0$ for some positive α_0 and $\Im c_0 > 0$. Assume that*

$$\partial_c E(\alpha_0, c_0) \neq 0.$$

Then, for sufficiently small viscosity ν, the profile U is also spectrally unstable for the Orr-Sommerfeld problem (linearized Navier-Stokes equations).

2 Statement of the results

2.1 Main results: bounds on the Green function of OS

Our main results in this paper are as follows:

Theorem 2.1. *Let U be a boundary layer profile. For each α, c, we denote by $G_{\alpha,c}(x,z)$ the Green kernel of the Orr-Sommerfeld problem, and set*

$$\mu_s = \alpha, \qquad m_f = \inf_z \Re\mu_f(z), \qquad M_f = \sup_z \Re\mu_f(z) \tag{2.1}$$

with $\mu_f(z) = \sqrt{\alpha^2 + \frac{U-c}{\epsilon}}$, taking the positive real part. Then, there are universal positive constants θ_0, C_0 so that there holds

$$\begin{aligned} |G_{\alpha,c}(x,z)| \le{}& C_0[D(\alpha,c)]^{-1}\Big(\frac{1}{\mu_s}e^{-\theta_0\mu_s(|z|+|x|)} + \frac{1}{m_f}e^{-\theta_0 m_f(|z|+|x|)}\Big) \\ &+ C_0\Big(\frac{1}{\mu_s}e^{-\theta_0\mu_s|x-z|} + \frac{1}{m_f}e^{-\theta_0 m_f|x-z|}\Big) \end{aligned} \tag{2.2}$$

uniformly in (α, c), within (1.13), and uniformly for all $x, z \ge 0$. In addition, there also hold the following derivative bounds

$$\begin{aligned} &|\partial_x^k \partial_z^\ell G_{\alpha,c}(x,z)| \\ &\le C_0[D(\alpha,c)]^{-1}\Big(\frac{\mu_s^{k+\ell}}{\mu_s}e^{-\theta_0\mu_s(|x|+|z|)} + \frac{M_f^{k+\ell}}{m_f}e^{-\theta_0 m_f(|x|+|z|)}\Big) \\ &\quad + C_0\Big(\frac{\mu_s^{k+\ell}}{\mu_s}e^{-\theta_0\mu_s|x-z|} + \frac{M_f^{k+\ell}}{m_f}e^{-\theta_0 m_f|x-z|}\Big) \end{aligned} \tag{2.3}$$

for all $x, z \ge 0$ and $k, \ell \ge 0$.

It is also interesting (see Section 2.2) to express the resolvent solutions to the linearized Navier-Stokes equations in term of the vorticity $\Delta_\alpha G_{\alpha,c}(x,z)$, which solves

$$\Big(-\epsilon\Delta_\alpha + U - c\Big)\Delta_\alpha G_{\alpha,c}(x,z) = \delta_x(z) + U'' G_{\alpha,c}(x,z).$$

This shows that at leading order, the vorticity $\Delta_\alpha G_{\alpha,c}(x,z)$ is governed by the fast dynamics of the operator $-\epsilon\Delta_\alpha + U - c$. To describe this, we write

$$\Delta_\alpha G_{\alpha,c}(x,z) = \mathcal{G}_a(x,z) + \mathcal{R}_G(x,z) \tag{2.4}$$

in which $\mathcal{G}_a(x,z)$ is the Green function of $-\epsilon\Delta_\alpha + U - c$ on the whole line, satisfying the Laplacian-like estimate

$$|\mathcal{G}_a(x,z)| \le |\epsilon m_f|^{-1} e^{-m_f|x-z|}. \tag{2.5}$$

See Section 3.3 for details of the construction, especially the equation (3.12). It then follows that the residual Green function satisfies

$$\mathcal{R}_G(x,z) = \int_0^\infty \mathcal{G}_a(y,z) U''(y) G_{\alpha,c}(x,y)\, dy.$$

We shall prove that the residual term $\mathcal{R}_G(x,z)$ is indeed an order of $(\epsilon m_f^2)^{-1}$ better than that of $G_{\alpha,c}(x,z)$. As $\epsilon\mu_f^2 = U - c + \alpha^2\epsilon$, such an estimate is useful, when c is away from the range of U. Precisely, we obtain the following corollary, whose proof is given at the end of Section 5.

Corollary 2.2. *Let $G_{\alpha,c}(x,z)$ be the Green kernel of the Orr-Sommerfeld problem. We write the vorticity of the Green function as in* (2.4). *Then,*

$$|\partial_x^\ell \partial_z^k \mathcal{G}_{\mathrm{a}}(x,z)| \le C|\epsilon m_f|^{-1} M_f^{k+\ell} e^{-m_f|x-z|}$$

and

$$\begin{aligned} |\partial_x^\ell \partial_z^k \mathcal{R}_G(x,z)| \le{}& \frac{C_0 M_f^k}{|\epsilon m_f^2|}\Big(\frac{\mu_s^\ell}{\mu_s} e^{-\theta_0\mu_s|x-z|} + \frac{M_f^\ell}{m_f} e^{-\theta_0 m_f|x-z|}\Big) \\ &+ \frac{C_0 M_f^k}{|D(\alpha,c)||\epsilon m_f^2|}\Big(\frac{\mu_s^\ell}{\mu_s} e^{-\theta_0\mu_s(|z|+|x|)} + \frac{M_f^\ell}{m_f} e^{-\theta_0 m_f(|z|+|x|)}\Big) \end{aligned}$$

for $k,\ell \ge 0$. Here, m_f, M_f are as in (2.1).

Our construction of the Green function for the Orr-Sommerfeld problem follows closely the analytical approach introduced by Zumbrun-Howard in the seminal paper [19] (see also [10]). The main difficulty is to construct independent solutions to the homogenous Orr-Sommerfeld equations, having uniform bounds with the parameters α, ϵ, c. Certainly, the standard conjugation method for ODEs (see, for instance, [19, 14]) does not apply directly due to the dependence on the various parameters in the equation. Our construction of slow modes of the OS equations is based on the recent iterative approach introduced in [7, 8], whereas the construction of fast modes is done via the standard diagonalization techniques ([11, 18]).

2.2 Main result: bounds on linearized Navier Stokes

Next, we give bounds on the semigroup of the linearized Navier-Stokes equations, using the Green function bounds obtained in our main theorem, Theorem 2.1. Let L be the linearized Navier Stokes operator around a time independent given shear layer profile $U_s = (U, 0)^{tr}$, namely,

$$L\omega = -(U_s \cdot \nabla)\omega - (v \cdot \nabla)\omega_s + \nu\Delta\omega, \tag{2.6}$$

where $\omega_s = U''$, $\omega = \partial_z v_1 - \partial_x v_2$, and $\nabla \cdot v = 0$, together with the boundary condition $v = 0$ on $z = 0$. The linearized Navier Stokes equation near U_s then reads

$$\partial_t \omega - L\omega = 0.$$

As the operator L is a compact perturbation of the Laplace operator, we can write an integral expression for e^{Lt}, namely

$$e^{Lt}\omega = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda - L)^{-1}\omega \, d\lambda \tag{2.7}$$

where Γ is a contour on the right of the spectrum of L. We now take the Fourier transform in the x variable, α being the dual discrete Fourier variable, which leads to

$$e^{Lt}\omega = \sum_{\alpha \in \mathbb{Z}} e^{i\alpha x} e^{L_\alpha t}\omega_\alpha \tag{2.8}$$

whose Fourier coefficients are computed by

$$e^{L_\alpha t}\omega_\alpha = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda - L_\alpha)^{-1}\omega_\alpha \, d\lambda. \tag{2.9}$$

In this formula, ω_α is the Fourier transform of ω in tangential variables and L_α is the Fourier transform of L. The following simple lemma links the resolvent solution to the Orr-Sommerfeld problem.

Lemma 2.3. *Let $G_{\alpha,c}(x,z)$ be the Green function of* $\mathrm{OS}(\cdot)$*. For each $\alpha \in \mathbb{Z}$, there hold the integral representation*

$$(\lambda - L_\alpha)^{-1}\omega_\alpha(z) = \frac{1}{i\alpha}\int_0^\infty \Delta_\alpha G_{\alpha,c}(x,z)\omega_\alpha(x)\,dx.$$

in which $c = i\alpha^{-1}\lambda$ and Γ_α can be chosen, depending on α and lying in the resolvent set of L_α.

Proof. Indeed, let us write the resolvent solutions in term of the Green function of the Orr-Sommerfeld equations. We write

$$\theta = (\lambda - L_\alpha)^{-1}\omega_\alpha$$

and introduce the stream function ψ of the vorticity θ through

$$\Delta_\alpha \psi = \theta$$

with $\Delta_\alpha = \partial_z^2 - \alpha^2$. Then ψ solves the Orr-Sommerfeld problem

$$\mathrm{OS}(\psi) = -\varepsilon\Delta_\alpha^2\psi + (U-c)\Delta_\alpha\psi - U''\psi = \frac{\omega_\alpha}{i\alpha}, \tag{2.10}$$

with the zero boundary conditions $\psi = \partial_z\psi = 0$ on $z = 0$. The lemma follows. □

We stress that the spectrum of L_α and of the Orr-Sommerfeld problem is linked through the relation $\lambda = -i\alpha c$. Thus, by view of Section 1.2, the unstable spectrum of L_α is also discrete with corresponding eigenfunctions being defined through the stream function of the form (1.15). In particular, we can define the maximal unstable eigenvalue λ_α so that $\Re\lambda_\alpha$ is maximum. We set $\lambda_\alpha = 0$, if no unstable eigenvalues exist. We also introduce the function space $\mathcal{X}^\eta$ for $\eta > 0$ by its finite norm

$$\|\omega\|_\eta = \sup_z |\omega(z)|e^{\eta z}. \tag{2.11}$$

Our next main theorem is as follows.

Theorem 2.4. *Let $\alpha \in \mathbb{Z}^*, \eta \in [0, \frac{1}{8}]$, and λ_α be the maximal unstable eigenvalue of L, with $\lambda_\alpha = 0$ if no unstable eigenvalues exist. Then, we can write*

$$e^{L_\alpha t} = S_\alpha + \mathcal{R}_\alpha$$

in which for any positive τ and $\omega_\alpha \in \mathcal{X}^\eta$, there is a constant C_τ so that

$$\|S_\alpha\omega_\alpha\|_\eta \le C_\tau\|\omega_\alpha\|_\eta e^{\tau t}e^{-\frac{1}{2}\alpha^2\nu t}$$

and

$$\|\mathcal{R}_\alpha \omega_\alpha\|_\eta \le C_\tau e^{(\Re\lambda_\alpha+\tau)t}\alpha^{-2}\log\alpha\|\omega_\alpha\|_\eta$$

uniformly in $t \ge 0$, small $\nu > 0$, and $\alpha \in \mathbb{Z}^*$. In summary, these two estimates combine into

$$\|e^{L_\alpha t}\omega_\alpha\|_\eta \le C_\tau e^{(\Re\lambda_\alpha+\tau)t}\|\omega_\alpha\|_\eta.$$

The standard energy estimates yields the uniform bound

$$\|e^{L_\alpha t}\omega_\alpha\|_\eta \le C_0 e^{C_0 t}\|\omega_\alpha\|_\eta$$

for $C_0 = \|U'\|_{L^\infty}$. Here, it is not clear whether C_0 remains close to $\Re\lambda_\alpha$. Nevertheless, such a rough bound was sufficient to derive an H^2 nonlinear instability from linear instability of boundary layers, via the nonlinear iterative scheme introduced in [5]. In order to improve the nonlinear instability, a sharp bound on the semigroup is needed; see, for instance, [1, 4, 6, 9]. Here, the sharpness is in the rate of the growth of the semigroup as compared to the growth from the maximal unstable eigenfunction.

The proof of Theorem 2.4 follows the Fourier-Laplace approach introduced by Zumbrun in [16, 17] in establishing the multi-dimensional time-asymptotic stability of viscous compressible shocks. For incompressible flows with a fixed viscosity, we point out the work [3] where the authors first use the spectral approach to obtain bounds on the semigroup for the spectrally stable boundary layers. To the best of our knowledge, Theorem 2.4 is the first to provide the uniform sharp bounds on the linearized semigroup of Navier-Stokes equations in the vanishing viscosity limit.

3 Independent Orr-Sommerfeld solutions

In this section, we shall construct all four independent solutions to the OS equations $\mathrm{OS}(\phi) = 0$, recalling

$$\mathrm{OS}(\phi) = -\epsilon\Delta_\alpha^2\phi + (U-c)\Delta_\alpha\phi - U''\phi$$

with $\Delta_\alpha = \partial_z^2 - \alpha^2$. For convenience, we also denote

$$\mathrm{Ray}_\alpha(\phi) = (U-c)\Delta_\alpha\phi - U''\phi$$

and

$$\mathrm{Airy}(\phi) = (U-c)\Delta_\alpha\phi - \varepsilon\Delta_\alpha^2\phi.$$

When c is in the range of U, $\mathrm{Airy}(\phi)$ is indeed similar to the classical Airy equations for $\Delta_\alpha \phi$. In this paper, we study the case when c is in fact away from the range of U; see (1.13). Nevertheless, we abuse the notation to name this operator as an Airy operator. Note also that all the operators depend on both α and c, and we occasionally make the dependence explicit, when needed.

The aim of this section is to develop a perturbative analysis to construct a solution (α, c, ϕ) to the Orr-Sommerfeld problem, when ϵ is sufficiently small, viewed as a perturbation of the Rayleigh problem. Our analysis is based on the iterative functional approach introduced in [7, 8] in which we treat the stable boundary layer profile (there, only decaying solutions are constructed and the ranges of (α, c) are limited to be sufficiently small). Our result in this section is as follows.

Proposition 3.1. *Let $E(\alpha, c)$ be the inviscid Evans function, defined as in* (3.5)*. For arbitrary (α, c) within* (1.13) *and for sufficiently large R so that*

$$\alpha m_f^{-2} E(\alpha, c)^{-1} \ll 1$$

(recalling $m_f \gtrsim \alpha + \sqrt{R}$), there are four independent solutions, two slow modes $\phi_{s,\pm}$ and two fast modes $\phi_{f,\pm}$, to the Orr-Sommerfeld equations. In addition, there exists some positive η so that there hold

$$\begin{aligned} \phi_{s,\pm}(z) &= e^{\pm \alpha z}\Big(1 + \psi_{s,\pm}\Big) \\ \phi_{f,\pm}(z) &= e^{\pm \int_0^z \tilde\mu_f(y)\, dy}\Big(1 + \sqrt{\epsilon}\psi_{f,\pm}\Big), \end{aligned}$$

with $\|e^{\eta z}\psi_{s,\pm}\|_{L^\infty} \lesssim 1$ and $\|e^{\eta z}\psi_{f,\pm}\|_{L^\infty} \lesssim 1$, uniformly in α, c and R. Here, $\tilde\mu_f$ is the solution with positive real part of

$$\tilde\mu_f^2 = \frac{1}{\epsilon}(U - c + \alpha^2 \epsilon)(1 + \mathcal{O}(\epsilon)).$$

3.1 Asymptotic behavior as $z \to +\infty$

As discussed in Section 1.2, there are two solutions $\phi_{s,\pm}$ of the Orr-Sommerfeld equations with a "slow behavior" $e^{\pm \mu_s z}$, with $\mu_s = \alpha$, and two solutions $\phi_{f,\pm}$ with a "fast behavior" $e^{\pm \mu_f z}$, with $\mu_f \sim 1/\sqrt{\epsilon}$. The first two slow-behavior solutions $\phi_{s,\pm}$ will be perturbations of eigenfunctions of the Rayleigh equation. The other two, $\phi_{f,\pm}$, are specific to the Orr-Sommerfeld equation and will be linked to the solutions of $\mathrm{Airy}(\phi) = 0$. The standard conjugation method for ODEs does not apply directly due to the ϵ-dependence in the equation, which makes the coefficients of the ODE decay exponentially only at a vanishing rate (in the fast variable) when tracking the fast modes. In fact, since the fast eigenvalues of the corresponding ODE system are well-separated from the slow ones, we will track these fast modes by diagonalization, keeping their variable-coefficients in our setting.

3.2 Rayleigh solutions

In this section we shall construct an exact inverse for the Rayleigh equation and so find the complete solution to

$$\mathrm{Ray}_\alpha(\phi) := (U-c)\Delta_\alpha\phi - U''\phi = f, \qquad z \ge 0, \tag{3.1}$$

for each fixed α, c, within (1.13). Since the complex number c is away from the range of U, we may divide (3.1) by $U - c$, which leads to

$$\Delta_\alpha\phi - \frac{U''}{U-c}\phi = \frac{f}{U-c}. \tag{3.2}$$

Hence, by the classical ODE theory, solutions ϕ are sufficiently smooth, if the source f is. We shall solve (3.1) via constructing the Green kernel of $\mathrm{Ray}_\alpha(\cdot)$.

Homogenous solutions.

In this paragraph we detail the construction of two independent solutions $\phi_{\alpha,\pm}$ to the Rayleigh operator. We will prove the following lemma.

Lemma 3.2. *There exist two independent solutions $\phi_{\alpha,\pm}$ to the homogenous equation $Ray_\alpha(\phi) = 0$ such that*

$$\phi_{\alpha,\pm}(z) = e^{\pm\alpha z}\Big[1 + \psi_{\alpha,\pm}\Big], \tag{3.3}$$

where $\psi_{\alpha,\pm} \in \mathcal{X}^\eta$, for some positive $\eta < \min\{\alpha, \eta_0\}$, with η_0 as in (1.4)*, and*

$$\|\psi_{\alpha,\pm}\|_\eta \le C.$$

Here, $\mathcal{X}^\eta$ denotes the weighted L^∞ function spaces; see (2.11)*.*

Proof. Let us denote $W = (\phi, \partial_z\phi)^{tr}$. The homogenous equation becomes

$$W' = AW, \qquad A = \begin{pmatrix} 0 & 1 \\ \alpha^2 + \frac{U''}{U-c} & 0 \end{pmatrix}. \tag{3.4}$$

We denote by A_+ the limit of $A(z)$ as $z \to \infty$. Note that

$$A_+ = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}.$$

Eigenvalues of A_+ are $\mu_\pm = \pm\alpha$. Let $V_\pm = (1, \pm\alpha)^{tr}$ be the associated eigenvectors. We now want to construct two independent solutions of (3.4) of the form

$$W_\pm = e^{\mu_\pm z} V(z), \qquad \lim_{z\to\infty} V(z) = V_\pm.$$

Let us first consider the $(-)$ case: $\mu_- = -\alpha$. Plugging the above Ansatz into (3.4), $V(z)$ solves

$$V' = (A_+ - \mu_-)V + (A - A_+)V.$$

Since the eigenvalues of $A_+ - \mu_-$ are 0 and 2α, a direct calculation leads to

$$\|e^{(A_+ - \mu_-)z}\| \le C_0 \alpha e^{-2\alpha|z|}, \qquad \forall\ z \le 0.$$

Here, $\|\cdot\|$ denotes the standard matrix norm. Thanks to the decay assumption on the boundary layer profile $U(z)$, there holds

$$\|A(z) - A_+\| \le C e^{-\eta_0 |z|}.$$

The Duhamel formula yields

$$V(z) = V_- - \int_z^\infty e^{(A_+ - \mu_-)(z-y)}(A(y) - A_+)V(y)\, dy,$$

the integral being taken over the half line $z + \mathbb{R}_+$. Denote by $TV(z)$ the right hand side of the above identity. We shall show that T is a contractive map from $\mathcal{X}^\eta$ into itself, defined on $[M, \infty)$, for sufficiently large M and for some $\eta < \min\{\alpha, \eta_0\}$. Indeed, for $V_1, V_2 \in \mathcal{X}^\eta$, we get

$$\begin{aligned}
|TV_1(z) - TV_2(z)| &\le \int_z^\infty \|e^{(A_+ - \mu_-)(z-y)}\| \|(A(y) - A_+)\| \|V_1 - V_2\|_\eta e^{-\eta y}\, dy \\
&\le C_0 \alpha \|V_1 - V_2\|_\eta \int_z^\infty e^{-2\alpha|y-z|} e^{-(\eta+\eta_0)y}\, dy \\
&\le C_0 \alpha e^{-(\eta+\eta_0)z} \|V_1 - V_2\|_\eta \int_z^\infty e^{-2\alpha|y-z|}\, dy \\
&\le C e^{-\eta z} e^{-\eta_0 M} \|V_1 - V_2\|_\eta
\end{aligned}$$

for all $z \ge M$. Taking M large enough so that $Ce^{-\eta_0 M} < 1/2$, the map T is contractive. Hence, there exists a solution $V \in \mathcal{X}^\eta$ such that $V = TV$, or equivalently $e^{-\alpha z}V(z)$ solves (3.4) on $[M,\infty)$. By the standard ODE theory, the solution V on $[M,\infty)$ can be extended to be a global solution on $[0,\infty)$, since the right-hand side of (3.4) is uniformly bounded.

Similarly, with the ansatz solution: $W_+ = e^{\mu_+ z}V(z)$, there holds

$$V' = (A_+ - \mu_+)V + (A - A_+)V,$$

in which the matrix $A_+ - \mu_+$ has only nonpositive eigenvalues. It follows that $\|e^{(A_+-\mu_+)z}\| \le C_0\alpha e^{-2\alpha|z|}$ for $z \ge 0$. The solution is then constructed via the Duhamel formula:

$$V(z) = V_+ + \int_M^z e^{(A_+-\mu_+)(z-y)}(A(y) - A_+)V(y)\, dy.$$

A similar argument as above shows that such a solution exists and satisfies the claimed bound. The lemma is proved. □

Exact Rayleigh solver.

Let us redefine the unstable solution $\phi_{\alpha,+}$ so that $\phi_{\alpha,+}(0) = 0$ and $\partial_z\phi_{\alpha,+}(0) = \alpha$. Hence, the Wronskian determinant is of order α. We introduce the (inviscid) Evans function

$$E(\alpha, c) := \alpha^{-1}W[\phi_{\alpha,-}, \phi_{\alpha,+}](0) = \phi_{\alpha,-}(0), \tag{3.5}$$

for each α, c, which is of order one. The last identity in the above is due to the normalization of the unstable solution $\phi_{\alpha,+}$. For real arguments x and z, a Green kernel $G_\alpha(x,z)$ of the Ray_α operator can be defined by

$$G_\alpha(x,z) = \frac{1}{\alpha E(\alpha,c)(U(x)-c)} \begin{cases} \phi_{\alpha,-}(z)\phi_{\alpha,+}(x), & \text{if} \quad z > x, \\ \phi_{\alpha,-}(x)\phi_{\alpha,+}(z), & \text{if} \quad z < x. \end{cases}$$

We denote by G_α^- the expression of G_α for $x < z$ and by G_α^+ the expression of G_α for $x > z$. The exact Rayleigh solver can then be defined by

$$Ray_\alpha^{-1}(f)(z) := \int_0^\infty G_\alpha(x,z)f(x)dx. \tag{3.6}$$

The following lemma gives bounds on this solution.

Lemma 3.3. *For any positive constant $\eta < \alpha$ and for each $f \in \mathcal{X}^\eta$, there holds on $z \ge 0$:*

$$|\partial_z^{\ell+k} Ray_\alpha^{-1}(f)(z)| \le C\alpha^{k-2}d_c^{-1}E(\alpha,c)^{-1}\|\partial_z^\ell f\|_\eta e^{-\eta z}, \tag{3.7}$$

for some positive constant C, and $k = 0, 1$, $\ell = 0, 1$, and d_c defined as in (1.13). *In addition, we have*

$$|\Delta_\alpha Ray_\alpha^{-1}(f)(z)| \le C\Big(1 + \alpha^{-2} d_c^{-1} E(\alpha, c)^{-1}\Big) d_c^{-1} \|f\|_\eta e^{-\eta z}. \tag{3.8}$$

Remark 3.4. Note that we "gain" bounds on two derivatives, which is natural since Rayleigh equation is a second order elliptic equation. We also gain α^{-2} for the Rayleigh solution, when α is sufficiently large.

Proof. By Lemma 3.2 and (1.13), we have

$$|\partial_z^k G_\alpha(x, z)| \le C\alpha^{k-1} d_c^{-1} E(\alpha, c)^{-1} e^{-\alpha|x-z|}, \tag{3.9}$$

for all $x, z \ge 0$, and for $k = 0$ or $k = 1$. This yields

$$\begin{aligned}\Big|\partial_z^k \int_0^\infty G_\alpha(x, z) f(x) dx\Big| &\le C\alpha^{k-1} d_c^{-1} E(\alpha, c)^{-1} \|f\|_\eta \int_0^\infty e^{-\alpha|x-z|} e^{-\eta x}\, dx \\ &\le C\alpha^{k-2} d_c^{-1} E(\alpha, c)^{-1} \|f\|_\eta e^{-\eta z}.\end{aligned}$$

This proves the first estimate stated in the lemma. For the second estimate, using the equation $Ray_\alpha(\phi) = 0$, we write

$$\Delta_\alpha Ray_\alpha^{-1}(f)(z) = \frac{f}{U - c} + \frac{U''}{U - c} Ray_\alpha^{-1}(f)(z).$$

This proves at once the second estimate in the lemma. □

Lemma 3.5. *For $f \in \mathcal{X}^\eta$, with $\eta = \alpha$, the Rayleigh solution $Ray_\alpha^{-1}(f)$ might not be in $\mathcal{X}^\alpha$. Precisely, we only have for $z \ge 0$:*

$$|\partial_z^{\ell+k} Ray_\alpha^{-1}(f)(z)| \le C\alpha^{k-1} d_c^{-1} E(\alpha, c)^{-1} \|\partial_z^\ell f\|_\alpha (1 + z) e^{-\alpha z},$$

with an extra linear growth in z.

Proof. The proof is similar to that of Lemma 3.3, upon noting that the linear growth in z is precisely due to the following estimate

$$\begin{aligned}\int_0^\infty e^{-\alpha|x-z|} e^{-\alpha x}\, dx &\le \int_0^z e^{-\alpha z}\, dx + \int_z^\infty e^{-2\alpha x} e^{\alpha z}\, dx \\ &\le z e^{-\alpha z} + \alpha^{-1} e^{-\alpha z}.\end{aligned}$$

□

Lemma 3.6. *For sufficiently large α, there are no nontrivial bounded solutions (α, c, ϕ) to the Rayleigh problem with $\Im c \gg e^{-\alpha}$.*

Proof. We study the Rayleigh equation when $\alpha \gg 1$. We introduce the change of variables

$$\tilde{z} = \alpha z, \qquad \phi(z) = \alpha^{-2}\tilde{\phi}(\alpha z).$$

Then, $\tilde{\phi}$ solves the scaled Rayleigh equation

$$\mathrm{Ray}_{sc}(\tilde{\phi}) := (U - c)(\partial_{\tilde{z}}^2 - 1)\tilde{\phi} - \alpha^{-2}U''\tilde{\phi} = 0$$

in which $U = U(\alpha^{-1}\tilde{z})$ and $U'' = U''(\alpha^{-1}\tilde{z})$. The inverse of $\mathrm{Ray}_{sc}(\cdot)$ is constructed by induction, starting from the inverse of $(U-c)(\partial_{\tilde{z}}^2 - 1)$. The iteration operator is defined by

$$T\tilde{\phi} := \alpha^{-2}\Big[(U - c)(\partial_{\tilde{z}}^2 - 1)\Big]^{-1} \circ U''\tilde{\phi}.$$

It suffices to show that T is contractive with respect to the $e^{\tilde{z}}$ or $e^{-\tilde{z}}$ weighted sup norm, denoted by $\|\cdot\|_{L^\infty_\pm}$, respectively for decaying or growing solutions. Indeed, recalling that $(\partial_{\tilde{z}}^2 - 1)^{-1} f = e^{-|\tilde{z}|} \star f$, we check that

$$\begin{aligned}
|T\tilde{\phi} - T\tilde{\psi}|(z) &\le \alpha^{-2}\int_0^\infty \Big|\frac{U''}{U-c}\Big| e^{-|\tilde{x}-\tilde{z}|}e^{\pm|\tilde{x}|}\|\tilde{\phi} - \tilde{\psi}\|_{L^\infty_\pm}\, d\tilde{x} \\
&\le \alpha^{-2}e^{\pm|\tilde{z}|}\|\tilde{\phi} - \tilde{\psi}\|_{L^\infty_\pm}\int_0^\infty \Big|\frac{U''(\alpha^{-1}\tilde{x})}{U(\alpha^{-1}\tilde{x}) - c}\Big|\, d\tilde{x} \\
&\le \alpha^{-1}e^{\pm|\tilde{z}|}\|\tilde{\phi} - \tilde{\psi}\|_{L^\infty_\pm}\int_0^\infty \Big|\frac{U''(x)}{U(x) - c}\Big|\, dx
\end{aligned}$$

in which the triangle inequality was used to deduce $e^{-|\tilde{x}-\tilde{z}|}e^{\pm|\tilde{x}|} \le e^{\pm|\tilde{z}|}$. As for the integral term, when c is away from the range of U, it follows that

$$\int_0^\infty \Big|\frac{U''(x)}{U(x) - c}\Big|\, dx \le C_0\int_0^\infty |U''|\, dx \le C.$$

In the case, when c is near the range of U, the integral of $1/(U-c)$ is bounded by $\log(U - c) \lesssim \log \Im c$. This proves that

$$|T\tilde{\phi} - T\tilde{\psi}|(z) \le C\alpha^{-1}|\log \Im c| e^{\pm|\tilde{z}|}\|\tilde{\phi} - \tilde{\psi}\|_{L^\infty_\pm}.$$

Hence, T is contractive as long as $\alpha^{-1}\log \Im c$ is sufficiently small. Similarly, the same bound holds for derivatives, since the derivative of the Green kernel of $\partial_{\tilde{z}}^2 - 1$ satisfies the same bound. This yields two independent Rayleigh solutions of the form

$$\partial_{\tilde{z}}^k \tilde{\phi}^0_{s,\pm}(\tilde{z}) = (-1)^k e^{\pm\tilde{z}}\Big(1 + \mathcal{O}(\alpha^{-1}\log \Im c)\Big), \qquad k = 0, 1. \tag{3.10}$$

In particular, this shows that there are no non-trivial bounded solutions to the Rayleigh problem with the zero boundary condition, as long as $\alpha^{-1}\log \Im c \ll 1$, since $\tilde{\phi}^0_{s,-}(0) \neq 0$. This yields the lemma. □

3.3 Airy operator

Let us focus in this section on the Airy type operator

$$\mathrm{Airy}(\phi) = (U - c)\Delta_\alpha \phi - \varepsilon \Delta_\alpha^2 \phi.$$

Recall that we consider the range of c so that $U - c$ never vanishes; see (1.13). We obtain the following.

Proposition 3.7. *For $\eta < \alpha$, there exists an operator Airy^{-1} defined on $\mathcal{X}^\eta$ such that for any $f \in \mathcal{X}^\eta$, the function*

$$\phi = \mathrm{Airy}^{-1}(f)$$

is a solution of

$$\mathrm{Airy}(\phi) = f.$$

Furthermore, for $k, \ell = 0, 1$, there hold the following uniform bounds

$$\|\partial_z^k \mathrm{Airy}^{-1}(\partial_z^\ell f)\|_\eta \le C|\epsilon m_f^2|^{-1}\alpha^{k-2} M_f^\ell \|f\|_\eta$$

and

$$\|\Delta_\alpha \mathrm{Airy}^{-1}(\partial_z^\ell f)\|_\eta \le C|\epsilon m_f^2|^{-1} M_f^\ell \|f\|_\eta. \tag{3.11}$$

Here, m_f, M_f are defined as in (2.1).

Proof. We write

$$\mathrm{Airy}(\phi) = \mathcal{P}\Delta_\alpha \phi, \qquad \mathcal{P} := -\epsilon \Delta_\alpha + U - c$$

Hence, four independent fundamental solutions to the Airy equation are two solutions $e^{\pm\alpha\phi}$ from $\Delta_\alpha \phi = 0$ and the other two from solving $\Delta_\alpha \phi = \psi$, with $\mathcal{P}\psi = 0$. Let $G_{\mathrm{a}}(x, z)$ be a Green function of $\mathrm{Airy}(\cdot)$, without taking care of the boundary conditions. Then, $\mathcal{G}_a(x, z) = \Delta_\alpha G_{\mathrm{a}}(x, z)$ is the Green function of the operator $\mathcal{P}$, and therefore

$$\mathcal{G}_{\mathrm{a}}(x, z) = \frac{1}{\varepsilon W[\psi_+, \psi_-](x)} \begin{cases} \psi_-(x)\psi_+(z), & \text{if} \quad x > z, \\ \psi_+(z)\psi_-(x), & \text{if} \quad x < z, \end{cases} \tag{3.12}$$

in which $\psi_\pm$ are growing and decaying solutions to $\mathcal{P}\psi = 0$. It is standard to construct such solutions $\psi_\pm$ (see, for instance, Section 3.5), yielding

$$\psi_\pm(z) = e^{\pm \int_0^z \mu_f(y)\, dy}\Big[1 + \mathcal{O}(\sqrt{\epsilon} e^{-\eta_0 z})\Big],$$

with functions $\mu_f(z)$ having positive real part and defined through $\epsilon\mu_f^2 = U(z) - c + \epsilon\alpha^2$. We note that the Wronskian determinant $W[\psi_+, \psi_-](z)$ is a constant, and satisfies

$$W[\psi_+, \psi_-](z) = \lim_{z\to\infty} W[\psi_+, \psi_-](z) = \lim_{z\to\infty} \mu_f(z).$$

Putting these bounds into (3.12), we obtain

$$|\partial_x^\ell \partial_z^k \mathcal{G}_{\rm a}(x,z)| \le \frac{C|\mu_f(z)|^k |\mu_f(x)|^\ell}{|\epsilon m_f|} e^{-|\int_x^z \Re \mu_f(y)\, dy|} \tag{3.13}$$

for $k, \ell = 0, 1$. For higher order derivatives, we use the fact that

$$\partial_z^2 \mathcal{G}_{\rm a}(x,z) = \mu_f^2(z)\mathcal{G}_{\rm a}(x,z), \qquad \partial_x^2 \mathcal{G}_{\rm a}(x,z) = \mu_f^2(x)\mathcal{G}_{\rm a}(x,z). \tag{3.14}$$

In addition, we remark that $\epsilon\partial_z\mu_f^2 = U'$, $1 \le \alpha \ll \mu_f$, and $\alpha|U-c| \gtrsim 1$. This yields

$$\alpha\epsilon\mu_f^2 \gtrsim 1.$$

In particular, $\partial_z\mu_f^2 \lesssim \epsilon^{-1} \le \mu_f^3(z)$. Hence, (3.13) holds for $k = 3$, by differentiating (3.14). Similar estimates hold for all $k, \ell \ge 0$. Together with the notation (2.1), the bound (3.13) becomes

$$|\partial_x^\ell \partial_z^k \mathcal{G}_{\rm a}(x,z)| \le C|\epsilon m_f|^{-1} M_f^{k+\ell} e^{-m_f|x-z|} \tag{3.15}$$

for $k, \ell \ge 0$.

We then construct the $\mathrm{Airy}^{-1}(\cdot)$ by

$$\mathrm{Airy}^{-1} f(z) = \Delta_\alpha^{-1}\mathcal{P}^{-1} f(z) = \Delta_\alpha^{-1}\int_0^\infty \mathcal{G}_{\rm a}(x,z) f(x)\, dx \tag{3.16}$$

in which Δ_α^{-1} is defined through its Green kernel, which is $\frac{1}{\alpha}e^{-\alpha|x-z|}$. Let us first give bounds on $\mathcal{P}^{-1}f$. For $f \in \mathcal{X}^\eta$ and for $k, \ell = 0, 1$, we have

$$\begin{aligned}
|\partial_z^k \mathcal{P}^{-1}\partial_x^\ell f(z)| &\le \int_0^\infty |\partial_x^\ell \partial_z^k \mathcal{G}_{\rm a}(x,z)||f(x)|\, dx \\
&\le C\|f\|_\eta \int_0^\infty |\epsilon m_f|^{-1} M_f^{k+\ell} e^{-m_f|x-z|} e^{-\eta|x|}\, dx \\
&\le C|\epsilon m_f^2|^{-1} M_f^{k+\ell}\|f\|_\eta e^{-\eta|z|}.
\end{aligned}$$

For $k, \ell = 0, 1$, using the bounds on $\mathcal{P}^{-1}f$ into (3.16) yields

$$\begin{aligned}
|\partial_z^k \mathrm{Airy}^{-1}\partial_z^\ell f(z)| &\le \int_0^\infty \alpha^{k-1} e^{-\alpha|x-z|} |\mathcal{P}^{-1}\partial_x^\ell f(x)|\, dx \\
&\le C|\epsilon\mu_f^2|^{-1} M_f^\ell \|f\|_\eta \int_0^\infty \alpha^{k-1} e^{-\alpha|x-z|} e^{-\eta x}\, dx \\
&\le C|\epsilon m_f^2|^{-1} M_f^\ell \alpha^{k-2}\|f\|_\eta e^{-\eta z}
\end{aligned}$$

in which we assumed $\eta < \alpha$. Finally, by definition, we note that

$$\Delta_\alpha \mathrm{Airy}^{-1} f = \mathcal{P}^{-1} f.$$

The proposition follows. $\square$

3.4 Slow modes

In this section, we iteratively construct "slow" modes $\phi_{s,\pm} \approx e^{\pm\alpha z}$ of the Orr-Sommerfeld equations, starting from the Rayleigh solutions: $\phi_{\alpha,\pm} \approx e^{\pm\alpha z}$ constructed in Section 3.2.

Proposition 3.8. *Let $\phi_{\alpha,\pm} = e^{\pm\alpha z}(1+\psi_{\alpha,\pm})$ be the two independent Rayleigh solutions constructed in Proposition 3.2, and let $E(\alpha, c)$ be the Wronskian determinant of the two solutions (which is independent of z). For sufficiently large Reynolds number R such that*

$$\alpha m_f^{-2}(1 + E(\alpha, c)^{-1}) \ll 1 \tag{3.17}$$

(recalling that $m_f^2 \gtrsim \alpha^2 + R$), there exist two independent solutions $\phi_{s,\pm}(z)$ which solve the Orr-Sommerfeld equations

$$\mathrm{OS}(\phi_{s,\pm}) = 0,$$

and are close to the Rayleigh solutions $\phi_{\alpha,\pm}$. Precisely, there is some constant $\eta > 0$ so that we can write

$$\phi_{s,\pm} = e^{\pm\alpha z}\Big[1 + \psi_{s,\pm}\Big]$$

with $\|\psi_{s,\pm} - \psi_{\alpha,\pm}\|_\eta \lesssim \alpha m_f^{-2} E(\alpha, c)^{-1}$, uniformly for all (α, c) within (1.13).

Proof. Let us search for solutions ϕ of the form

$$\phi = e^{\pm\alpha z}\psi.$$

for each separate $+$ and $-$ case. We set

$$\Delta_{\alpha,\pm}\psi := e^{\mp\alpha z}\Delta_\alpha(e^{\pm\alpha z}\psi) = (\partial_z^2 \pm 2\alpha\partial_z)\psi$$

and introduce the conjugated Orr-Sommerfeld operator

$$\mathrm{OS}_\pm(\psi) := -\epsilon\Delta_{\alpha,\pm}^2\psi + (U - c)\Delta_{\alpha,\pm}\psi - U''\psi.$$

It follows that $\phi = e^{\pm\alpha z}\psi$ solves the Orr-Sommerfeld equations if and only if ψ solves the conjugated equation $\mathrm{OS}_\pm(\psi) = 0$.

We shall construct a solution to $\mathrm{OS}_\pm(\psi) = 0$ in the function space $1+\mathcal{X}^\eta$, for some positive η. We start our construction from a Rayleigh solution. That is, let $\psi_0 \in 1 + \mathcal{X}^{\eta_0}$ be as in Proposition 3.2, solving

$$\mathrm{Ray}_{\alpha,\pm}(\psi_0) = (U - c)\Delta_{\alpha,\pm}\psi_0 - U''\psi_0 = 0.$$

By definition, $\mathrm{OS}_\pm(\cdot) = \mathrm{Ray}_{\alpha,\pm}(\cdot) - \epsilon\Delta^2_{\alpha,\pm}$, and hence we have

$$\begin{aligned}\mathrm{OS}_\pm(\psi_0) &= -\epsilon\Delta^2_{\alpha,\pm}\psi_0 = -\epsilon\Delta_{\alpha,\pm}\Big[\frac{U''\psi_0}{U-c}\Big]\\ &= -\epsilon(\partial_z^2 \pm 2\alpha\partial_z)\Big[\frac{U''\psi_0}{U-c}\Big] =: E_0(z)\end{aligned}$$

in which the error term E_0 on the right is of order $\epsilon(1+\alpha)d_c^{-1}(1+d_c^{-2})$ with respect to $\mathcal{X}^{\eta_0}$ norm. Here, d_c is the distance from c to the range of U.

Next, by induction, let us assume that we have constructed ψ_n so that

$$\mathrm{OS}_\pm(\psi_n) = E_n(z),$$

with a sufficiently small error term E_n in X^η. We first correct the error term, using the Rayleigh operator. Define

$$\Psi_n := -\mathrm{Ray}^{-1}_{\alpha,\pm}(E_n) = -e^{\mp\alpha z}\mathrm{Ray}^{-1}_\alpha(e^{\pm\alpha z}E_n) + C_\pm \tag{3.18}$$

for some constants $C_\pm$ added to ensure that Ψ_n decays at infinity. Here, the inverse of Ray_α is constructed as in Lemma 3.3. We prove that

Lemma 3.9. *There are $C_\pm$ (depending on E_n) so that*

$$\begin{aligned}\|\partial_z^k\mathrm{Ray}^{-1}_{\alpha,\pm}(E_n)\|_\eta &\le C\alpha^k E(\alpha,c)^{-1}\|E_n\|_\eta,\\ \|\Delta_{\alpha,\pm}\mathrm{Ray}^{-1}_{\alpha,\pm}(E_n)\|_\eta &\le C(1+E(\alpha,c)^{-1})d_c^{-1}\|E_n\|_\eta,\end{aligned} \tag{3.19}$$

for $k = 0, 1$.

Proof. Recall from (3.9) the Green function estimate

$$|\partial_z^k G_\alpha(x,z)| \le C\alpha^{k-1}d_c^{-1}E(\alpha,c)^{-1}e^{-\alpha|x-z|}.$$

Using this and (3.6), we have

$$\begin{aligned}&|\partial_z^k\mathrm{Ray}^{-1}_\alpha(e^{\alpha z}E_n)|\\ &\le C\alpha^{k-1}d_c^{-1}E(\alpha,c)^{-1}\|f\|_\eta\Big(\int_0^z + \int_z^\infty\Big)e^{-\alpha|x-z|}e^{-\eta x}e^{\alpha x}\,dx\\ &\le C\alpha^{k-1}d_c^{-1}E(\alpha,c)^{-1}\|f\|_\eta e^{\alpha z}e^{-\eta z},\end{aligned}$$

in which we stress that we do not gain an extra factor of α^{-1} from the integration, as $e^{\alpha z}$ is in the kernel of the Laplacian Δ_α. In particular, $e^{-\alpha z}\mathrm{Ray}_\alpha^{-1}(e^{\alpha z}E_n)$ decays exponentially at infinity, and hence the constant C_+ in (3.18) is taken to be simply zero. This proves

$$\|\partial_z^k \mathrm{Ray}_{\alpha,+}^{-1}(E_n)\|_\eta \le C\alpha^{k-1}d_c^{-1}E(\alpha,c)^{-1}\|E_n\|_\eta.$$

The first estimate in (3.19) for $\mathrm{Ray}_{\alpha,+}^{-1}(E_n)$ follows, upon using the fact that $\alpha d_c \gtrsim 1$ for all (α, c) satisfying (1.13).

Similarly, we compute

$$\begin{aligned}
&|\mathrm{Ray}_\alpha^{-1}(e^{-\alpha z}E_n)| \\
&\le C\alpha^{-1}d_c^{-1}E(\alpha,c)^{-1}\|f\|_\eta\Big(\int_0^z + \int_z^\infty\Big)e^{-\alpha|x-z|}e^{-\eta x}e^{-\alpha x}\,dx \\
&\le C\alpha^{-1}d_c^{-1}E(\alpha,c)^{-1}\|f\|_\eta e^{-\alpha z}(1+e^{-\eta z}).
\end{aligned}$$

In particular, $e^{\alpha z}\mathrm{Ray}_\alpha^{-1}(e^{-\alpha z}E_n)$ converges to a nonzero constant C_-, as $z\to\infty$. Thus, as above, we have

$$\|\partial_z^k \mathrm{Ray}_{\alpha,-}^{-1}(E_n)\|_\eta \le C\alpha^{k-1}d_c^{-1}E(\alpha,c)^{-1}\|E_n\|_\eta.$$

Finally, as for the second estimate in (3.19), we recall that $\Delta_{\alpha,\pm}\mathrm{Ray}_{\alpha,\pm}^{-1}(\cdot)$ is the conjugated operator of $\Delta_\alpha \mathrm{Ray}_\alpha^{-1}(\cdot)$. Thus, by the definition of Ray_α^{-1}, we compute

$$\Delta_{\alpha,\pm}\mathrm{Ray}_{\alpha,\pm}^{-1}(E_n) = \frac{E_n}{U-c} + \frac{U''}{U-c}\mathrm{Ray}_{\alpha,\pm}^{-1}(E_n).$$

The second estimate in (3.19) follows. □

By definition, we then have

$$\mathrm{OS}_\pm(\Psi_n) = -E_n + \varepsilon\Delta_{\alpha,\pm}^2\Psi_n = -E_n + \varepsilon\Delta_{\alpha,\pm}^2\mathrm{Ray}_{\alpha,\pm}^{-1}(E_n)$$

and so

$$\mathrm{OS}_\pm(\psi_n+\Psi_n) = \varepsilon\Delta_{\alpha,\pm}^2\mathrm{Ray}_{\alpha,\pm}^{-1}(E_n)$$

which leads to a loss of two derivatives in the estimates. We now need to correct this error term, using the two highest derivative terms in the Orr-Sommerfeld operator $\mathrm{OS}_\pm(\cdot)$. Let us introduce a new corrector Φ_n defined by

$$\Phi_n := -\varepsilon\mathrm{Airy}_\pm^{-1}\Delta_{\alpha,\pm}^2\mathrm{Ray}_{\alpha,\pm}^{-1}(E_n), \tag{3.20}$$

in which $\mathrm{Airy}_\pm$ is defined by $\mathrm{Airy}_\pm(\Phi_n) := \mathrm{OS}_\pm(\Phi_n) + U''\Phi_n$, the two highest derivative terms in $\mathrm{OS}_\pm$. By definition, $\mathrm{Airy}_\pm(\Phi_n) = e^{\mp\alpha z}\mathrm{Airy}(e^{\pm\alpha z}\Phi_n)$. Proposition 3.7 yields

$$\begin{aligned} \|\partial_z^k \mathrm{Airy}_\pm^{-1}(\partial_z^\ell f)\|_\eta &\le C|\epsilon m_f^2|^{-1}\alpha^{k-2}M_f^\ell \|f\|_\eta \\ \|\partial_z^{k+2} \mathrm{Airy}_\pm^{-1}(\partial_z^\ell f)\|_\eta &\le C|\epsilon m_f^2|^{-1}\alpha^{k}M^{k+\ell} \|f\|_\eta \end{aligned} \tag{3.21}$$

for $k, \ell = 0, 1, 2$. This introduction of Φ_n leads to

$$\mathrm{OS}_\pm(\psi_n + \Psi_n + \Phi_n) = -U''\Phi_n = -\varepsilon U'' \mathrm{Airy}_\pm^{-1}\Delta_{\alpha,\pm}^2 \mathrm{Ray}_{\alpha,\pm}^{-1}(E_n).$$

That is, we can continue our inductive construction by defining

$$\psi_{n+1} = \psi_n + \Psi_n + \Phi_n.$$

This approximately solves the Orr-Sommerfeld equations:

$$\mathrm{OS}_\pm(\psi_{n+1}) = E_{n+1}$$

leaving the error

$$E_{n+1} = \mathrm{Iter}(E_n), \qquad \mathrm{Iter}(\cdot) := -\varepsilon U'' \mathrm{Airy}_\pm^{-1}\Delta_{\alpha,\pm}^2 \mathrm{Ray}_{\alpha,\pm}^{-1}(\cdot).$$

We claim that the Iter operator is contractive in suitable function space. Precisely, we prove the following.

Lemma 3.10. *There holds*

$$\|\mathrm{Iter}(f)\|_{\eta,2} \le C\alpha m_f^{-2}(1 + E(\alpha,c)^{-1})\|f\|_{\eta,2} \tag{3.22}$$

in which $\|\cdot\|_{\eta,2}$ *is defined by*

$$\|f\|_{\eta,2} = \sum_{k=0}^{2} \alpha^{-k}\|\partial_z^k f\|_\eta. \tag{3.23}$$

Proof. We shall use repeatedly that $\alpha d_c \gtrsim 1$ for all (α, c) satisfying (1.13). By definition and the estimates (3.21), we obtain

$$\begin{aligned} \|\partial_z^k \mathrm{Iter}(f)\|_\eta &\le C\alpha^{k-2}m_f^{-2}\|\Delta_{\alpha,\pm}^2 \mathrm{Ray}_{\alpha,\pm}^{-1}(f)\|_\eta \\ &\le C\alpha^{k-2}m_f^{-2}\Big\|\Delta_{\alpha,\pm}\Big[\frac{f}{U-c} + \frac{U''\mathrm{Ray}_{\alpha,\pm}^{-1}(f)}{U-c}\Big]\Big\|_\eta. \end{aligned} \tag{3.24}$$

We compute

$$
\begin{aligned}
&\Big\|\Delta_{\alpha,\pm}\Big[\frac{f}{U-c}\Big]\Big\|_\eta \\
&\quad\le C\Big[(d_c^{-3}+\alpha d_c^{-2})\|f\|_\eta + (d_c^{-2}+\alpha d_c^{-1})\|\partial_z f\|_\eta + d_c^{-1}\|\partial_z^2 f\|_\eta\Big] \\
&\quad\le C\Big[d_c^{-3}+\alpha d_c^{-2}+\alpha^2 d_c^{-1}\Big]\|f\|_{\eta,2}.
\end{aligned}
$$

Using $\alpha d_c \gtrsim 1$, we thus obtain

$$
\Big\|\Delta_{\alpha,\pm}\Big[\frac{f}{U-c}\Big]\Big\|_\eta \le C\alpha^3\|f\|_{\eta,2}. \tag{3.25}
$$

Similarly, we compute

$$
\begin{aligned}
&\Big\|\Delta_{\alpha,\pm}\Big[\frac{U''\mathrm{Ray}_{\alpha,\pm}^{-1}(f)}{U-c}\Big]\Big\|_\eta \\
&\quad\le C\Big[(d_c^{-3}+\alpha d_c^{-2})\|\mathrm{Ray}_{\alpha,\pm}^{-1}(f)\|_\eta + (d_c^{-2}+\alpha d_c^{-1})\|\partial_z \mathrm{Ray}_{\alpha,\pm}^{-1}(f)\|_\eta \\
&\qquad + d_c^{-1}\|\Delta_{\alpha,\pm}\mathrm{Ray}_{\alpha,\pm}^{-1}(f)\|_\eta\Big].
\end{aligned}
$$

Using the estimates in (3.19), we obtain

$$
\begin{aligned}
&\Big\|\Delta_{\alpha,\pm}\Big[\frac{U''\mathrm{Ray}_{\alpha,\pm}^{-1}(f)}{U-c}\Big]\Big\|_\eta \\
&\quad\le C\Big[(d_c^{-3}+\alpha d_c^{-2}) + \alpha(d_c^{-2}+\alpha d_c^{-1}) + d_c^{-2}\Big](1+E(\alpha,c)^{-1})\|f\|_\eta \\
&\quad\le C\alpha^3(1+E(\alpha,c)^{-1})\|f\|_\eta.
\end{aligned}
$$

Putting this and the estimate (3.25) into (3.24), we obtain

$$
\|\partial_z^k \mathrm{Iter}(f)\|_\eta \le C\alpha^{k+1} m_f^{-2}(1+E(\alpha,c)^{-1})\|f\|_{\eta,2}
$$

for $k=0,1,2$. This proves the lemma, by using the definition of $\|\cdot\|_{\eta,2}$. □

By view of the assumption (3.17) that $\alpha m_f^{-2}(1+E(\alpha,c)^{-1}) \ll 1$, Lemma 3.10 thus proves that Iter operator is indeed contractive with respect to the norm $\|\cdot\|_{\eta,2}$. In particular, E_n converges to zero in $\mathcal{X}^\eta$ and thus the series $\phi_n = e^{\pm\alpha z}\psi_n$ also converges to an exact solution of the Orr-Sommerfeld equations. This proves Proposition 3.8. □

3.5 Fast modes

In this section, we shall construct two independent solutions, which asymptotically behave as $e^{\pm\mu_f z}$, of the Orr-Sommerfeld equation: $\mathrm{OS}(\phi) = 0$.

Proposition 3.11. *For sufficiently large R, there exist two independent solutions $\phi_{f,\pm}(z)$ which solve the Orr-Sommerfeld equations*

$$\mathrm{OS}(\phi_{f,\pm}) = 0,$$

so that $\phi_{f,\pm}$ satisfy

$$\phi_{f,\pm}(z) = e^{\pm\int_0^z \hat\mu_f(y)\,dy}\Big[1 + \mathcal{O}(\sqrt{\epsilon})e^{-\eta_0 z}\Big],$$

in which

$$\hat\mu_f(z)^2 = \frac{1}{\epsilon}(U - c + 2\alpha^2\epsilon)(1 + \mathcal{O}(\epsilon)).$$

Proof. We rewrite the Orr-Sommerfeld equation as

$$\mathrm{OS}(\phi) := -\epsilon\partial_z^4\phi + b(z)\partial_z^2\phi - a(z)\phi = 0,$$

in which for convenience we have denoted $a(z) := \alpha^2(\epsilon\alpha^2 + U - c) + U''$ and $b(z) := (U - c + 2\epsilon\alpha^2)$. We make a change of variables:

$$\tilde z = \frac{z}{\sqrt{\epsilon}}, \qquad \phi(z) = \tilde\phi(\frac{z}{\sqrt{\epsilon}}).$$

The equation for $\tilde\phi$ reads

$$-\partial_{\tilde z}^4\tilde\phi + b(\sqrt{\epsilon}\tilde z)\partial_{\tilde z}^2\tilde\phi - \epsilon a(\sqrt{\epsilon}\tilde z)\tilde\phi = 0.$$

It is convenient to write the above equation as a first order ode system. We introduce

$$W = (\tilde\phi, \tilde\phi', \tilde\phi'', \tilde\phi'''),$$

with $' = \partial_{\tilde z}$. The system for W reads

$$\partial_{\tilde z} W = \mathcal{A}(\tilde z)W, \qquad \mathcal{A}(\tilde z) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\epsilon a(\sqrt{\epsilon}\tilde z) & 0 & b(\sqrt{\epsilon}\tilde z) & 0 \end{pmatrix}. \tag{3.26}$$

We note that within (1.13), $\alpha b(\sqrt{\epsilon}\tilde z)$ is bounded away from zero, for $\tilde z \ge 0$. Let us set

$$\tilde\mu_s^2(\tilde z) := \frac{\epsilon a(\sqrt{\epsilon}\tilde z)}{b(\sqrt{\epsilon}\tilde z)}(1 + \mathcal{O}(\epsilon)), \qquad \tilde\mu_f^2(\tilde z) := b(\sqrt{\epsilon}\tilde z)(1 + \mathcal{O}(\epsilon))$$

to be the eigenvalues of the matrix $\mathcal{A}$. By convention, we shall always take $\tilde\mu_s, \tilde\mu_f$ to be the eigenvalues with positive real part. Observe that

$$\tilde\mu_s^2 \approx \epsilon\alpha^2$$

and

$$\tilde\mu_f^2 \approx \epsilon\alpha^2 + U - c = \epsilon\Big(\alpha^2 + \frac{U-c}{\epsilon}\Big) = \epsilon\Big(\alpha^2 + \frac{\alpha(U-c)}{\alpha\epsilon}\Big).$$

Recall that $\alpha|U-c|$ is bounded below away from zero and $i\alpha\epsilon = \nu \ll 1$. Hence, in particular, the above yields

$$\alpha\tilde\mu_f^2 \gtrsim 1, \qquad \tilde\mu_s \ll \tilde\mu_f. \tag{3.27}$$

The last inequality implies that the slow and fast modes are well separated. We let $\tilde\mu_{f,\pm} = \pm\tilde\mu_f$ and set

$$\mathcal{T}_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde\mu_{f,\pm} & 1 & 0 & 0 \\ \tilde\mu_{f,\pm}^2 & 0 & 1 & 0 \\ \tilde\mu_{f,\pm}^3 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{T}_\pm^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\tilde\mu_{f,\pm} & 1 & 0 & 0 \\ -\tilde\mu_{f,\pm}^2 & 0 & 1 & 0 \\ -\tilde\mu_{f,\pm}^3 & 0 & 0 & 1 \end{pmatrix}$$

and for each $+/-$ case, we introduce the "partial" diagonalization

$$W = \mathcal{T}_\pm V.$$

The function V then solves

$$\partial_{\tilde z} V = \begin{pmatrix} \tilde\mu_{f,\pm} & 0 \\ 0 & \mathcal{A}_1 \end{pmatrix} V - \mathcal{T}_\pm^{-1}\partial_{\tilde z}\mathcal{T}_\pm V,$$

in which $\mathcal{A}_1$ is defined as the lower 3×3 block of the matrix $\mathcal{T}_\pm^{-1}\mathcal{A}\mathcal{T}_\pm$, and the first row of the matrix $\mathcal{T}_\pm^{-1}\partial_{\tilde z}\mathcal{T}_\pm$ is zero. In addition, $\partial_{\tilde z}\tilde\mu_f^2 = \sqrt{\epsilon}U'(\sqrt{\epsilon}\tilde z)$, with $|U'(z)| \lesssim e^{-\eta_0|z|}$. Thus,

$$\mathcal{T}_\pm^{-1}\partial_{\tilde z}\mathcal{T}_\pm V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \partial_z\tilde\mu_{f,\pm} & 0 & 0 & 0 \\ \partial_z\tilde\mu_{f,\pm}^2 & 0 & 0 & 0 \\ \partial_z\tilde\mu_{f,\pm}^3 & 0 & 0 & 0 \end{pmatrix} V = \mathcal{O}(\sqrt{\epsilon}e^{-\eta_0\sqrt{\epsilon}\tilde z}V) \begin{pmatrix} 0 \\ \tilde\mu_f^{-1} \\ 1 \\ \tilde\mu_f \end{pmatrix}.$$

In particular, this proves

$$\mathcal{T}_\pm^{-1}\partial_{\tilde z}\mathcal{T}_\pm = \mathcal{O}(\sqrt{\epsilon}e^{-\eta_0\sqrt{\epsilon}\tilde z}V)(\tilde\mu_f + \tilde\mu_f^{-1}). \tag{3.28}$$

We further introduce

$$V = e^{\int_0^{\tilde z} \tilde\mu_{f,\pm}(y)\,dy} Z$$

in order to substract the natural growth of V. Then, Z solves

$$\partial_{\tilde z} Z = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{A}_1 - \tilde\mu_{f,\pm} I \end{pmatrix} Z - \mathcal{T}_\pm^{-1}\partial_{\tilde z}\mathcal{T}_\pm Z. \tag{3.29}$$

We write $Z = (Z_1, \tilde Z)$. Then, the above system yields $Z_1(\tilde z) = 1$ and

$$\tilde Z(\tilde z) = \tilde Z(M_\pm) + \int_{M_\pm}^{\tilde z} e^{(\mathcal{A}_1 - \tilde\mu_{f,\pm})(\tilde z - y)} \mathcal{T}_\pm^{-1}\partial_{\tilde z}\mathcal{T}_\pm Z\, dy, \tag{3.30}$$

for $M_\pm$ depending on each $+/-$ case.

The $-$ case. We observe that the eigenvalues of $\mathcal{A}_1$ are the same as those of $\mathcal{A}$ except for the eigenvalue $\tilde\mu = \tilde\mu_{f,-}$. Thus, the eigenvalues of $\mathcal{A}_1 - \tilde\mu_{f,-}$ have real parts that are all positive and bounded away from zero by $\theta_0 \tilde m_f$, with $\tilde m_f = \inf_z \tilde\mu_f(z)$ (since all the eigenvalues are well separated, with $\tilde\mu_s \ll \tilde\mu_f$). This proves that there is a positive constant θ_0 so that

$$\|e^{(\mathcal{A}_1 - \tilde\mu_{f,-})y}\| \le C e^{-\theta_0 \tilde m_f |y|}, \qquad y \le 0.$$

Taking $M_- = \infty$ in (3.30), we get

$$\tilde Z(\tilde z) = \tilde Z_- - \int_{\tilde z}^{\infty} e^{(\mathcal{A}_1 - \tilde\mu_{f,-})(\tilde z - y)} \mathcal{O}(\sqrt{\epsilon} e^{-\eta_0\sqrt{\epsilon}\tilde z} |\tilde Z|)(\tilde\mu_f + \tilde\mu_f^{-1})\, dy.$$

Now, let us denote by $T\tilde Z$ the right-hand side of the above integral form. We shall show that T is contractive from $L^\infty([0,\infty))$ into itself. We consider two cases.

Case 1: $\epsilon\alpha^2 \ll 1$. In this case, we note that $\tilde\mu_f$ is bounded. Hence, by definition of T, we compute for any $\tilde Z, \tilde Y \in L^\infty([0,\infty))$,

$$\begin{aligned} |(T\tilde Z - T\tilde Y)(\tilde z)| &\le \int_{\tilde z}^{\infty} \|e^{(\mathcal{A}_1 - \tilde\mu_{f,-})(\tilde z - y)}\| \mathcal{O}(\sqrt{\epsilon}\tilde\mu_f^{-1} e^{-\eta_0\sqrt{\epsilon} y}) |\tilde Z - \tilde Y|(y)\, dy \\ &\le C\sqrt{\epsilon}\tilde m_f^{-1} e^{-\eta_0\sqrt{\epsilon}\tilde z} \int_{\tilde z}^{\infty} e^{-\theta_0 \tilde m_f |y - \tilde z|} |\tilde Z - \tilde Y|(y)\, dy \\ &\le C\sqrt{\epsilon}\tilde m_f^{-2} e^{-\eta_0\sqrt{\epsilon}\tilde z} \|Z - Y\|_{L^\infty}. \end{aligned}$$

Now recall from (3.27) that $\alpha\tilde{\mu}_f^2 \gtrsim 1$. Thus, $\sqrt{\epsilon}\tilde{m}_f^{-2} \lesssim \sqrt{\epsilon\alpha^2} \ll 1$. This proves that T is a contractive map, and thus there exists a (unique) solution $Z \in L^\infty([0,\infty))$ solving (3.29) in the $-$ case. Furthermore, there holds

$$|Z(\tilde{z}) - Z_-| \le C\sqrt{\epsilon}\tilde{m}_f^{-2} e^{-\eta_0\sqrt{\epsilon}\tilde{z}}, \qquad \tilde{z} \ge 0.$$

Case 2: $\epsilon\alpha^2 \gtrsim 1$. In this case, $\tilde{\mu}_f \gtrsim 1$. Hence, we compute

$$\begin{aligned} |(T\tilde{Z} - T\tilde{Y})(\tilde{z})| &\le \int_{\tilde{z}}^\infty \|e^{(\mathcal{A}_1 - \tilde{\mu}_{f,-})(\tilde{z}-y)}\| \mathcal{O}(\sqrt{\epsilon}\tilde{\mu}_f e^{-\eta_0\sqrt{\epsilon}y}) |\tilde{Z} - \tilde{Y}|(y)\, dy \\ &\le C\sqrt{\epsilon} e^{-\eta_0\sqrt{\epsilon}\tilde{z}} \int_{\tilde{z}}^\infty \tilde{m}_f e^{-\theta_0 \tilde{m}_f |y-\tilde{z}|} |\tilde{Z} - \tilde{Y}|(y)\, dy \\ &\le C\sqrt{\epsilon} e^{-\eta_0\sqrt{\epsilon}\tilde{z}} \|Z - Y\|_{L^\infty}. \end{aligned}$$

The contraction of T thus follows.

The + case. In this case, the eigenvalues of $\mathcal{A}_1 - \tilde{\mu}_{f,+}$ have real parts that are all negative and bounded away from zero by Cm_f. This yields

$$\|e^{(\mathcal{A}_1 - \tilde{\mu}_{f,+})y}\| \le C e^{-\theta_0 m_f |y|}, \qquad y \ge 0,$$

for some $\theta_0 > 0$. Taking $M_+ = 0$ in (3.30), we get

$$\tilde{Z}(\tilde{z}) = \tilde{Z}_+ + \int_0^{\tilde{z}} e^{(\mathcal{A}_1 - \tilde{\mu}_{f,+})(\tilde{z}-y)} \mathcal{O}(\sqrt{\epsilon} e^{-\eta_0\sqrt{\epsilon}y}) Z(y)(\tilde{\mu}_f + \tilde{\mu}_f^{-1})\, dy.$$

The same argument as done in the $-$ case again yields the existence of solution $Z \in L^\infty([0,\infty))$ solving (3.29).

Going back to the variable W, we obtain two independent solutions $W_\pm$ to (3.26) such that

$$W_\pm(\tilde{z}) = e^{\int_0^{\tilde{z}} \tilde{\mu}_{f,\pm}(y)\, dy} \mathcal{T}_\pm \Big[Z_\pm + \mathcal{O}(\sqrt{\epsilon} e^{-\theta_0\sqrt{\epsilon}\tilde{z}}) \Big],$$

for arbitrary constants $Z_\pm$. By rescaling to the original variables, the proposition follows at once. □

3.6 Proof of Proposition 1.1

We are ready to construct a solution to the Orr-Sommerfeld problem, with $\Im c > 0$ and prove our instability result, assuming that the background profile U is spectrally unstable to the corresponding Rayleigh problem. Indeed, we let ϕ_s and ϕ_f be two slow and fast modes that decay at infinity. Then, a bounded solution to the Orr-Sommerfeld problem is constructed as a linear combination of these two decaying modes:

$$\phi = A\phi_s + B\phi_f.$$

The zero boundary conditions on ϕ yield the linear dispersion relation:

$$\frac{\phi_s}{\phi_s'}\Big|_{z=0} = \frac{\phi_f}{\phi_f'}\Big|_{z=0}. \tag{3.31}$$

We need to show that there are choices of α, c so that the above relation holds for ϵ sufficiently small.

From the construction of fast decaying modes, Proposition 3.11, there holds the following estimate

$$\frac{\phi_f}{\phi_f'}\Big|_{z=0} \approx \frac{1}{\mu_f(0)},$$

with $\epsilon\mu_f(z)^2 = U - c + \alpha^2\epsilon$. Whereas, Proposition 3.8 yields the following asymptotic expansion for the slow mdoes:

$$\phi_s = \phi_{\alpha,c} + \mathcal{O}(\alpha m_f^{-2} E(\alpha,c)^{-1})$$

as long as $\alpha m_f^{-2}E(\alpha,c)^{-1} \ll 1$, in which $\phi_{\alpha,c}$ denotes the decaying Rayleigh solution, normalized so that $\|\phi_{\alpha,c}\|_{L^\infty} = 1$. By definition, the inviscid Evans function (see (1.16) and (3.5)) is defined by

$$E(\alpha,c) = \phi_{\alpha,c}(0).$$

This proves that

$$\frac{\phi_s}{\phi_s'}\Big|_{z=0} \approx \frac{1}{\phi_{\alpha,c}'(0)}\Big(E(\alpha,c) + \mathcal{O}(\alpha m_f^{-2}E(\alpha,c)^{-1})\Big).$$

The dispersion relation (3.31) then reads

$$E(\alpha,c) + \mathcal{O}(\alpha m_f^{-2}E(\alpha,c)^{-1}) \approx \frac{\phi_{\alpha,c}'(0)}{\mu_f(0)}. \tag{3.32}$$

We now show that as $\nu \ll 1$, there exists a pair (α, c), with $\Im c > 0$, so that the dispersion relation holds. By assumption, there exists a pair (α_0, c_0) so that $E(\alpha_0, c_0) = \phi_{\alpha_0,c_0}(0) = 0$, and

$$\partial_c E(\alpha_0, c_0) \neq 0.$$

This and the relation 3.32 prove that for all sufficiently small ν, there is a sequence c_ν so that $c_\nu \to c_0$ and $E(\alpha_0, c_\nu)$ satisfies (3.32). In particular, there holds

$$E(\alpha_0, c_\nu) \approx \frac{\phi'_{\alpha_0,c_0}(0)}{\mu_f(0)} + \mathcal{O}(\alpha m_f^{-2} E(\alpha_0, c_\nu)^{-1}).$$

Here, we note that $\phi'_{\alpha_0,c_0}(0) \neq 0$, since otherwise $\phi_{\alpha_0,c_0} \equiv 0$ identically, since by the Rayleigh equation, all derivatives of ϕ_0 vanish on the boundary. This proves that $E(\alpha_0, c_\nu)$ is of order $1/\mu_f(0)$. Observe that $\mu_f(0) \sim \epsilon^{-1/2}\sqrt{-c_0 + \alpha_0^2\epsilon}$. In particular, $\alpha \ll \mu_f$ and $\alpha\mu_f^{-2}E(\alpha_0, c_\nu)^{-1}$ remains sufficiently small. The instability for the Navier-Stokes problem follows for sufficiently small ν, since $\Im c_\nu \to \Im c_0 > 0$, as $\nu \to 0$.

4 Green function for 4^{th} order ODEs

In this section, we construct the Green kernel for the following 4^{th} order operator

$$L\phi := -\epsilon\partial_z^4\phi + b(z)\partial_z^2\phi + a(z)\phi. \tag{4.1}$$

on the half-line $z \geq 0$, together with zero boundary conditions

$$\phi_{|z=0} = \partial_z\phi_{|z=0} = 0. \tag{4.2}$$

The construction follows closely that of Zumbrun-Howard [10, 19] in their study of stability of viscous shock profiles for conservation laws.

We assume that the coefficients $a(z), b(z)$ converge to their corresponding limit a_+, b_+, with $b_+ \neq 0$, exponentially as $z \to \infty$. We also assume that the algebraic equation

$$-\epsilon\mu^4 + b_+\mu^2 + a_+ = 0$$

has two distinct solutions μ_s, μ_f with positive real part, with $\Re\mu_s \ll \Re\mu_f$. This assumption is clearly verified, whenever $a_+\epsilon$ is not real (which is of course the case for Orr-Sommerfeld equations). Let ϕ_s, ϕ_f be two independent and decaying solutions of $L\phi = 0$, with $\|\phi_s\|_{L^\infty} = \|\phi_f\|_{L^\infty} = 1$, whose behavior at infinity is $e^{-\mu_j z}$, $j = s, f$, respectively. Set

$$D_L := \mu_f^{-1} W[\phi_s, \phi_f]_{|z=0}. \tag{4.3}$$

in which $W[f, g] = f\partial_z g - g\partial_z f$ denotes the the Wronskian determinant. The factor μ_f^{-1} is to assure that $|D_L|$ remains bounded as $\epsilon \to 0$.

Proposition 4.1. *Let $G(x,z)$ be the Green function of L with zero boundary conditions* (4.2)*, and let D_L be the normalized Wronskian determinant* (4.3)*. There holds*

$$\begin{aligned} |G(x,z)| \le \frac{C_0}{D_L}\Big(\frac{1}{\mu_s}e^{-\mu_s(|z|+|x|)} + \frac{1}{\mu_f}\sum_{(j,k)\neq(s,s)} e^{-\mu_j|z|}e^{-\mu_k|x|}\Big) \\ + C_0\Big(\frac{1}{\mu_s}e^{-\mu_s|x-z|} + \frac{1}{\mu_f}e^{-\mu_f|x-z|}\Big) \end{aligned} \tag{4.4}$$

uniformly in small ϵ, for all $x,z\ge 0$, and for some universal constant C_0. Here, the ϵ-dependence is hidden in μ_s and μ_f, which satisfy

$$\mu_s^2 = -\frac{a_+}{b_+}(1+\mathcal{O}(\epsilon)), \qquad \mu_f^2 = \frac{b_+}{\epsilon}(1+\mathcal{O}(\epsilon)) \tag{4.5}$$

as $\epsilon\to 0$.

4.1 The Green kernel on the whole line

We first study the whole line problem $L\phi = 0$ on $\mathbb{R}$, with

$$a_\pm = \lim_{z\to\pm\infty} a(z), \qquad b_\pm = \lim_{z\to\pm\infty} b(z)$$

with $b_\pm \neq 0$. Asymptotically, solutions to this homogenous ODEs behave as $e^{\pm\mu_{s,\pm}z}$ and $e^{\pm\mu_{f,\pm}z}$, as $z\to\pm\infty$, in which $\mu_{j,\pm}$ are the solutions, with nonnegative real part, of

$$\mu_{s,\pm}^2 = -\frac{a_\pm}{b_\pm}(1+\mathcal{O}(\epsilon)), \qquad \mu_{f,\pm}^2 = \frac{b_\pm}{\epsilon}(1+\mathcal{O}(\epsilon))$$

defined for each $+$ and $-$, respectively. We denote by $\phi_{j,\pm}$ and $\psi_{j,\pm}$ the stable (decaying) and unstable (growing) solutions, respectively, near each infinity $z\to\pm\infty$. That is, $\phi_{j,\pm}\approx e^{-\mu_{j,\pm}|z|}$ and $\psi_{j,\pm}\approx e^{\mu_{j,\pm}|z|}$. Similarly, we denote by $\phi^*_{j,\pm}$ and $\psi^*_{j,\pm}$ the corresponding solutions to the adjoint equation

$$L^*\psi := -\epsilon\partial_x^4\psi + \partial_x^2(b(x)\psi) + a(x)\psi = 0.$$

that behave as $e^{\mu_{j,\pm}|x|}$ and $e^{-\mu_{j,\pm}|x|}$ at the infinities, respectively.

Green kernels of L and L^*.

Let $G(x,z)$ be the Green kernel of L. That is, $G(x,z)$ is a bounded function and satisfies

$$LG(x,\cdot) = \delta_x(\cdot)$$

for each fixed x. As a consequence, $G(x,z)$ and its derivatives up to the second order are continuous at $z = x$ and its third derivative satisfies the jump condition

$$[-\epsilon\partial_z^3 G(x,z)]_{|z=x} = 1.$$

Similarly, let $H(x,z)$ be the Green kernel of the corresponding adjoint operator L^*, solving the adjoint equation $L^* H(\cdot,z) = \delta_z(\cdot)$ for each z. Let us compute

$$\begin{aligned} G(x,z) &= \langle \delta_z(\cdot), G(x,\cdot)\rangle_{L^2} = \langle L^* H(\cdot,z), G(x,\cdot)\rangle_{L^2} \\ &= \langle H(\cdot,z), LG(x,\cdot)\rangle_{L^2} = \langle H(\cdot,z), \delta_x(\cdot)\rangle_{L^2} \\ &= H(x,z), \end{aligned}$$

for all x, z. This proves that for each z, $L^* G(\cdot,z) = \delta_z(\cdot)$ and so $G(x,z)$ is also the Green kernel of L^*. Equivalently, let ϕ and ψ be two arbitrary functions, and let $f = L\phi$ and $g = L^*\psi$. Then, we compute

$$\begin{aligned} \iint G(x,z) f(x) g^*(z)\, dz dx &= \int \phi(z) g^*(z)\, dz = \int \phi(z)(L^*\psi(z))^* dz \\ &= \int L\phi(z)\psi^*(z) dz = \int f(z)\psi^*(z) dz \\ &= \int f(x)\Big(\int G^*(x,z) g(z) dz\Big)^* dx. \end{aligned}$$

Since f was arbitrary, we have

$$\psi(x) = \Big(\int G^*(x,z) g(z) dz\Big)^*$$

and so $G(x,z)$ is also the Green kernel of L^*.

Green kernel's representation.

Hence, as a function of x, G is a solution of L^*, provided $x \neq z$. For $x > z$ and for $x < z$, the Green kernel is therefore a linear combination of decaying solutions $\phi_{j,\pm}(z)$ and $\psi^*_{k,\pm}(x)$. We can therefore write

$$G(x,z) = \begin{cases} \displaystyle\sum_{j,k=s,f} d_{jk}\phi_{j,+}(z)\psi^*_{k,-}(x), & z > x \\ \displaystyle -\sum_{j,k=s,f} e_{jk}\phi_{j,-}(z)\psi^*_{k,+}(x), & z < x, \end{cases} \tag{4.6}$$

for constants d_{jk} and e_{jk} to be computed, using the jump conditions of $G(x,z)$ across $z = x$.

For convenience, we introduce the Green kernel 4×4 matrix $\mathcal{G}(x,z)$, whose component (j,k) is defined by $\partial_z^{j-1}\partial_x^{k-1}G(x,z)$. Namely,

$$\mathcal{G}(x,z) = \Big[\partial_z^{j-1}\partial_x^{k-1}G(x,z)\Big]_{1\le j\le 4, 1\le k\le 4}.$$

We also denote by Φ the column vectors $(\phi,\phi',\phi'',\phi''')^t$ and Φ^* the row vector $(\phi,\phi',\phi'',\phi''')$. We set $\Phi_\pm = [\Phi_{s,\pm},\Phi_{f,\pm}]$, which are 4×2 matrices. Similar notations are introduced for Ψ,Ψ^*. We may rewrite (4.6) in the matrix form:

$$\mathcal{G}(x,z) = \begin{cases} \Phi_+(z)M_1\Psi_-^*(x), & z > x \\ -\Phi_-(z)M_2\Psi_+^*(x), & z < x, \end{cases} \tag{4.7}$$

for 2×2 constant matrices M_1, M_2.

Jump conditions.

To determine M_1, M_2, we compute the jump conditions for $\mathcal{G}(x,z)$. By definition,

$$[\partial_z^j G(x,z)]_{|z=x} = \lim_{z\to x^+}\partial_z^j G(x,z) - \lim_{z\to x^-}\partial_z^j G(x,z)$$

and hence, for $0\le j\le 3$,

$$\begin{aligned} 0 = \partial_x[\partial_z^j G(x,z)]_{|z=x} &= \partial_x \lim_{z\to x^+}\partial_z^j G(x,z) - \partial_x \lim_{z\to x^-}\partial_z^j G(x,z) \\ &= [\partial_z^j\partial_x G(x,z)]_{|z=x} + [\partial_z^{j+1}G(x,z)]_{|z=x} \end{aligned}$$

This gives

$$[\partial_z^j\partial_x G(x,z)]_{|z=x} = -[\partial_z^{j+1}G(x,z)]_{|z=x}.$$

Similarly, we have

$$\begin{aligned} 0 = \partial_x^2[\partial_z^j G(x,z)]_{|z=x} &= [\partial_z^j\partial_x^2 G(x,z)]_{|z=x} + 2[\partial_z^{j+1}\partial_x G(x,z)]_{|z=x} \\ &\quad + [\partial_z^{j+2}G(x,z)]_{|z=x} \\ 0 = \partial_x^3[\partial_z^j G(x,z)]_{|z=x} &= [\partial_z^j\partial_x^3 G(x,z)]_{|z=x} + 3[\partial_z^{j+1}\partial_x^2 G(x,z)]_{|z=x} \\ &\quad + 3[\partial_z^{j+2}\partial_x G(x,z)]_{|z=x} + [\partial_z^{j+3}G(x,z)]_{|z=x}. \end{aligned}$$

Similar computations hold for $\partial_z^j[\partial_x^k G(x,z)]_{|x=z}$, $0 \le k \le 3$, yielding

$$\begin{aligned}
0 = \partial_z[\partial_x^k G(x,z)]_{|x=z} &= [\partial_z\partial_x^k G(x,z)]_{|x=z} + [\partial_x^{k+1}G(x,z)]_{|x=z} \\
0 = \partial_z^2[\partial_x^k G(x,z)]_{|x=z} &= [\partial_z^2\partial_x^k G(x,z)]_{|x=z} + 2[\partial_z\partial_x^{k+1}G(x,z)]_{|x=z} \\
&\quad + [\partial_x^{k+2}G(x,z)]_{|x=z} \\
0 = \partial_z^3[\partial_x^k G(x,z)]_{|x=z} &= [\partial_z^3\partial_x^k G(x,z)]_{|x=z} + 3[\partial_z^2\partial_x^{k+1}G(x,z)]_{|x=z} \\
&\quad + 3[\partial_z\partial_x^{k+2}G(x,z)]_{|x=z} + [\partial_x^{k+3}G(x,z)]_{|x=z}.
\end{aligned}$$

In addition, a direct use of the identity $LG(x,z) = L^*G(x,z) = 0$ for $z \ne x$, we get $[\partial_z^4 G(x,z)]_{|z=x} = [\partial_x^4 G(x,z)]_{|x=z} = 0$ and

$$\begin{aligned}
[\partial_z^5 G(x,z)]_{|z=x} &= -\frac{1}{\epsilon^2}b(x), \qquad [\partial_z^6 G(x,z)]_{|z=x} = -\frac{2}{\epsilon^2}b'(x) \\
[\partial_x^5 G(x,z)]_{|x=z} &= \frac{1}{\epsilon^2}b(z), \qquad [\partial_x^6 G(x,z)]_{|x=z} = \frac{2}{\epsilon^2}b'(z).
\end{aligned}$$

Putting these together yields

$$[\mathcal{G}(x,z)]_{|z=x} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\epsilon} \\ 0 & 0 & -\frac{1}{\epsilon} & 0 \\ 0 & \frac{1}{\epsilon} & 0 & \frac{1}{\epsilon^2}b(x) \\ -\frac{1}{\epsilon} & 0 & -\frac{1}{\epsilon^2}b(x) & \frac{1}{\epsilon^2}b'(x) \end{pmatrix} = \frac{1}{\epsilon^2}\begin{pmatrix} 0 & \epsilon J \\ \epsilon J & B \end{pmatrix},$$

with $J := [-e_2, e_1]$ for $e_1 = (1,0)^t$ and $e_2 = (0,1)^t$. It is useful to compute

$$[\mathcal{G}(x,z)]^{-1}_{|z=x} = \begin{pmatrix} b'(x) & b(x) & 0 & -\epsilon \\ -b(x) & 0 & \epsilon & 0 \\ 0 & -\epsilon & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{B}(x) & -\epsilon J \\ -\epsilon J & 0 \end{pmatrix}. \tag{4.8}$$

Coefficients of the Green kernel.

By a view of (4.7), we have

$$[\mathcal{G}(x,z)]_{|z=x} = [\Phi_+, \Phi_-]\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\begin{pmatrix} \Psi_-^* \\ \Psi_+^* \end{pmatrix}(x).$$

This yields constant matrices M_1, M_2, and in particular, for any given two solutions ϕ and ψ^* to the equation $L\phi = 0$ and $L^*\psi^* = 0$, respectively, there holds

$$\Psi^*(x)[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi(x) \quad \equiv \quad \text{const.} \tag{4.9}$$

Hence, the adjoint solutions will be constructed so that

$$\begin{aligned}\Phi^*_{j,\pm}[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_{k,\pm} &= \delta_{jk}, \qquad \Phi^*_{j,\pm}[\mathcal{G}(x,z)]^{-1}_{|z=x}\Psi_{k,\pm} = 0,\\ \Psi^*_{j,\pm}[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_{k,\pm} &= 0, \qquad \Psi^*_{j,\pm}[\mathcal{G}(x,z)]^{-1}_{|z=x}\Psi_{k,\pm} = \delta_{jk},\end{aligned} \tag{4.10}$$

for each $\pm$ and for each $j,k = s,f$. Such a construction is straightforward when all the independent solutions $\Phi_{k,\pm}, \Psi_{k,\pm}$ of the equation $L\phi = 0$ have already been constructed.

We now compute the constant matrices M_1, M_2. We have

$$\begin{aligned}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} &= [\Phi_+, \Phi_-]^{-1}[\mathcal{G}(x,z)]_{|z=x}\begin{pmatrix}\Psi^*_- \\ \Psi^*_+\end{pmatrix}^{-1}(x)\\ &= \left(\begin{pmatrix}\Psi^*_- \\ \Psi^*_+\end{pmatrix}[\mathcal{G}(x,z)]^{-1}_{|z=x}[\Phi_+, \Phi_-]\right)^{-1}\\ &= \begin{pmatrix}\Psi^*_-[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_+ & 0 \\ 0 & \Psi^*_+[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_-\end{pmatrix}^{-1}\end{aligned}$$

in which we have used the constraints (4.10). Let Π_1 denote the projection on the first two components of a vector in $\mathbb{R}^4$. Together with (4.8), the above yields

$$\begin{aligned}M_1 &= \left(\Pi_1\Psi^*_-\tilde{B}\Pi_1\Phi_+\right)^{-1}_{|z=0} + \mathcal{O}(\epsilon),\\ M_2 &= \left(\Pi_1\Psi^*_+\tilde{B}\Pi_1\Phi_-\right)^{-1}_{|z=0} + \mathcal{O}(\epsilon).\end{aligned} \tag{4.11}$$

Putting the calculations on M_1, M_2 back into (4.7) yields the representation for the Green kernel $\mathcal{G}(x,z)$ in term of solutions to $L\phi = 0$ and their adjoints.

4.2 The half-line problem

Expression of the Green kernel.

We first extend the problem to the whole line $z \in \mathbb{R}$, with the extension $a(z) = a(-z)$ and $b(z) = b(-z)$ for negative z. Note that a and b may not be smooth at $z = 0$. The previous construction yields the Green kernel $G(x,z)$ of L on the whole line. In particular, we now consider the case $x, z \geq 0$. We may write the $(-)$ solutions as a linear combination of the $(+)$ solutions. We drop the $(+)$ subscript. This leads to the following presentation of the Green kernel:

$$\mathcal{G}(x,z) = \begin{cases} \Phi(z)M_{11}\Psi^*(x) + \Phi(z)M_{12}\Phi^*(x), & z > x > 0\\ -\Phi(z)M_{22}\Psi^*(x) - \Psi(z)M_{21}\Psi^*(x), & 0 < z < x,\end{cases} \tag{4.12}$$

in which M_{jk} are constant 2×2 matrices and $\Phi = [\Phi_s, \Phi_f]$ are the two decaying solutions, whereas $\Psi = [\Psi_s, \Psi_f]$ are the two growing solutions. The adjoint solutions Φ^*, Ψ^* are constructed through the orthogonality relation (4.10).

Jump relations.

To determine the coefficient matrices, we compute

$$[\mathcal{G}(x,z)]_{|z=x} = \Phi[M_{11}, M_{12}] \begin{pmatrix} \Psi^* \\ \Phi^* \end{pmatrix} + [\Psi, \Phi] \begin{pmatrix} M_{21} \\ M_{22} \end{pmatrix} \Psi^*,$$

in which the functions on the right-hand side are evaluated at x. Thus, as above, using the orthogonality relation (4.10), we immediately obtain

$$M_{12} = I_2, \qquad M_{21} = I_2, \qquad M_{11} + M_{22} = 0.$$

Putting these back into (4.12), we thus obtain the following general representation for the Green kernel $G(x,z)$ of L:

$$G(x,z) = \begin{cases} \displaystyle\sum_{j,k=s,f} d_{jk}\phi_j(z)\psi_k^*(x) + \sum_{k=s,f} \phi_k(z)\phi_k^*(x), & z > x > 0 \\ \displaystyle\sum_{j,k=s,f} d_{jk}\phi_j(z)\psi_k^*(x) - \sum_{k=s,f} \psi_k(z)\psi_k^*(x), & 0 < z < x \end{cases} \tag{4.13}$$

for some constants d_{jk}. Clearly, the function $G(x,z)$ defined as in (4.13) is a Green kernel of L on the half-line $z \geq 0$. It remains to determine the constant coefficients d_{jk} so that $G(x,z)$ satisfies the zero boundary condition (4.2) at $z = 0$. Written in the vector form, the boundary condition forces

$$0 = \Pi_{2\times 2}(\mathcal{G}(x,0)) = \Pi_1\Big(\Phi(0)M - \Psi(0)\Big)\Psi^*(x),$$

for all $x \geq 0$, with $M = (d_{jk})$ being the 2×2 constant matrix as in (4.13). This yields

$$M = (d_{jk}) = [\Pi_1\Phi(0)]^{-1}\Pi_1\Psi(0). \tag{4.14}$$

By a view of $\Phi = [\Phi_s, \Phi_f]$, the vanishing of the determinant of $\Pi_1\Phi(0)$ is equivalent to the existence of a nontrivial decaying solution ϕ of $L\phi = 0$, satisfying the zero boundary condition. That is,

$$\det(\Pi_1\Phi(0)) = 0 \tag{4.15}$$

is the usual dispersion relation in the literature, with Π_1 being the projection on the first two components of a vector in $\mathbb{R}^4$.

Bounds on the Green kernel.

Let us now give some bounds on $G(x,z)$. We recall that $\phi_j(z) \sim e^{-\mu_j|z|}$ and $\psi_j(z) \sim e^{\mu_j|z|}$, with the convention that $\Re\mu_j \geq 0$, for each $j = s, f$. We construct the adjoint solutions through

$$\Phi_j^*[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_k = \delta_{jk}, \qquad \Phi_j^*[\mathcal{G}(x,z)]^{-1}_{|z=x}\Psi_k = 0,$$
$$\Psi_j^*[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_k = 0, \qquad \Psi_j^*[\mathcal{G}(x,z)]^{-1}_{|z=x}\Psi_k = \delta_{jk},$$

in which the matrix $[\mathcal{G}(x,z)]^{-1}_{|z=x}$ is computed as in (4.8). We write

$$[\mathcal{G}(x,z)]^{-1}_{|z=x} = \begin{pmatrix} \tilde{B}(x) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon) & 0 \end{pmatrix}, \qquad \tilde{B}(x) = \begin{pmatrix} b'(x) & b(x) \\ -b(x) & 0 \end{pmatrix}.$$

Since $\tilde{B}(x)$ is of order one, it follows at once that asymptotically at $z = \infty$, the adjoint solutions decay and grow at the same exponential rate as those of ϕ and ψ. Precisely, we have $\phi_j^*(z) \sim c_j e^{\mu_j|z|}$ and $\psi_j^*(z) \sim d_j e^{-\mu_j|z|}$, for some constants c_j, d_j, and for each $j = s, f$. Let us compute it more carefully their amplitude in term of ϵ. By definition, $\mu_f \gg \mu_s$ (typically, $\mu_s \sim 1$ and $\mu_f \sim \frac{1}{\sqrt{\epsilon}}$). Let Π_1, Π_2 be the projection on the first two and the last two components of a vector in $\mathbb{C}^4$. From the relation $\Phi_f^*[\mathcal{G}(x,z)]^{-1}_{|z=x}\Phi_f = 1$, we first compute

$$1 \sim \Pi_1\Phi_f^*\tilde{B}(x)\Pi_1\Phi_f = b'\phi_f\phi_f^* - b(\phi_f\partial_z\phi_f^* - \phi_f^*\partial_z\phi_f).$$

Since z-derivative of the fast modes ϕ_f, ϕ_f^* yields a factor $\mu_f \gg 1$, this shows that $\phi_f\partial_z\phi_f^* - \phi_f^*\partial_z\phi_f \sim 1$ and so

$$\phi_f^* \sim \frac{1}{\mu_f}e^{\mu_f|z|}.$$

Similarly, we have

$$\phi_s^* \sim \frac{1}{\mu_s}e^{\mu_s|z|}, \qquad \psi_j^* \sim \frac{1}{\mu_j}e^{-\mu_j|z|}.$$

Hence, by the construction (4.13), there holds

$$\begin{aligned}|G(x,z)| \leq \frac{C_0\mu_f}{\det(\Pi_1\Phi(0))}\Big(\frac{1}{\mu_s}e^{-\mu_s(|z|+|x|)} + \frac{1}{\mu_f}\sum_{(j,k)\neq(s,s)} e^{-\mu_j|z|}e^{-\mu_k|x|}\Big) \\ + C_0\Big(\frac{1}{\mu_s}e^{-\mu_s|x-z|} + \frac{1}{\mu_f}e^{-\mu_f|x-z|}\Big)\end{aligned} \tag{4.16}$$

for all $x, z \geq 0$, and for some universal constant C_0. This yields Proposition 4.1.

5 Proof of Theorem 2.1: Green function of OS

In this section, we construct the Green kernel of the Orr-Sommerfeld operator OS(ϕ) with the zero boundary conditions, proving our main theorem. We write the Orr-Sommerfeld equation as

$$\mathrm{OS}(\phi) = -\epsilon\partial_z^4\phi + b(z)\partial_z^2\phi - a(z)\phi = 0,$$

in which

$$b(z) = (U - c + 2\epsilon\alpha^2), \qquad a(z) = \alpha^2(\epsilon\alpha^2 + U - c) + U''.$$

Let $\alpha > 0$ and $c \in \mathbb{C}$, and let $G_{\alpha,c}(x,z)$ be the Green function of the Orr-Sommerfeld equations with the zero boundary conditions:

$$G_{\alpha,c}(x,0) = \partial_z G_{\alpha,c}(x,0) = 0.$$

In addition, we let $\phi_{s,\pm}, \phi_{f,\pm}$ be fast and slow modes of the Orr-Sommerfeld equations, respectively, and $\phi^*_{s,\pm}, \phi^*_{f,\pm}$ be the corresponding adjoint solutions, constructed through the algebraic relations (4.10). Precisely, in the present case, the adjoint solutions satisfy

$$\begin{aligned} \Phi^*_{j,+}\mathcal{B}\Phi_{k,-} &= \delta_{jk}, \qquad & \Phi^*_{j,+}\mathcal{B}\Phi_{k,+} &= 0,\\ \Phi^*_{j,-}\mathcal{B}\Phi_{k,-} &= 0, \qquad & \Phi^*_{j,-}\mathcal{B}\Phi_{k,+} &= \delta_{jk}, \end{aligned} \tag{5.1}$$

for $j,k = s,f$, in which we have used the column vector notation

$$\Phi = [\phi, \phi', \phi'', \phi''']^t,$$

respectively for $\phi = \phi_{j,\pm}$ with $j = s,f$. The same notation applies for the adjoint solutions Φ^*, defined as a row vector. Here, the matrix

$$\mathcal{B} := [\mathcal{G}_{\alpha,c}(x,z)]^{-1}_{|z=x}$$

is computed as in (4.8), yielding

$$\mathcal{B} = \begin{pmatrix} B(x) & \mathcal{O}(\epsilon) \\ \mathcal{O}(\epsilon) & 0 \end{pmatrix}, \qquad B(x) = \begin{pmatrix} U'(x) & b(x;\epsilon) \\ -b(x;\epsilon) & 0 \end{pmatrix}.$$

Following the construction introduced in Section 4, we obtain the following representation for the Green kernel $G_{\alpha,c}(x,z)$

$$G_{\alpha,c}(x,z) = \begin{cases} \displaystyle\sum_{j,k=s,f} d_{jk}\phi_{j,-}(z)\phi^*_{k,-}(x) + \sum_{k=s,f} \phi_{k,-}(z)\phi^*_{k,+}(x), & z > x > 0 \\ \displaystyle\sum_{j,k=s,f} d_{jk}\phi_{j,-}(z)\phi^*_{k,-}(x) - \sum_{k=s,f} \phi_{k,+}(z)\phi^*_{k,-}(x), & 0 < z < x \end{cases}$$

in which the coefficient matrix (d_{jk}) is defined by

$$M = (d_{jk}) = \begin{pmatrix} \phi_{s,-} & \phi_{f,-} \\ \phi'_{s,-} & \phi'_{f,-} \end{pmatrix}^{-1} \begin{pmatrix} \phi_{s,+} & \phi_{f,+} \\ \phi'_{s,+} & \phi'_{f,+} \end{pmatrix}_{|z=0}.$$

It remains to give bounds on M. We recall that the slow and fast modes of Orr-Sommerfeld equations constructed in Proposition 3.8 and Proposition 3.11 read

$$\phi_{s,\pm}(z) \approx e^{\pm\mu_s z}, \qquad \phi_{f,\pm} \approx e^{\pm\int_0^z \mu_f(y)\,dy}$$

with $\mu_s = \alpha$ and $\mu_f \gg \mu_s$. We also recall the normalized Evans function

$$D(\alpha, c) := \mu_f^{-1} \det \begin{pmatrix} \phi_{s,-} & \phi_{f,-} \\ \phi'_{s,-} & \phi'_{f,-} \end{pmatrix}_{|z=0}.$$

By a direct computation, the constant matrix M, appearing in the representation for the Green function satisfies

$$\begin{aligned} |M| &\sim C_0 \mu_f^{-1} D(\alpha,c)^{-1} \begin{pmatrix} W[\phi_{f,-},\phi_{s,+}] & W[\phi_{f,-},\phi_{f,+}] \\ W[\phi_{s,-},\phi_{s,+}] & W[\phi_{f,+},\phi_{s,-}] \end{pmatrix}_{|z=0} \\ &\sim C_0 \mu_f^{-1} D(\alpha,c)^{-1} \begin{pmatrix} \mu_f & \mu_f \\ \mu_s & \mu_f \end{pmatrix} \\ &\le C_0 [D(\alpha,c)]^{-1}, \end{aligned}$$

which is bounded, provided $E(\alpha, c) \neq 0$. As for the adjoint solutions, following the calculations done in Section 4, we get

$$\phi^*_{s,\pm}(z) \sim \mu_s^{-1} e^{\pm\mu_s |z|}, \qquad \phi^*_{f,\pm}(z) \sim \mu_f^{-1} e^{\pm\int_0^z \mu_f\,dy}.$$

Putting these all together into the representation formula for the Green function, we obtain at once the Green function bounds, which complete the proof of our main theorem.

Proof of Corollary 2.2. Finally, we derive bounds on $\Delta_\alpha G_{\alpha,c}(x,z)$. We recall that $\Delta_\alpha G_{\alpha,c}(x,z)$ solves

$$\Big(-\epsilon\Delta_\alpha + U - c\Big)\Delta_\alpha G_{\alpha,c}(x,z) = U'' G_{\alpha,c}(x,z) + \delta_x(z).$$

We write

$$\Delta_\alpha G(x,z) = \mathcal{G}_{\mathrm{a}}(x,z) + \mathcal{R}_G(x,z)$$

in which $\mathcal{G}_{\rm a}(x,z)$ is constructed as in (3.12). Hence, the residual Green function satisfies

$$\mathcal{R}_G(x,z) = \int_0^\infty \mathcal{G}_{\rm a}(y,z)U''(y)G_{\alpha,c}(x,y)\,dy.$$

In particular, we recall from (3.15) that

$$|\partial_x^\ell \partial_z^k \mathcal{G}_{\rm a}(x,z)| \le C|\epsilon m_f|^{-1} M_f^{k+\ell} e^{-m_f|x-z|}$$

for $k,\ell \ge 0$. Together with the Green function bound on $G_{\alpha,c}(x,z)$ obtained in Theorem 2.1, we obtain

$$\begin{aligned}
|\mathcal{R}_G(x,z)| \le{}& C|\epsilon m_f|^{-1}[D(\alpha,c)]^{-1}\int_0^\infty e^{-m_f|y-z|}\frac{1}{\mu_s}e^{-\theta_0\mu_s(|x|+|y|)}e^{-\eta_0|y|}\,dy\\
&+ C|\epsilon m_f|^{-1}[D(\alpha,c)]^{-1}\int_0^\infty e^{-m_f|y-z|}\frac{1}{m_f}e^{-\theta_0 m_f(|x|+|y|)}e^{-\eta_0|y|}\,dy\\
&+ C|\epsilon m_f|^{-1}\int_0^\infty e^{-m_f|y-z|}\frac{1}{\mu_s}e^{-\theta_0\mu_s|x-y|}e^{-\eta_0|y|}\,dy\\
&+ C|\epsilon m_f|^{-1}\int_0^\infty e^{-m_f|y-z|}\frac{1}{m_f}e^{-\theta_0 m_f|x-y|}e^{-\eta_0|y|}\,dy.
\end{aligned}$$

We recall that $\mu_s \ll m_f$. Hence, we have

$$\begin{aligned}
\int_0^\infty e^{-m_f|y-z|}e^{-\theta_0\mu_s(|x|+|y|)}e^{-\eta_0|y|}\,dy &\le Cm_f^{-1}e^{-\theta_0\mu_s(|x|+|z|)}\\
\int_0^\infty e^{-m_f|y-z|}e^{-\theta_0\mu_s(|x-y|)}e^{-\eta_0|y|}\,dy &\le Cm_f^{-1}e^{-\theta_0\mu_s(|x-z|)}
\end{aligned}$$

and, recalling that $\theta_0 < 1$,

$$\begin{aligned}
\int_0^\infty e^{-m_f|y-z|}e^{-\theta_0 m_f(|x|+|y|}e^{-\eta_0|y|}\,dy &\le Cm_f^{-1}e^{-\theta_0 m_f(|x|+|z|)}\\
\int_0^\infty e^{-m_f|y-z|}e^{-\theta_0 m_f(|x-y|)}e^{-\eta_0|y|}\,dy &\le Cm_f^{-1}e^{-\theta_0 m_f(|x-z|)}.
\end{aligned}$$

This yields the claimed bound for $\Delta_\alpha G_{\alpha,c}(x,z)$. As for the derivatives, we note that

$$\partial_x^\ell\partial_z^k\mathcal{R}_G(x,z) = \int_0^\infty \partial_z^k\mathcal{G}_{\rm a}(y,z)U''(y)\partial_x^\ell G_{\alpha,c}(x,y)\,dy.$$

By using the derivative bounds on $\mathcal{G}_a(x,z)$ and on $G_{\alpha,c}(x,z)$, the derivative bounds on the residual $\mathcal{R}_G(x,z)$ follows at once. The proof of the corollary is complete. $\square$

6 Proof of Theorem 2.4: linearized Navier-Stokes

In this section, we prove Theorem 2.4 on the semigroup of the linearized Navier-Stokes equations about a boundary layer profile. The proof relies on the Green function bounds, obtained in Theorem 2.1. We recall the following link to the Orr-Sommerfeld Green function, obtained in Lemma 2.3,

$$(\lambda - L_\alpha)^{-1}\omega_\alpha(z) = \frac{1}{i\alpha}\int_0^\infty \Delta_\alpha G_{\alpha,c}(x,z)\omega_\alpha(x)\, dx.$$

Hence, the semigroup $e^{L_\alpha t}$ is computed by

$$e^{L_\alpha t}\omega_\alpha(z) = \frac{-1}{2\pi\alpha}\int_{\Gamma_\alpha}\int_0^\infty e^{\lambda t}\Delta_\alpha G_{\alpha,c}(x,z)\omega_\alpha(x)\, dx d\lambda, \tag{6.1}$$

in which $c = i\alpha^{-1}\lambda$ and Γ_α can be chosen, depending on α and lying in the resolvent set of L_α.

6.1 Spectrum of L

The aim of this section is to change the contour integral to a more suitable one. To do this we need to study the spectum of L and L_α. Since $U'(z)$ decays exponentially fast to zero as $z \to \infty$, the convection term $v \cdot \nabla\omega_s$ is relatively compact to the Stokes operator $\nu\Delta$. Hence, the unstable spectrum of L in L^2 consists of possible point spectrum or eigenvalues.

Next, applying the standard energy estimate to the resolvent equation for velocity $(\lambda - A_\alpha)v = h$, we get at once

$$\Re\lambda\|v\|_{L^2}^2 + \nu\|\nabla_\alpha v\|_{L^2}^2 \le \|h\|_{L^2}\|v\|_{L^2} + \int_0^\infty |U'v_1v_2|\, dz.$$

Note that

$$|v_2(z)| \le \int_0^z |\partial_z v_2|\, dy \le \sqrt{z}\|\partial_z v_2\|_{L^2}.$$

Using the divergence-free condition

$$\partial_z v_2 = -i\alpha v_1,$$

we get

$$\int_0^\infty |U'v_1v_2|\, dz \le |\alpha|\|zU'\|_{L^2}\|v_1\|_{L^2}^2.$$

In addition, taking the imaginary part of the L^2 energy estimate (in which there is no contribution from the dissipation), we get

$$|\Im\lambda|\|v\|_{L^2}^2 \le |\alpha|\|U\|_{L^\infty}\|v\|_{L^2}^2 + \|h\|_{L^2}\|v\|_{L^2} + \int_0^\infty |U'v_1v_2|\,dz.$$

This yields

$$\begin{aligned} \Re\lambda\|v\|_{L^2}^2 + \nu\|\nabla_\alpha v\|_{L^2}^2 &\le C_0|\alpha|\|v\|_{L^2}^2 + \|h\|_{L^2}\|v\|_{L^2} \\ |\Im\lambda|\|v\|_{L^2}^2 &\le C_1|\alpha|\|v\|_{L^2}^2 + \|h\|_{L^2}\|v\|_{L^2} \end{aligned} \tag{6.2}$$

in which $C_0 := \|zU'\|_{L^2}$ and $C_1 = \|U\|_{L^\infty} + \|zU'\|_{L^2}$. As a direct consequence of the second inequality, the spectrum of L is contained in the half strip

$$\mathcal{S}_\alpha := \Big\{\lambda \in \mathbb{C} \ : \ \Re\lambda \le -\alpha^2\nu + C_0|\alpha|, \quad |\Im\lambda| \le C_1|\alpha|\Big\}. \tag{6.3}$$

Here, the bound of $\Re\lambda$ by $-\alpha^2\nu$ is due to the gradient term $\nabla_\alpha = [i\alpha, \partial_z]^{tr}$ in the first inequality of (6.2).

In addition, by a view of Lemma 3.6, for any fixed positive constant δ, the spectrum of L_α does not intersect with $\{\Re\lambda \ge \delta\}$ for sufficiently large α. This proves that the maximal unstable eigenvalue λ_0, if exists, is associated with at most finitely many spatial frequencies $\alpha_j \in \mathbb{Z}$. That is, the unstable eigenfunction is a linear combination of a finite number of eigenmodes of the form

$$e^{\lambda_0 t}e^{i\alpha_j x}\Delta_\alpha\psi_j(z). \tag{6.4}$$

6.2 Semigroup decomposition

As in Theorem 2.1, we write

$$\Delta_\alpha G_{\alpha,c}(x,z) = \mathcal{G}_a(x,z) + \mathcal{R}_G(x,z) \tag{6.5}$$

and decompose the semigroup as

$$e^{L_\alpha t} = S_\alpha + \mathcal{R}_\alpha \tag{6.6}$$

with

$$\begin{aligned} S_\alpha\omega_\alpha(z) :&= \frac{1}{2\pi i}\int_{\Gamma_\alpha}\int_0^\infty e^{\lambda t}\mathcal{G}_a(x,z)\omega_\alpha(x)\,\frac{dxd\lambda}{i\alpha}, \\ \mathcal{R}_\alpha\omega_\alpha(z) :&= \frac{1}{2\pi i}\int_{\Gamma_\alpha}\int_0^\infty e^{\lambda t}\mathcal{R}_G(x,z)\omega_\alpha(x)\,\frac{dxd\lambda}{i\alpha}. \end{aligned}$$

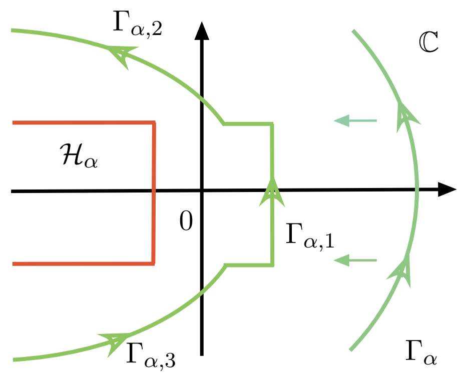

Figure 1: *Shown the decomposition of contour* Γ_α *of integration.*

We shall estimate these operators, using the pointwise bounds on the Green functions and appropriately choosing the contour Γ_α of integration. Here, we remark that by construction $S_\alpha \omega_\alpha$ solves the equation

$$(\partial_t + i\alpha U(z))\omega - \nu \Delta_\alpha \omega = 0$$

with the initial data

$$\omega_{|t=0} = \omega_\alpha(z).$$

6.3 Bounds on S_α.

Let us start with S_α. By the explicit construction of the Green function, $\mathcal{G}_a(x,z)$ is holomorphic in λ, except on the complex half strip:

$$\mathcal{H}_\alpha(x,z) := \Big\{ \lambda = -k - \alpha^2\nu - i\alpha U(y), \qquad k \in \mathbb{R}_+, \quad y \in [x,z] \Big\}.$$

Here and in what follows, we shall consider the case when $x < z$; the other case is similar. In our choice of contour of integration below, we shall avoid to enter this complex strip. Recalling $\epsilon = \frac{\nu}{i\alpha}$, we note that the eigenvalues

$$\hat\mu_f(x) = \sqrt{\alpha^2 + \frac{U-c}{\epsilon}} = \nu^{-1/2}\sqrt{\lambda + i\alpha U(x) + \alpha^2\nu}$$

changes it sign when crossing the half line $\lambda = -k - \alpha^2\nu - i\alpha U(x)$, with $k \in \mathbb{R}_+$.

From its construction (3.12), the Green function satisfies the following explicit bound:

$$\mathcal{G}_a(x,z) = \frac{1}{\epsilon \hat\mu_f(x)} e^{-\int_x^z \hat\mu_f(y)\,dy}\Big(1 + \mathcal{O}(\epsilon)\Big).$$

We set

$$\Gamma_{\alpha,1} := \Big\{\lambda = \gamma - \alpha^2\nu - i\alpha c, \qquad \min_{y\in[x,z]} U(y) \le c \le \max_{y\in[x,z]} U(y)\Big\}$$
$$\Gamma_{\alpha,2} := \Big\{\lambda = (a^2 - k^2 - \frac{1}{2}\alpha^2)\nu - i\alpha \min_{[x,z]} U + 2\nu iak, \qquad k \ge 0\Big\}$$
$$\Gamma_{\alpha,3} := \Big\{\lambda = (a^2 - k^2 - \frac{1}{2}\alpha^2)\nu - i\alpha \max_{[x,z]} U + 2\nu iak, \qquad k \le 0\Big\}$$

in which we take

$$a = \frac{|x-z|}{2\nu t}.$$

See Figure 1. The choice of the parabolic contours $\Gamma_{\alpha,2}$ and $\Gamma_{\alpha,3}$ is necessary to avoid singularities in small time. We stress that they never meet the complex strip $\mathcal{H}_\alpha(x,z)$. We consider two cases: $a^2\nu \gtrsim 1$ and $a^2\nu \ll 1$. In the former case, we shall take

$$\gamma := a^2\nu + \frac{1}{2}\alpha^2\nu \tag{6.7}$$

and thus

$$\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,2} \cup \Gamma_{\alpha,3}.$$

In the case when $a^2\nu \le \theta_0$, for an arbitrarily small, but fixed, positive constant θ_0, we set

$$\gamma := \theta_0 + \frac{1}{2}\alpha^2\nu. \tag{6.8}$$

In this case, we decompose $\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,2} \cup \Gamma_{\alpha,3} \cup \Gamma_{\alpha,4} \cup \Gamma_{\alpha,5}$, with

$$\Gamma_{\alpha,4} := \Big\{\lambda = k - \frac{1}{2}\alpha^2\nu - i\alpha \min_{[x,z]} U, \qquad a^2\nu \le k \le \theta_0\Big\}$$
$$\Gamma_{\alpha,5} := \Big\{\lambda = k - \frac{1}{2}\alpha^2\nu - i\alpha \max_{[x,z]} U, \qquad a^2\nu \le k \le \theta_0\Big\}.$$

We note that in both cases, we have

$$\gamma \ge \theta_0 + \frac{1}{2}\alpha^2\nu. \tag{6.9}$$

Bounds on $\Gamma_{\alpha,1}$.

We start our computation with the integral on $\Gamma_{\alpha,1}$. For $\lambda \in \Gamma_{\alpha,1}$ and $y \in [x,z]$, we have

$$\Re\hat\mu_f(y) = \nu^{-1/2}\Re\sqrt{\gamma + i\alpha(U(y)-c)} \ge \nu^{-1/2}\sqrt{\gamma}$$

and

$$|\epsilon\hat{\mu}_f(x)| = \nu^{1/2}\alpha^{-1}|\sqrt{\gamma + i\alpha(U(x)-c)}|$$

Hence, we have

$$\begin{aligned}
&\Big|\int_{\Gamma_{\alpha,1}} e^{\lambda t}\mathcal{G}_a(x,z)\,\frac{d\lambda}{i\alpha}\Big| \\
&\le e^{\gamma t}e^{-\alpha^2\nu t}\int_{\Gamma_{\alpha,1}}\frac{1}{|\epsilon\hat{\mu}_f(x)|}e^{-\Re\int_x^z \hat{\mu}_f(y)\,dy}\Big(1+\mathcal{O}(\epsilon)\Big)\frac{|d\lambda|}{\alpha} \\
&\le e^{\gamma t}e^{-\alpha^2\nu t}e^{-\nu^{-1/2}\sqrt{\gamma}|x-z|}\Big(1+\mathcal{O}(\epsilon)\Big)\int_{\min_{[x,z]}U}^{\max_{[x,z]}U}\frac{\alpha dc}{\nu^{1/2}|\sqrt{\gamma+i\alpha(U(x)-c)}|} \\
&\le C_0 e^{\gamma t}e^{-\alpha^2\nu t}e^{-\nu^{-1/2}\sqrt{\gamma}|x-z|}\nu^{-1/2}|\sqrt{\gamma+i\alpha(U(x)-c)}|\Big|_{\min_{[x,z]}U}^{\max_{[x,z]}U} \\
&\le C_0 e^{\gamma t}e^{-\alpha^2\nu t}e^{-\nu^{-1/2}\sqrt{\gamma}|x-z|}\nu^{-1/2}\sqrt{\gamma+\alpha|x-z|\|U'\|_{L^\infty}}.
\end{aligned}$$

In the case that γ is defined as in (6.8), we have

$$e^{\gamma t}e^{-\alpha^2\nu t}e^{-\nu^{-1/2}\sqrt{\gamma}|x-z|} \le e^{\theta_0 t}e^{-\frac{1}{2}\alpha^2\nu t}e^{-\nu^{-1/2}\sqrt{\gamma}|x-z|}.$$

When γ is defined by (6.7), using $\sqrt{\gamma}\ge a\sqrt{\nu}$ and the definition of a, we compute

$$\begin{aligned}
e^{\gamma t}e^{-\alpha^2\nu t}e^{-\nu^{-1/2}\sqrt{\gamma}|x-z|} &\le e^{\gamma t}e^{-\alpha^2\nu t}e^{-\frac{1}{2}a|x-z|}e^{-\frac{1}{2}\nu^{-1/2}\sqrt{\gamma}|x-z|} \\
&\le e^{-\frac{1}{2}\alpha^2\nu t}e^{-\frac{1}{2}\nu^{-1/2}\sqrt{\gamma}|x-z|}.
\end{aligned}$$

In addition, using the fact that by definition (6.9) of γ, $\alpha\sqrt{\nu}\le 2\sqrt{\gamma}$, and the inequality $Xe^{-\frac{X}{2}}\le 2e^{-\frac{X}{4}}$, the above yields

$$\begin{aligned}
\Big|\int_{\Gamma_{\alpha,1}} e^{\lambda t}\mathcal{G}_a(x,z)\,\frac{d\lambda}{i\alpha}\Big| &\le C_0 e^{\theta_0 t}e^{-\frac{1}{2}\alpha^2\nu t}e^{-\frac{1}{2}\nu^{-1/2}\sqrt{\gamma}|x-z|}\nu^{-1/2}\sqrt{\gamma+\nu^{-1/2}\sqrt{\gamma}|x-z|} \\
&\le C_0 e^{\theta_0 t}e^{-\frac{1}{2}\alpha^2\nu t}e^{-\frac{1}{4}\nu^{-1/2}\sqrt{\gamma}|x-z|}\sqrt{\gamma}\nu^{-1/2}.
\end{aligned}$$

Certainly, the identical bound as above holds for $x\ge z$.

For $\omega_\alpha\in\mathcal{X}^\eta$, with

$$|\omega_\alpha(z)|\le\|\omega_\alpha\|_\eta e^{-\eta z},$$

we can now estimate

$$\begin{aligned}
&\Big|\int_0^\infty\int_{\Gamma_{\alpha,1}} e^{\lambda t}\mathcal{G}_a(x,z)\omega_\alpha(x)\,\frac{d\lambda}{i\alpha}dx\Big| \\
&\le C_0 e^{\theta_0 t}e^{-\frac{1}{2}\alpha^2\nu t}\|\omega_\alpha\|_\eta\int_0^\infty e^{-\frac{1}{4}\nu^{-1/2}\sqrt{\gamma}|x-z|}\nu^{-1/2}\sqrt{\gamma}e^{-\eta|x|}\,dx.
\end{aligned}$$

To recover the exponential decay in z, we note that for $\eta \le \frac{1}{4}\nu^{-1/2}\sqrt{\gamma}$. Using $|z| \le |x| + |z-x|$, namely $|x| \ge |z| - |z-x|$,

$$e^{-\frac{1}{8}\nu^{-1/2}\sqrt{\gamma}|x-z|}e^{-\eta|x|} \le e^{-\eta|z|}e^{-\frac{1}{8}\nu^{-1/2}\sqrt{\gamma}|x-z|}.$$

Hence,

$$\begin{aligned}
&\Big|\int_0^\infty \int_{\Gamma_{\alpha,1}} e^{\lambda t}\mathcal{G}_a(x,z)\omega_\alpha(x)\ \frac{d\lambda}{i\alpha} dx\Big| \\
&\le C_0 e^{\theta_0 t}e^{-\frac{1}{2}\alpha^2\nu t}\|\omega_\alpha\|_\eta e^{-\eta|z|}\int_0^\infty e^{-\frac{1}{8}\nu^{-1/2}\sqrt{\gamma}|x-z|}\nu^{-1/2}\sqrt{\gamma}\ dx \\
&\le C_0 e^{\theta_0 t}e^{-\frac{1}{2}\alpha^2\nu t}\|\omega_\alpha\|_\eta e^{-\eta|z|}.
\end{aligned}$$

Bounds on $\Gamma_{\alpha,2}$

Next, we focus on $\Gamma_{\alpha,2}$; clearly, the integral on $\Gamma_{\alpha,3}$ is identical. For $\lambda \in \Gamma_{\alpha,2}$ and $y \in [x,z]$, we compute

$$\begin{aligned}
\Re\hat{\mu}_f(y) &= \nu^{-1/2}\Re\sqrt{(a^2-k^2)\nu + i\alpha(U - \min_{[x,z]} U) + 2i\nu ak} \\
&\ge \nu^{-1/2}\Re\sqrt{(a^2-k^2)\nu + 2i\nu ak} \\
&\ge a.
\end{aligned}$$

So, together with the definition of a,

$$e^{\Re\lambda t}e^{-\int_x^z \Re\hat{\mu}_f(y)\ dy} \le e^{-\nu k^2 t - \alpha^2\nu t}e^{-\frac{|x-z|^2}{4\nu t}}$$

and

$$\Big|\frac{d\lambda}{\alpha\epsilon\hat{\mu}_f(x)}\Big| = \Big|\frac{d\lambda}{\sqrt{\nu}\sqrt{(a+ik)^2\nu + i\alpha(U - \min_{[x,z]} U)}}\Big| \le \frac{2|a-ik|dk}{|a+ik|} \le 2dk.$$

Hence, we can estimate

$$\begin{aligned}
\Big|\int_{\Gamma_{\alpha,2}} e^{\lambda t}\mathcal{G}_a(x,z)\ \frac{d\lambda}{i\alpha}\Big| &\le 2\int_{\mathbb{R}_+} e^{-\nu k^2 t - \alpha^2\nu t}e^{-\frac{|x-z|^2}{4\nu t}} dk \\
&\le C_0(\nu t)^{-1/2}e^{-\alpha^2\nu t}e^{-\frac{|x-z|^2}{4\nu t}}.
\end{aligned} \tag{6.10}$$

We thus obtain

$$\begin{aligned}
&\Big|\int_0^\infty \int_{\Gamma_{\alpha,4}} e^{\lambda t}\mathcal{G}_a(x,z)\omega_\alpha(x)\ \frac{d\lambda}{i\alpha} dx\Big| \\
&\le C_0 e^{-\alpha^2\nu t}\|\omega_\alpha\|_\eta \int_0^\infty (\nu t)^{-1/2}e^{-\frac{|x-z|^2}{4\nu t}}e^{-\eta|x|}\ dx.
\end{aligned} \tag{6.11}$$

In the case when $|x-z| \geq 8\eta\nu t$, it is clear that

$$e^{-\frac{|x-z|^2}{8\nu t}} e^{-\eta|x|} \leq e^{-\eta|z|} e^{-|x-z|\left(\frac{|x-z|}{8\nu t} - \eta\right)} \leq e^{-\eta|z|}.$$

Whereas, for $|x-z| \leq 8\eta\nu t$, we note that

$$e^{-\frac{1}{2}\alpha^2\nu t} e^{-\eta|x|} \leq e^{-\frac{1}{2}\nu t} e^{-\eta|x|} \leq e^{-\frac{1}{16\eta}|x-z|} e^{-\eta|x|} \leq e^{-\eta|z|}$$

for $\eta \leq \frac{1}{4}$. In both cases, we obtain

$$\begin{aligned} &\Big| \int_0^\infty \int_{\Gamma_{\alpha,4}} e^{\lambda t} \mathcal{G}_a(x,z)\omega_\alpha(x) \, \frac{d\lambda}{i\alpha} dx \Big| \\ &\leq C_0 e^{-\frac{1}{2}\alpha^2\nu t} \|\omega_\alpha\|_\eta e^{-\eta|z|} \int_0^\infty (\nu t)^{-1/2} e^{-\frac{|x-z|^2}{8\nu t}} \, dx \\ &\leq C_0 \|\omega_\alpha\|_\eta e^{-\eta|z|} e^{-\frac{1}{2}\alpha^2\nu t} \end{aligned}$$

for $\alpha \in \mathbb{Z}^*$.

Bounds on $\Gamma\alpha, 4$ and $\Gamma_{\alpha,5}$.

Finally, we give estimates on $\Gamma_{\alpha,4}$ and $\Gamma_{\alpha,5}$. It suffices to focus on $\Gamma_{\alpha,4}$. In this case, $a^2\nu \leq \theta_0$, with, up to a change in the parametrization k,

$$\lambda = k - \alpha^2\nu - i\alpha \min_{[x,z]} U, \qquad a^2\nu + \frac{1}{2}\alpha^2\nu \leq k \leq \theta_0 + \frac{1}{2}\alpha^2\nu.$$

We note

$$\Re\hat{\mu}_f(y) = \nu^{-1/2} \Re\sqrt{k + i\alpha(U - \min_{[x,z]} U)} \geq \nu^{-1/2}\sqrt{k}$$

and, recalling the definition of $a = \frac{|x-z|}{2\nu t}$,

$$\begin{aligned} \Re\lambda t - \int_x^z \Re\hat{\mu}_f(y)\, dy &\leq -\alpha^2\nu t + kt - \nu^{-1/2}\sqrt{k}|x-z| \\ &\leq -\frac{1}{2}\alpha^2\nu t + \theta_0 t - a|x-z| \\ &= -\frac{1}{2}\alpha^2\nu t + \theta_0 t - \frac{|x-z|^2}{2\nu t}. \end{aligned}$$

Hence, we can estimate

$$\begin{aligned} \Big| \int_{\Gamma_{\alpha,4}} e^{\lambda t} \mathcal{G}_a(x,z) \, \frac{d\lambda}{i\alpha} \Big| &\leq \int_{a^2\nu + \frac{1}{2}\alpha^2\nu}^{\theta_0 + \frac{1}{2}\alpha^2\nu} e^{\theta_0 t - \frac{1}{2}\alpha^2\nu t} e^{-\frac{|x-z|^2}{2\nu t}} dk \\ &\leq C_0 e^{\theta_0 t - \frac{1}{2}\alpha^2\nu t} e^{-\frac{|x-z|^2}{2\nu t}} \end{aligned}$$

using $a^2\sqrt{\nu} \le \theta_0$, in which we could insert a factor $(\nu t)^{-1/2}$, when $\nu t \gg 1$, thanks to the exponentially decaying term $e^{-\frac{1}{2}\alpha^2 \nu t}$. That is, we have obtained the same bound for $\Gamma_{\alpha,4}$ as that for $\Gamma_{\alpha,2}$; see (6.10). This completes the proof of the bounds for S_α as stated in Theorem 2.4.

6.4 Bounds on $\mathcal{R}_\alpha$.

Let us now bound the $\mathcal{R}_\alpha$ integral. Again, we need to choose the contour Γ_α in (6.1). Let M be a number so that $1 + \|U\|_{L^\infty} \le M$, and let

$$\gamma = \Re\lambda_0 + \tau \tag{6.12}$$

for arbitrary, but fixed, constant $\tau > 0$. By the Cauchy's theory, we can decompose the contour Γ_α, for each $\alpha \in \mathbb{N}^*$, as

$$\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,2} \tag{6.13}$$

with

$$\Gamma_{\alpha,1} := [\gamma - i\alpha M, \gamma + i\alpha M], \qquad \Gamma_{\alpha,2} := \Big\{ \gamma - k \pm i\alpha M, \quad k \in \mathbb{R}_+ \Big\}$$

in which $[\cdot,\cdot]$ denotes a segment in the complex plane. Since Γ_α remains in the resolvent set of L_α, the normalized Evans function $D(\alpha, c)$ never vanishes, and so there holds

$$|[D(\alpha,c)]^{-1}| \le C_\tau$$

for some C_τ that depends on τ in (6.12). By Theorem 2.1, the residual Green function $\mathcal{R}_G(x,z)$ satisfies

$$\begin{aligned} |\mathcal{R}_G(x,z)| \le \frac{C_0}{|\epsilon m_f^2|}\Big(\frac{1}{\mu_s} e^{-\theta_0 \mu_s |x-z|} + \frac{1}{m_f} e^{-\theta_0 m_f |x-z|}\Big) \\ + \frac{C_\tau}{|\epsilon m_f^2|}\Big(\frac{1}{\mu_s} e^{-\theta_0 \mu_s (|z|+|x|)} + \frac{1}{m_f} e^{-\theta_0 m_f (|z|+|x|)}\Big). \end{aligned}$$

We estimate term by term by first integrating the convolution in x and then the λ-integral. Again, we assume $\|\omega_\alpha\|_\eta = 1$. We have

$$\begin{aligned} &\int_0^\infty \mu_s^{-1}\Big(e^{-\theta_0\mu_s(|x|+|z|)} + e^{-\theta_0\mu_s|x-z|}\Big)|\omega_\alpha(x)|\,dx \\ &\le \|\omega_\alpha\|_\eta \int_0^\infty \mu_s^{-1}\Big(e^{-\theta_0\mu_s(|x|+|z|)} + e^{-\theta_0\mu_s|x-z|}\Big)e^{-\beta x}\,dx \\ &\le C_0 \|\omega_\alpha\|_\eta e^{-\beta z} \mu_s^{-2} \end{aligned} \tag{6.14}$$

in which we have used $\theta_0 \mu_s > \beta$, yielding the spatial decay at the rate of $e^{-\beta z}$. Certainly, the same computation holds for the fast behavior terms in the Green function. To summarize, we have obtained the following convolution estimate

$$\Big| \int_0^\infty \mathcal{R}_G(x,z)\omega_\alpha(x)\, \frac{dx}{i\alpha} \Big| \le C_0 |\alpha\epsilon m_f^2|^{-1} \alpha^{-2} e^{-\beta z} \|\omega_\alpha\|_\eta, \tag{6.15}$$

using the fact that $m_f \gg \mu_s$ and $\mu_s = \alpha$ to simplify the bound. Using this, we can now integrate the above with respect to λ along the contour Γ_α of integration that are chosen as in (6.13). We recall that

$$i\alpha\epsilon m_f^2 = \lambda + i\alpha \inf_{\mathbb{R}_+} U + \alpha^2 \nu.$$

We integrate

$$\begin{aligned}
\int_{\Gamma_{\alpha,1}} \frac{e^{\Re\lambda t} |d\lambda|}{|\lambda + i\alpha \inf_{\mathbb{R}_+} U + \alpha^2\nu|} &\le \int_{-M}^{M} \frac{e^{\gamma t} \alpha dk}{\sqrt{\gamma^2 + \alpha^2 (k + \inf_{\mathbb{R}_+} U)^2}} \\
&\le \int_{-M + \inf_{\mathbb{R}_+} U}^{M + \inf_{\mathbb{R}_+} U} \frac{e^{\gamma t} dk}{\sqrt{\gamma^2 \alpha^{-2} + k^2}} \\
&\le C_0 \int_{-M + \inf_{\mathbb{R}_+} U}^{M + \inf_{\mathbb{R}_+} U} \frac{e^{\gamma t} dk}{k + \gamma\alpha^{-1}} \\
&\le C_0 e^{\gamma t} \log \alpha.
\end{aligned}$$

Similarly, we compute the integral on $\Gamma_{\alpha,2}$. We start with the case when $\alpha t \gtrsim 1$. We have

$$\begin{aligned}
\int_{\Gamma_{\alpha,2}} \frac{e^{\Re\lambda t} |d\lambda|}{|\lambda + i\alpha \inf_{\mathbb{R}_+} U + \alpha^2\nu|} &\le \int_{\mathbb{R}_+} \frac{e^{\gamma t} e^{-kt} dk}{\sqrt{(\gamma - k)^2 + \alpha^2 M^2}} \\
&\le C_0 \int_{\mathbb{R}_+} \frac{e^{\gamma t} e^{-l} dl}{\sqrt{l^2 + \alpha^2 t^2}} \\
&\le C_0 e^{\gamma t},
\end{aligned}$$

in which the assumption $\alpha t \gtrsim 1$ was used.

Finally, in the case when $\alpha t \ll 1$, to avoid the singularity in small time, we shall need to perform the λ-integration first. We decompose the contour of integration as follows:

$$\begin{aligned}
\Gamma_{\alpha,1} &:= \{|\lambda| = \alpha M\} \cap \{\Re\lambda \ge 0\} \\
\Gamma_{\alpha,2} &:= \Big\{ \lambda = -|k| + ik \pm i\alpha M, \quad k \in \mathbb{R} \Big\}.
\end{aligned}$$

We start with the fast behavior in the Green function:

$$\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\Re\lambda t} e^{-\theta_0 m_f |x-z|} |\omega_\alpha(x)|}{\alpha|\epsilon m_f^2||m_f|} \, |d\lambda| dx. \tag{6.16}$$

On $\Gamma_{\alpha,1}$, since $m_f \geq \sqrt{\alpha}\nu^{-1/2}$, we compute

$$\begin{aligned}\int_{\Gamma_{\alpha,1}} \frac{e^{\Re\lambda t} e^{-\theta_0 m_f |x-z|}}{\alpha|\epsilon m_f^2||m_f|} \, |d\lambda| &\leq \int_{|\lambda|=\alpha M\}} \sqrt{\nu}\alpha^{-3/2} e^{\alpha M t} e^{-\theta_0 \sqrt{\alpha}\nu^{-1/2}|x-z|} |d\lambda| \\ &\leq C_0 \sqrt{\nu}\alpha^{-3/2} e^{-\theta_0\sqrt{\alpha}\nu^{-1/2}|x-z|}\end{aligned}$$

upon recalling that we are in the case when $\alpha t \ll 1$. Now on $\Gamma_{\alpha,2}$, we note that

$$m_f \geq \nu^{-1/2}\sqrt{k+\alpha}$$

with $\lambda = -|k| + ik \pm i\alpha M$. Hence,

$$\begin{aligned}\int_{\Gamma_{\alpha,2}} \frac{e^{\Re\lambda t} e^{-\theta_0 m_f |x-z|}}{\alpha|\epsilon m_f^2||m_f|} \, |d\lambda| &\leq C_0 \int_{\mathbb{R}} \sqrt{\nu}(k+\alpha)^{-3/2} e^{-kt} e^{-\theta_0\sqrt{\alpha}\nu^{-1/2}|x-z|} dk \\ &\leq C_0 \sqrt{\nu} e^{-\theta_0\sqrt{\alpha}\nu^{-1/2}|x-z|} \int_{\mathbb{R}} (k+\alpha)^{-3/2} dk \\ &\leq C_0 \sqrt{\nu} e^{-\theta_0\sqrt{\alpha}\nu^{-1/2}|x-z|}.\end{aligned}$$

The convolution estimate thus follows straightforwardly as above. A similar estimate holds for the fast behavior in the Green function involving the term $e^{-\theta_0 m_f(|x|+|z|)}$. We now turn to the slow behavior, treating the integral

$$\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\Re\lambda t} e^{-\theta_0 \mu_s |x-z|} |\omega_\alpha(x)|}{\alpha|\epsilon m_f^2||\mu_s|} \, |d\lambda| dx. \tag{6.17}$$

In this case, since $|\epsilon m_f^2|^{-1}\mu_s^{-1}$ is no longer integrable for large λ, we are obliged to use the fast behavior from the Green function $\mathcal{G}_a(x,z)$. Indeed, we recall that the residual Green function $\mathcal{R}_G(x,z)$ is defined by

$$\mathcal{R}_G(x,z) = \int_0^\infty \mathcal{G}_a(y,z) U''(y) G_{\alpha,c}(x,y) \, dy$$

and so the integral (6.17) was in fact the following integral

$$\int_0^\infty \int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\Re\lambda t} e^{-m_f|y-z|} e^{-\theta_0\mu_s|x-y|} e^{-\eta_0|y|} |\omega_\alpha(x)|}{\alpha|\epsilon m_f||\mu_s|} \, |d\lambda| dy dx. \tag{6.18}$$

Let us now estimate (6.18). We now take the following contour of integration

$$\Gamma_\alpha = \Gamma_{\alpha,1} \cup \Gamma_{\alpha,\pm}$$

with

$$\begin{aligned}\Gamma_{\alpha,1} &= [-\alpha^2\nu + a^2 - i\alpha M, -\alpha^2\nu + a^2 + i\alpha M]\\ \Gamma^{\pm}_{\alpha,2} &= \Big\{\lambda = -\alpha^2\nu + a^2 - k^2 + 2aik \pm i\alpha M, \quad k \in \mathbb{R}_\pm\Big\}\end{aligned}$$

in which we take

$$a := \frac{|y-z|}{2\sqrt{\nu t}} + \sqrt{\alpha M}.$$

We start with $\Gamma_{\alpha,1}$. With $\lambda = -\alpha^2\nu + a^2 - i\alpha c$, we compute

$$\hat\mu_f = \nu^{-1/2}\sqrt{\lambda + i\alpha U + \alpha^2\nu} = \nu^{-1/2}\sqrt{a^2 + i\alpha(U-c)}$$

and hence

$$\Re\hat\mu_f \ge \nu^{-1/2}a.$$

Using this, and the assumption that $\alpha t \ll 1$, we have

$$\begin{aligned}\int_{\Gamma_{\alpha,1}} \frac{e^{\Re\lambda t}e^{-m_f|y-z|}}{\alpha^2|\varepsilon m_f|}\,|d\lambda| &\le C_0 e^{a^2t - \nu^{-1/2}a|y-z|}e^{-\alpha^2\nu t}\int_{-M}^{M}\frac{dc}{\sqrt{\nu}|\sqrt{a^2+i\alpha(U-c)}|}\\ &\le C_0 e^{-\frac{|y-z|^2}{4\nu t}}e^{-\alpha^2\nu t}e^{\alpha M t}\int_{-M}^{M}\frac{dc}{\sqrt{\nu}a}\\ &\le C_0\alpha^{-1/2}\nu^{-1/2}e^{-\frac{|y-z|^2}{4\nu t}}e^{-\alpha^2\nu t}\\ &\le C_0\alpha^{-1/2}t^{1/2}(\nu t)^{-1/2}e^{-\frac{|y-z|^2}{4\nu t}}e^{-\alpha^2\nu t}\\ &\le C_0\alpha^{-1}(\nu t)^{-1/2}e^{-\frac{|y-z|^2}{4\nu t}}e^{-\alpha^2\nu t}.\end{aligned}$$

Let us turn to the integral on $\Gamma^{\pm}_{\alpha,2}$. We compute

$$\begin{aligned}\Re\hat\mu_f(x) &= \nu^{-1/2}\Re\sqrt{a^2 - k^2 + 2aik \pm i\alpha M + i\alpha U(x)}\\ &\ge \nu^{-1/2}\Re\sqrt{a^2-k^2+2aik} = \nu^{-1/2}a\end{aligned}$$

and so

$$\begin{aligned}\Re\lambda t - m_f|y-z| &\le -\alpha^2\nu t + (a^2-k^2)t - \nu^{-1/2}a|y-z|\\ &\le -\alpha^2\nu t - k^2 t - \frac{|y-z|^2}{4\nu t} + M\alpha t.\end{aligned}$$

Recalling $\alpha t \ll 1$, we thus have

$$\int_{\Gamma^{\pm}_{\alpha,2}} \frac{e^{\Re\lambda t}e^{-m_f|y-z|}}{\alpha^2|\varepsilon m_f|}\,|d\lambda| \le C_0 e^{-\frac{|y-z|^2}{4\nu t}} e^{-\alpha^2\nu t}\int_{\mathbb{R}} \alpha^{-1}\nu^{-1/2}e^{-k^2 t}dk$$
$$\le C_0\alpha^{-1}(\nu t)^{-1/2}e^{-\frac{|y-z|^2}{4\nu t}}e^{-\alpha^2\nu t}$$

which is the same bound as that of $\Gamma_{\alpha,1}$. Thus, we can estimate the y-integration in (6.18). The above yields

$$\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\Re\lambda t}e^{-m_f|y-z|}e^{-\theta_0\mu_s|x-y|}e^{-\eta_0|y|}}{\alpha|\varepsilon m_f||\mu_s|}\,|d\lambda|dy$$
$$\le C_0\int_0^\infty \alpha^{-1}(\nu t)^{-1/2}e^{-\frac{|y-z|^2}{4\nu t}}e^{-\alpha^2\nu t}e^{-\theta_0\mu_s|x-y|}dy.$$

We consider two cases. First, when $|y-z| \ge 8\theta_0\mu_s\nu t$, with $\theta_0 \le \frac{1}{4}$, we have

$$e^{-\frac{|y-z|^2}{8\nu t}}e^{-\theta_0\mu_s|x-y|} \le e^{-|y-z|\left(\frac{|y-z|}{8\nu t}-\theta_0\mu_s\right)}e^{-\theta_0\mu_s|x-z|} \le e^{-\theta_0\mu_s|x-z|}.$$

Whereas when $|y-z| \le 8\theta_0\mu_s\nu t$, we bound

$$e^{-\frac{1}{2}\alpha^2\nu t}e^{-\theta_0\mu_s|x-y|} \le e^{-\frac{1}{2}\alpha^2\nu t}e^{\theta_0^2\mu_s^2\nu t}e^{-\theta_0\mu_s|x-z|} \le e^{-\theta_0\mu_s|x-z|}$$

upon recalling that $\mu_s = \alpha$. Combining these estimates, we obtain

$$\int_0^\infty \int_{\Gamma_\alpha} \frac{e^{\Re\lambda t}e^{-m_f|y-z|}e^{-\theta_0\mu_s|x-y|}e^{-\eta_0|y|}}{\alpha|\varepsilon m_f||\mu_s|}\,|d\lambda|dy$$
$$\le C_0 e^{-\theta_0\mu_s|x-z|}\int_0^\infty \mu_s^{-1}(\nu t)^{-1/2}e^{-\frac{|y-z|^2}{8\nu t}}e^{-\frac{1}{2}\alpha^2\nu t}dy$$
$$\le C_0\mu_s^{-1}e^{-\theta_0\mu_s|x-z|}e^{-\frac{1}{2}\alpha^2\nu t}.$$

The x-convolution estimate is exactly that of (6.14). The above estimate clearly holds for the case when $e^{-\mu_s|y-z|}$ is replaced by $e^{-\mu_s(|y|+|z|)}$. This completes the proof of the bounds on $\mathcal{R}_\alpha$ as stated in Theorem 2.4.

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