Spectral instability of general symmetric shear flows in a two-dimensional channel

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Abstract

In this paper, we prove the spectral instability of general symmetric shear flows of the incompressible Navier-Stokes equations at a high Reynolds number in a two-dimensional channel. This includes shear flows that are spectrally stable to the corresponding Euler equations, and thus for the first time, provides a complete mathematical proof of the viscous destabilization phenomenon, pointed out by Heisenberg (1924), C.C. Lin and Tollmien (1940s), among others. Precisely, we construct exact unstable eigenvalues and eigenfunctions of the linearized Navier-Stokes equations around symmetric shear flows, showing that the solution could grow slowly at the rate of $e^{t/\sqrt{\alpha R}}$, where R is the sufficiently large Reynolds number and α is the small spatial frequency that remains between lower and upper marginal stability curves: $\alpha_{\rm low}(R) \approx R^{-1/7}$ and $\alpha_{\rm up}(R) \approx R^{-1/11}$. We introduce a new, operator-based approach, which avoids to deal with matching inner and outer asymptotic expansions, but instead involves a careful study of singularity in the critical layers by deriving pointwise bounds on the Green function of the corresponding Rayleigh and Airy operators.

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1 Introduction

Study of hydrodynamic stability and the inviscid limit of viscous fluids is one of the most classical subjects in fluid dynamics, going back to the most prominent physicists including Lord Rayleigh, Orr, Sommerfeld, Heisenberg, among many others. It is documented in the physical literature (see, for instance, [10, 1]) that laminar viscous fluids are unstable, or become turbulent, in a small viscosity or high Reynolds number limit. In particular, shear flows other than the linear Couette flow in a two-dimensional channel are linearly unstable for sufficiently large Reynolds numbers. In the present work, we provide a complete mathematical proof of these physical results in a channel.

Specifically, let $u_0 = (U(z), 0)^{tr}$ be a stationary plane shear flow in a two-dimensional channel: $(y, z) \in \mathbb{R} \times [0, 2]$; see Figure 1. We are interested in the linearization of the incompressible Navier-Stokes equations about the shear profile:

$$v_t + u_0 \cdot \nabla v + v \cdot \nabla u_0 + \nabla p = \frac{1}{R}\Delta v \tag{1.1a}$$

$$\nabla \cdot v = 0 \tag{1.1b}$$

posed on $\mathbb{R} \times [0, 2]$, together with the classical no-slip boundary conditions on the walls:

$$v_{|_{z=0,2}} = 0. \tag{1.2}$$

Here v denotes the usual velocity perturbation of the fluid, and p denotes the corresponding pressure. Of interest is the Reynolds number R sufficiently large, and whether the linearized problem is spectrally unstable: the existence of unstable modes of the form $(v, p) = (e^{\lambda t}\tilde{v}(y, z), e^{\lambda t}\tilde{p}(y, z))$ for some λ with $\Re\lambda > 0$.

The spectral problem is a very classical issue in fluid mechanics. A huge literature is devoted to its detailed study. We in particular refer to [1, 15] for the major works of Heisenberg, C.C. Lin, Tollmien, and Schlichting. The studies began around 1930, motivated by the study of the boundary layer around wings. In airplanes design, it is crucial to study the boundary layer around the wing, and more precisely the transition between the laminar and turbulent regimes, and even more crucial to predict the point where boundary layer splits from the boundary. A large number of papers has been devoted to the estimation of the critical Reynolds number of classical shear flows (plane Poiseuille flow, Blasius profile, exponential suction/blowing profile, among others).

It were Sommerfeld and Orr [16, 12] who initiated the study of the spectral problem via the Fourier normal mode theory. They search for the unstable solutions of the form $e^{i\alpha(y-ct)}(\hat{v}(z), \hat{p}(z))$, with $\alpha \in \mathbb{R}$ and $c \in \mathbb{C}$, and derive the well-known Orr-Sommerfeld equations for linearized viscous fluids:

$$-i\epsilon(\partial_z^2 - \alpha^2)^2\phi = (U - c)(\partial_z^2 - \alpha^2)\phi - U''\phi, \tag{1.3}$$

with i the complex number $i^2 = -1$ and $\epsilon = \frac{1}{\alpha R}$, where $\phi(z)$ denotes the corresponding stream function, with ϕ and $\partial_z\phi$ vanishing at the boundaries $z = 0, 2$. When $\epsilon = 0$,

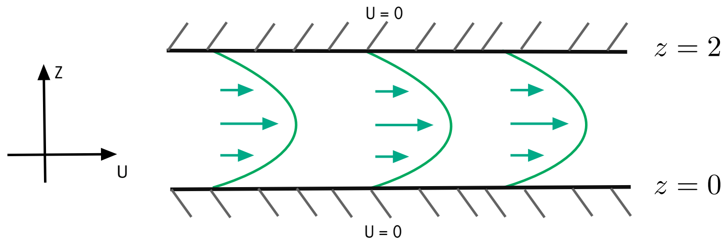

Figure 1: *Shown is the graph of an inviscid stable shear profile.*

(1.3) reduces to the classical Rayleigh equation, which corresponds to inviscid flows. The singular perturbation theory was developed to construct Orr-Sommerfeld solutions from those of Rayleigh solutions.

Inviscid unstable profiles. If the profile is unstable for the Rayleigh equation, then there exist a spatial frequency α_∞, an eigenvalue c_∞ with Im $c_\infty > 0$, and a corresponding eigenvalue ϕ_∞ that solve (1.3) with $\epsilon = 0$ or $R = \infty$. We can then make a perturbative analysis to construct an unstable eigenmode ϕ_R of the Orr-Sommerfeld equation with an eigenvalue Im $c_R > 0$ for any large enough R. This can be done by adding a boundary sublayer to the inviscid mode ϕ_∞ to correct the boundary conditions for the viscous problem. In fact, we can further check that

$$c_R = c_\infty + \mathcal{O}(R^{-1}), \tag{1.4}$$

as $R \to \infty$. Thus, the time growth is of order $e^{\theta_0 t}$, for some $\theta_0 > 0$. Such a perturbative argument for the inviscid unstable profiles is well-known; see, for instance, Grenier [4] where he rigorously establishes the nonlinear instability of inviscid unstable profiles.

Inviscid stable profiles. There are various criteria to check whether a shear profile is stable to the Rayleigh equation. The most classical one was due to Rayleigh [13]: *A necessary condition for instability is that $U(z)$ must have an inflection point*, or its refined version by Fjortoft [1]: *A necessary condition for instability is that $U''(U - U(z_0)) < 0$ somewhere in the flow, where z_0 is a point at which $U''(z_0) = 0$.* For instance, the plane Poiseuille flow: $U(z) = 1 - (z-1)^2$, or the sin profile: $U(z) = \sin(\frac{\pi z}{2})$ are stable to the Rayleigh equation.

For such a stable profile, all the spectrum of the Rayleigh equation is imbedded on the imaginary axis: Re $(-i\alpha c_\infty) = \alpha$Im $c_\infty = 0$, and thus it is not clear whether a perturbative argument to construct solutions (c_R, ϕ_R) to (1.3) would yield stability (Im $c_R < 0$) or instability (Im $c_R > 0$). Except the case of the linear Couette flow $U(z) = z$, which is proved to be linearly stable for all Reynolds numbers by Romanov [14], *all other profiles (including those which are inviscid stable) are physically shown to be linearly unstable for large Reynolds numbers.* Heisenberg [5, 6], and then Tollmien [17] and C. C. Lin [9, 10] were among the first physicists to use asymptotic expansions to study the instability; see also Drazin and Reid [1] for a complete account of the physical literature on the subject. There, it

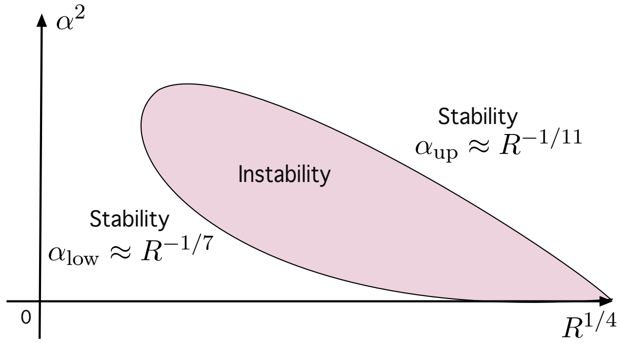

Figure 2: *Illustrated are the marginal stability curves, defined as in* (1.5).

is documented that there are lower and upper marginal stability branches $\alpha_{\text{low}}(R), \alpha_{\text{up}}(R)$ so that whenever $\alpha \in [\alpha_{\text{low}}(R), \alpha_{\text{up}}(R)]$, there exist an unstable eigenvalue c_R and an eigenfunction $\phi_R(z)$ to the Orr-Sommerfeld problem. In the case of symmetric Poiseuille profile: $U(z) = 1 - (z-1)^2$, the marginal stability curves are

$$\alpha_{\text{low}}(R) = A_{1c} R^{-1/7} \qquad \text{and} \qquad \alpha_{\text{up}}(R) = A_{2c} R^{-1/11}, \tag{1.5}$$

for some critical constants A_{1c}, A_{2c}. Their formal analysis has been compared with modern numerical computations and also with experiments, showing a very good agreement; see Figure 2 (see also [1, Figure 5.5]) for a sketch of the marginal stability curves for plane Poiseuille flow, which is an exact steady state solution to the Navier-Stokes equations.

In his works [19, 20, 21], Wasow developed the turning point theory to rigorously validate the formal asymptotic expansions used by the physicists in a full neighborhood of the turning points (or the critical layers in our present paper). Wasow wrote ([19, pp. 868–870]): *"It also turns out that the formal theory alone does not give sufficient information about the actual asymptotic behavior. We are not going to apply our theory to the stability problem proper, but we shall mention two points which are left somewhat obscure in previous investigations..."*. In his book ([21, Chapter 1]), Wasow pointed out again the need of a complete mathematical justification of the linear stability theory. Even though Drazin and Reid ([1]) indeed provide many delicate asymptotic analysis in different regimes with different matching conditions near the critical layers, it is mathematically unclear how to combine their "local" analysis into a single convergent "global expansion" to produce an exact growing mode for the Orr-Sommerfeld equation. To our knowledge, remarkably, after all these efforts, a complete rigorous construction of an unstable growing mode is still elusive for such a fundamental problem.

Our present paper rigorously establishes the spectral instability of generic shear flows. The main theorem reads as follows.

Theorem 1.1. *Let $U(z)$ be an arbitrary shear profile that is analytic and symmetric about $z = 1$ with $U(0) = 0$, $U'(0) > 0$ and $U'(1) = 0$. Let $\alpha_{\text{low}}(R)$ and $\alpha_{\text{up}}(R)$ be defined as in* (1.5). *Then, there is a critical Reynolds number R_c so that for all $R \geq R_c$ and all*

$\alpha \in (\alpha_{\rm low}(R), \alpha_{\rm up}(R))$, there exist a triple $c(R), \hat v(z;R), \hat p(z;R)$, with Im $c(R) > 0$*, such that $v_R := e^{i\alpha(y-ct)}\hat v(z;R)$ and $p_R := e^{i\alpha(y-ct)}\hat p(z;R)$ solve the problem* (1.1a)-(1.1b) *with the no-slip boundary conditions. In the case of instability, there holds the following estimate for the growth rate of the unstable solutions:*

$$\alpha Im\ c(R) \quad \approx \quad (\alpha R)^{-1/2},$$

as $R \to \infty$. In addition, the horizontal component of the unstable velocity v_R is odd in z, whereas the vertical component is even in z.

Theorem 1.1 allows general shear profiles. It is worth recalling that stationary shear flows $u_0 = (U(z), 0)^{tr}$ are exact solutions to the Euler equations, and thus approximate solutions to the Navier-Stokes equations at sufficiently high Reynolds numbers. For instance, the time-dependent shear flows $u_{sh}(t,z) = (U(R^{-1/2}t, z), 0)^{tr}$, with $U(\tau, z)$ solving the heat equation with initial data $U(z)$, are exact solutions to the Navier-Stokes equations. Since the instability growth, obtained in the theorem, is of order $e^{t/\sqrt{\alpha R}}$, which becomes significant only after the the instability time of order $t_R \approx \sqrt{\alpha R} \to \infty$, the time-dependent shear flows can be approximated, within the instability time t_R, by the corresponding stationary ones: $u_{sh}(t,z) = u_0(z) + \mathcal{O}(\sqrt{\alpha})$, as $\alpha \to 0$. The instability of u_0 indicates that of general shear flows u_{sh}, which are exact solutions to the Navier-Stokes equations.

The instability is found for general shear flows, including in particular inviscid stable flows such as plane Poiseuille flows or flows without an inflection point, and thus the instability is due to the presence of small viscosity. It is worth noting that the growth rate is vanishing in the inviscid limit: $R \to \infty$, which is expected as the Euler instability is necessary in the inviscid limit for the instability with non-vanishing growth rate; for the latter result, see [3] in which general stationary profiles are considered. Linear to nonlinear instability is a delicate issue, primarily due to the fact that there is no available, comparable bound on the linearized solution operator as compared to the maximal growing mode. Available analyses (for instance, [2, 4]) do not appear applicable in the inviscid limit.

As mentioned earlier, we construct the unstable solutions via the Fourier normal mode method. Precisely, let us introduce the stream function ψ through

$$v = \nabla^\perp \psi = (\partial_z, -\partial_y)\psi, \qquad \psi(t,y,z) := \phi(z) e^{i\alpha(y-ct)}, \tag{1.6}$$

with $y \in \mathbb{R}$, $z \in [0,2]$, the spatial frequency $\alpha \in \mathbb{R}$ and the temporal eigenvalue $c \in \mathbb{C}$. As our main interest is to study symmetric profiles, we will construct solutions that are also symmetric with respect to the line $z = 1$. The equation for vorticity $\omega = \Delta\psi$ becomes the classical Orr–Sommerfeld equation for ϕ

$$-i\epsilon(\partial_z^2 - \alpha^2)^2\phi = (U - c)(\partial_z^2 - \alpha^2)\phi - U''\phi, \qquad z \in [0,1], \tag{1.7}$$

with $\epsilon = \frac{1}{\alpha R}$. The no-slip boundary condition on v then becomes

$$\alpha\phi = \partial_z\phi = 0 \quad \text{at} \quad z = 0, \tag{1.8}$$

whereas the symmetry about $z = 1$ requires

$$\partial_z \phi = \partial_z^3 \phi = 0 \quad \text{at} \quad z = 1. \tag{1.9}$$

Clearly, if $\phi(z)$ solves the Orr-Sommerfeld problem (1.7)-(1.9), then the velocity v defined as in (1.6) solves the linearized Navier-Stokes problem with the pressure p solving

$$-\Delta p = \nabla U \cdot \nabla v, \qquad \partial_z p_{|_{z=0,2}} = -\partial_z^2 \partial_y \psi_{|_{z=0,2}}.$$

Throughout the paper, we study the Orr-Sommerfeld problem.

Delicacy in the construction is primarily due to the formation of critical layers. To see this, we first observe that $\alpha_0 = 0, c_0 = 0$, and $\phi_0 = U(z)$ solve the Rayleigh problem. We shall construct exact Orr-Sommerfeld solutions as a perturbation of the Rayleigh solution ϕ_0, as $\alpha, \epsilon \to 0$. Next, we extend $U(z)$ analytically in a neighborhood of $z = 0$ in $\mathbb{C}$. Then, for any complex number c in a small neighborhood of 0, since $U(0) = 0$ and $U'(0) \neq 0$, there is a unique $z_c \in \mathbb{C}$, so that $z_c \approx c$ and

$$U(z_c) = c. \tag{1.10}$$

Now, since the coefficient of the highest-order derivative in the Rayleigh equation vanishes at $z = z_c$, the Rayleigh solution $\phi_\alpha(z)$ has a singularity of the form: $1 + (z - z_c)\log(z - z_c)$. A perturbation analysis to construct an Orr-Sommerfeld solution ϕ_ϵ out of ϕ_α will face a singular source $\epsilon(\partial_z^2 - \alpha^2)^2 \phi_\alpha$ at $z = z_c$. To deal with the singularity, we need to introduce the critical layer ϕ_{cr} that solves

$$-i\epsilon \partial_z^4 \phi_{\mathrm{cr}} = (U - c)\partial_z^2 \phi_{\mathrm{cr}}. \tag{1.11}$$

When z is near z_c, by (1.10), we may replace $U - c$ by $U_c'(z - z_c)$ and so the above equation for the critical layer becomes the classical Airy equation for $\partial_z^2 \phi_{\mathrm{cr}}$. This shows that the critical layer mainly depends on the fast variable: $\phi_{\mathrm{cr}} = \phi_{\mathrm{cr}}(Y)$ with $Y = (z - z_c)/\delta$, in which the critical layer thickness is defined by

$$\delta = \Big(\frac{\epsilon}{iU'(z_c)}\Big)^{1/3} = e^{-i\pi/6}\Big(\frac{\epsilon}{U'(z_c)}\Big)^{1/3} \tag{1.12}$$

in which we have taken $i^{1/3} = e^{i\pi/6}$.

In the literature, the point z_c is occasionally referred to as a turning point, since the eigenvalues of the associated first-order ODE system cross at $z = z_c$, and therefore it is delicate to construct asymptotic solutions that are analytic across different regions near the turning point. In his work, Wasow fixed the turning point to be zero, and were able to construct asymptotic solutions in a full neighborhood of the turning point. It is also interesting to point out that the authors in [8] recently revisit the analysis near turning points, and are able to construct unstable solutions in the context of gas dynamics, via WKB-type asymptotic techniques.

In the present paper, we introduce a new, operator-based approach, which avoids dealing with inner and outer asymptotic expansions, but instead constructs the Green's function, and therefore the inverse, of the corresponding Rayleigh and Airy operators. The Green's function of the critical layer (Airy) equation is complicated by the fact that we have to deal with the second primitive Airy functions, not to mention that the argument Y is complex. The basic principle of our construction, for instance, of a slow decaying solution, will be as follows. We start with an exact Rayleigh solution ϕ_α (solving (1.7) with $\epsilon = 0$). This solution then solves (1.7) approximately up to the error term $\epsilon(\partial_z^2 - \alpha^2)^2\phi_\alpha$, which is singular at $z = z_c$ since ϕ_α is of the form $1 + (z - z_c)\log(z - z_c)$ inside the critical layer. We then correct ϕ_α by adding a critical layer profile $\phi_{\rm cr}$ constructed by convoluting the Green's function of the primitive Airy operator against the singular error $\epsilon(\partial_z^2 - \alpha^2)^2\phi_\alpha$. The resulting solution $\phi_\alpha + \phi_{\rm cr}$ solves (1.7) up to a smaller error that consists of no singularity. An exact slow mode of (1.7) is then constructed by inductively continuing this process. For a fast mode, we start the induction with a second primitive Airy function.

Small parameters. Throughout the paper, there are three small independent parameters (α, c, ϵ):

$$(\alpha, c, \epsilon) \quad \approx \quad (0, 0, 0) \tag{1.13}$$

in which α, ϵ are real numbers and c is a complex number. Two other small parameters are the critical layer z_c, defined through the relation $U(z_c) = c$, and the critical layer thickness $\delta = (\epsilon / iU'(z_c))^{1/3}$, defined as in (1.12). In Section 8, through the dispersion relation, we shall prove the existence of a small complex parameter $c = c(\alpha, \epsilon)$, for each small numbers (α, ϵ). Motivated by the physical literature, we then restrict to the range of (α, ϵ) so that $\alpha^{10} \lesssim \epsilon \lesssim \alpha^6$, with which we establish the instability theorem. The relation of the spatial frequency α and the Reynolds number R (as stated in the main theorem) then follows from the definition $\epsilon = \frac{1}{\alpha R}$. Except in Section 8, estimates carried out in the paper are obtained without a priori knowledge of the restricted range of smallness of α, ϵ, and c.

Notation. We shall use C_0 to denote a universal constant that may change from line to line, but is independent of the smallness of α, c and ϵ. We also use the notation $f = \mathcal{O}(g)$ or $f \lesssim g$ to mean that $|f| \le C_0|g|$, for some constant C_0. Similarly, $f \approx g$ if and only if $f \lesssim g$ and $g \lesssim f$. Finally, we shall use the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$.

2 Strategy of proof

2.1 Operators

For our convenience, let us introduce the following operators. Let us denote by Orr the Orr-Sommerfeld operator

$$Orr(\phi) := (U - c)(\partial_z^2 - \alpha^2)\phi - U''\phi + i\varepsilon(\partial_z^2 - \alpha^2)^2\phi, \tag{2.1}$$

by Ray_α the Rayleigh operator

$$Ray_\alpha(\phi) := (U - c)(\partial_z^2 - \alpha^2)\phi - U''\phi, \tag{2.2}$$

by $Diff$ the diffusive part of the Orr-Sommerfeld operator,

$$Diff(\phi) := i\varepsilon(\partial_z^2 - \alpha^2)^2\phi, \tag{2.3}$$

by $Airy$ the modified Airy equation

$$Airy(\phi) := -i\varepsilon\partial_z^4\phi - (U - c - 2i\varepsilon\alpha^2)\partial_z^2\phi, \tag{2.4}$$

and finally, by Reg the regular zeroth order part of the Orr-Sommerfeld operator

$$Reg(\phi) := \Big[i\varepsilon\alpha^4 - U'' - \alpha^2(U - c)\Big]\phi. \tag{2.5}$$

Clearly, there hold identities

$$Orr = Ray_\alpha + Diff = -Airy + Reg. \tag{2.6}$$

2.2 Outline of the construction

Let us outline the strategy of the proof before going into the technical details and computations. Our ultimate goal is to construct four independent solutions of the fourth order differential equation (1.7) and then combine them in order to satisfy boundary conditions (1.8) and (1.9), yielding the linear dispersion relation. The unstable eigenvalues are then found by carefully studying the dispersion relation.

The idea of the proof is to start from a mode of Rayleigh equation, or from an Airy function ϕ_0. This function is not an exact solutions of Orr Sommerfeld equations, but leads to an error

$$E_0 = Orr(\phi_0).$$

We correct it by adding ϕ_0^{Ray} defined by

$$Ray_\alpha(\phi_0^{Ray}) = -Orr(\phi_0).$$

Again $\phi_0 + \phi_0^{Ray}$ is not an exact solution of Orr Sommerfeld equations

$$Orr(\phi_0 + \phi_0^{Ray}) = Diff(\phi_0^{Ray}).$$

It turns out that, even if ϕ_0 is smooth, ϕ_0^{Ray} is not smooth and contains a singularity of the form $(z - z_c)\log(z - z_c)$. As a consequence, $Diff(\phi_0^{Ray})$ contains terms like $1/(z - z_c)^3$. To smooth out this singularity we use Airy operator and introduce ϕ_0^A defined by

$$Airy(\phi_0^A) = -Diff(\phi_0^{Ray}).$$

Then

$$\phi_1 = \phi_0 + \phi_0^{Ray} + \phi_0^A$$

satisfies

$$E_1 = Orr(\phi_1) = Reg(\phi_0^A).$$

Note that in some sense ϕ_0^A replaces the $(z - z_c)\log(z - z_c)$ singular term by a smoother one.

We then iterate the construction. Note that

$$E_1 = Reg\Big(Airy^{-1}\Big(Diff(Ray^{-1}(E_0))\Big)\Big).$$

The main problem is to check the convergence of this process, and more precisely to prove that

$$Reg \circ Airy^{-1} \circ Diff \circ Ray^{-1}$$

has a norm strictly smaller than 1 in suitable functional spaces. Note that our approach avoids to deal with inner and outer expansions, but requires a careful study of the singularities and delicate estimates on the resolvent solutions.

We introduce two families of function spaces, X_p and Y_p which turn out to be very well fitted to describe functions which are singular near z_c. Here, the critical layer z_c is defined through $U(z_c) = c$. First, the the function spaces X_p are defined by their norms:

$$\|f\|_{X_p} := \sup_{z\in[0,1]} \sum_{k=0}^{p} |(z - z_c)^k \partial_z^k f(z)|.$$

We also introduce the function spaces Y_p defined by: $f \in Y_p$ if there exists a constant C such that

$$|f(z)| \le C \quad \forall 0 \le z \le 1,$$
$$|\partial_z f(z)| \le C(1 + |\log(z - z_c)|) \quad \forall 0 \le z \le 1,$$
$$|\partial_z^l f(z)| \le C(1 + |z - z_c|^{1-l})$$

for every $0 \le z \le 1$ and every $l \le p$. The best constant C in the previous bounds is by definition the norm $\|f\|_{Y_p}$.

Let us now sketch the key estimates of the paper. The first point is, thanks to almost explicit computations, we can construct an inverse operator Ray^{-1} for Ray_α. Note that if $Ray_\alpha(\phi) = f$, then

$$(\partial_z^2 - \alpha^2)\phi = \frac{U''}{U - c}\phi + \frac{f}{U - c}. \tag{2.7}$$

Hence, provided $U - c$ does not vanish (which is the case when c is complex), using classical elliptic regularity we see that if $f \in C^k$ then $\phi \in C^{k+2}$. We thus gain two derivatives. However the estimates on the derivatives degrade as $z - z_c$ goes smaller. The main point is that the weight $(z - z_c)^l$ is enough to control this singularity. Moreover, deriving l times (2.7) we see that $\partial_z^{2+l}\phi$ is bounded by $C/(z - z_c)^{l+1}$ if $f \in X_k$. Hence we gain one $z - z_c$ factor in the derivative estimates between f and ϕ. Hence if f lies in X_p, ϕ lies in Y_{p+2}, with a gain of two derivatives and of an extra $z - z_c$ weight. As a matter of fact we will construct an inverse Ray^{-1} which is continuous from X_k to Y_{k+2} for any k.

Using Airy functions, their double primitves, and a special variable and unknown transformation known in the literature as Langer transformation, we can construct an almost explicit inverse $Airy^{-1}$ to our $Airy$ operator. We then have to investigate $Airy^{-1} \circ Diff$. Formally it is of order 0, however it is singular, hence to control it we need to use two derivatives, and to make it small we need a $z - z_c$ factor in the norms. After tedious computations on almost explicit Green functions we prove that $Airy^{-1} \circ Diff$ has a small norm as an operator from Y_{k+2} to X_k.

Last, Reg is bounded from X_k to X_k, since it is a simple multiplication by a bounded function. Combining all these estimates we are able to construct exact solutions of Orr Sommerfeld equations, starting from solutions of Rayleigh equations of from Airy equations. This leads to the construction of four independent solutions. Each such solution is defined as a convergent series, which gives its expansion. It then remains to combine all the various terms of all these solutions to get the dispersion relation of Orr Sommerfeld. The careful analysis of this dispersion relation gives our instability result.

3 Rayleigh equation

In this part, we shall construct an exact inverse for the Rayleigh operator Ray_α for small α and so find a complete solution to

$$Ray_\alpha(\phi) = (U-c)(\partial_z^2 - \alpha^2)\phi - U''\phi = f \tag{3.1}$$

in accordance with the boundary condition: $\partial_z\phi_{|_{z=1}} = 0$. Note that as we do not prescribe a boundary condition at $z=0$ there is not one unique inverse for Ray_α. We only construct one possible inverse. To do so, we first invert the Rayleigh operator Ray_0 when $\alpha = 0$ by exhibiting an explicit Green function. We then use this inverse to inductively construct the Green function and the inverse of the Ray_α operator. Precisely, we will prove in this section the following Proposition.

Proposition 3.1. *Assume that Im $c \neq 0$ and that α is sufficiently small. Then, there exists a bounded operator $RaySolver_\alpha(\cdot)$ so that*

$$\begin{aligned} Ray_\alpha(RaySolver_\alpha(f))(z) &= f(z), \qquad \forall\ z \in [0,1] \\ \partial_z RaySolver_\alpha(f)_{|_{z=1}} &= 0 \end{aligned} \tag{3.2}$$

Morevover this operator is bounded from X_k to Y_{k+2} for every interger $k \geq 0$, with

$$\|RaySolver_\alpha(f)\|_{Y_{k+2}} \leq C_0 \|f\|_{X_k},$$

for some universal constants C_k.

Remark 3.2. If we assume further in Proposition 3.1 that $f'(1) = 0$, the equation (3.2) yields

$$\partial_z^3 RaySolver_\alpha(f)_{|_{z=1}} = 0.$$

This implies that the inviscid solution $RaySolver_\alpha(f)$ automatically satisfies the boundary condition (1.9) at $z=1$, and hence no boundary layer correctors are needed in vicinity of the boundary point $z=1$.

Remark 3.3. Away from z_c, Rayleigh equation is elliptic, hence it is natural to gain the control on two derivatives. Near z_c, $\partial_z^l\phi$ behaves like $\partial_z^{l-2}(f/(z-z_c))$ if $l \geq 2$, which is coherent with the definitions of X_k and Y_k spaces.

3.1 Case $\alpha = 0$

As mentioned, we begin with the Rayleigh operator Ray_0 when $\alpha = 0$. We will find the inverse of Ray_0. More precisely, we will construct the Green function of Ray_0 and solve

$$Ray_0(\phi) = (U-c)\partial_z^2\phi - U''\phi = f \tag{3.3}$$

with boundary condition: $\partial_z\phi_{|_{z=1}} = 0$. We recall that z_c is defined by solving the equation $U(z_c) = c$. We first prove the following lemma.

Lemma 3.4. *Assume that Im $c \neq 0$. There are two independent solutions $\phi_{1,0}, \phi_{2,0}$ of $Ray_0(\phi) = 0$ with the Wronskian determinant*

$$W(\phi_{1,0}, \phi_{2,0}) := \partial_z \phi_{2,0} \phi_{1,0} - \phi_{2,0} \partial_z \phi_{1,0} = 1.$$

Furthermore, there are analytic functions $P_1(z), P_2(z), Q(z)$ with $P_1(z_c), P_2(z_c), Q(z_c) \neq 0$ so that the asymptotic descriptions

$$\phi_{1,0}(z) = (z - z_c) P_1(z), \qquad \phi_{2,0}(z) = P_2(z) + Q(z)(z - z_c) \log(z - z_c) \tag{3.4}$$

hold for z near z_c. Here when $z - z_c$ is on the negative real axis, we take the value of $\log(z - z_c)$ *to be* $\log |z - z_c| - i\pi$*. In particular, $\phi_{1,0}$ is a smooth C^∞ function, whereas $\phi_{2,0} \in Y_k$ for every $k \geq 0$.*

Proof. First, we observe that

$$\phi_{1,0}(z) = U(z) - c$$

is an exact solution of $Ray_0(\phi) = 0$. In addition, the claimed asymptotic description for $\phi_{1,0}$ clearly holds for z near z_c since $U(z_c) = c$. We then construct a second particular solution $\phi_{2,0}$, imposing the Wronskian determinant to be one:

$$W[\phi_{1,0}, \phi_{2,0}] = \partial_z \phi_{2,0} \phi_{1,0} - \phi_{2,0} \partial_z \phi_{1,0} = 1.$$

From this, the variation-of-constant method $\phi_{2,0}(z) = \phi_{1,0}(z) u(z)$ then yields

$$\phi_{1,0} u \partial_z \phi_{1,0} + \ \phi_{1,0}^2 \partial_z u - \partial_z \phi_{1,0} u \phi_{1,0} = 1.$$

This gives $\partial_z u(z) = 1/\phi_{1,0}^2(z)$ and therefore

$$\phi_{2,0}(z) = (U(z) - c) \int_{1/2}^{z} \frac{1}{(U(y) - c)^2} dy. \tag{3.5}$$

Note that $\phi_{2,0}$ is well defined if the denominator does not vanishes, hence if Im $c \neq 0$ or if Im $c = 0$ and $0 \leq z < z_c$. More precisely, with denoting $U_c' = U'(z_c)$,

$$\frac{1}{(U(z) - U(z_c))^2} = \frac{1}{(U_c'(z - z_c) + U_c''(z - z_c)^2/2 + ...)^2}$$

$$= \frac{1}{(U_c')^2 (z - z_c)^2} - \frac{U_c''}{(U_c')^3} \frac{1}{z - z_c} + \text{ holomorphic.}$$

Hence

$$\phi_{2,0} = -\frac{U(z) - c}{(U_c')^2 (z - z_c)} - \frac{U_c''}{(U_c')^3} (U(z) - c) \log(z - z_c) + \text{ holomorphic.} \tag{3.6}$$

As $\phi_{2,0}$ is not properly defined for $z < z_c$ when $z_c \in \mathbb{R}^-$, it is coherent to choose the determination of the logarithm which is defined on $\mathbb{C} - \mathbb{R}^-$.

With such a choice of the logarithm, $\phi_{2,0}$ is holomorphic in $\mathbb{C} - \{z_c + \mathbb{R}^-\}$. In particular if Im $z_c = 0$, $\phi_{2,0}$ is holomorphic in z excepted on the half line $z_c + \mathbb{R}^-$. For $z \in \mathbb{R}$, $\phi_{2,0}$ is holomorphic as a function of c excepted if $z - z_c$ is real and positive, namely excepted if $z < z_c$. For a fixed z, $\phi_{2,0}$ is an holomorphic function of c provided z_c does not cross $\mathbb{R}^-$, and provided $z - z_c$ does not cross $\mathbb{R}^-$. The lemma then follows from the explicit expression (3.6) of $\phi_{2,0}$. $\square$

Let $\phi_{1,0}, \phi_{2,0}$ be constructed as in Lemma 3.4. Then the Green function $G_{R,0}(x,z)$ of the Ray_0 operator, taking into account of the boundary conditions, can be defined by

$$G_{R,0}(x,z) = \begin{cases} (U(x)-c)^{-1}\phi_{1,0}(z)\phi_{2,0}(x), & \text{if} \quad z > x, \\ (U(x)-c)^{-1}\phi_{1,0}(x)\phi_{2,0}(z), & \text{if} \quad z < x. \end{cases}$$

Here we note that c is complex with Im $c \neq 0$ and so the Green function $G_{R,0}(x,z)$ is a well-defined function in (x,z), continuous across $x = z$, and its first derivative has a jump across $x = z$. Let us now introduce the inverse of Ray_0 as

$$RaySolver_0(f)(z) := \int_0^1 G_{R,0}(x,z)f(x)dx. \tag{3.7}$$

Lemma 3.5. *For any $f \in X_0 = Y_0$, the function $RaySolver_0(f)$ is a solution to the Rayleigh problem* (3.3)*, with $\partial_z RaySolver_0(f)(1) = 0$. In addition, $RaySolver_0(f) \in Y_2$, and there holds*

$$\|RaySolver_0(f)\|_{Y_2} \leq C\|f\|_{X_0},$$

for some universal constant.

Note that Y_k spaces are somehow better adapted to Rayleigh equation, since the singularity comes from $(z - z_c)\log(z - z_c)$ which appears only after taking two derivatives.

Proof. By definition, we have

$$RaySolver_0(f)(z) = \phi_{1,0}(z)\int_0^z \phi_{2,0}(x)\frac{f(x)}{U(x)-c}dx + \phi_{2,0}(z)\int_z^1 \phi_{1,0}(x)\frac{f(x)}{U(x)-c}dx.$$

Using the definition of the function space X_0 and the asymptotic expansion of $\phi_{2,0}(z)$ for z near z_c obtained in Lemma 3.4, we have

$$\begin{aligned}\Big|\phi_{1,0}(z)\int_0^z \phi_{2,0}(x)\frac{f(x)}{U(x)-c}dx\Big| &\leq C\|f\|_{X_0}|z-z_c|\int_0^z \frac{1}{|x-z_c|}\,dx \\ &\leq C\|f\|_{X_0}|z-z_c|\Big(1 + |\log(z-z_c)|\Big), \\ &\leq C\|f\|_{X_0},\end{aligned}$$

and similarly,

$$\Big|\phi_{2,0}(z)\int_z^1 \phi_{1,0}(x)\frac{f(x)}{U(x)-c}dx\Big| = \Big|\phi_{2,0}(z)\int_z^1 f(x)dx\Big| \leq C\|f\|_{X_0}.$$

Hence

$$\|RaySolver_0(f)\|_{Y_0} \le C\|f\|_{X_0}. \tag{3.8}$$

Next, we write

$$\partial_z RaySolver_0(f)(z) = \partial_z\phi_{1,0}(z)\int_0^z \phi_{2,0}(x)\frac{f(x)}{U(x)-c}dx + \partial_z\phi_{2,0}(z)\int_z^1 \phi_{1,0}(x)\frac{f(x)}{U(x)-c}dx.$$

The boundary condition follows directly by the assumption that $\partial_z\phi_{1,0}(1) = U'(1) = 0$. Now, near $z = z_c$, $\partial_z\phi_{2,0}(z) = \mathcal{O}(\log(z - z_c))$, and so a similar estimate to those given just above yields

$$|\partial_z RaySolver_0(f)(z)| \le C\|f\|_{X_0}(1 + |\log(z - z_c)|).$$

Hence

$$\|RaySolver_0(f)\|_{Y_1} \le C\|f\|_{X_0}.$$

For the second derivative, we write

$$\partial_z^2(RaySolver_0(f)) = \frac{U''}{U-c}RaySolver_0(f) + \frac{f}{U-c}, \tag{3.9}$$

which proves at once $\|RaySolver_0(f)\|_{Y_2} \le C\|f\|_{X_0}$. ☐

The following lemma is then straightforward and will be of use in the latter sections.

Lemma 3.6. *Let $k \ge 2$. For any $f \in X_k$, the function $RaySolver_0(f)$ belongs to Y_{k+2}, and there holds*

$$\|RaySolver_0(f)\|_{Y_{k+2}} \le C\|f\|_{X_k}$$

for some universal constants C_k.

Proof. The lemma follows directly from taking derivatives of the identity (3.9), and using the estimates obtained in Lemma 3.5, since each time we derive, we lose an $(U-c)$ factor. ☐

3.2 Case $\alpha \neq 0$: the exact Rayleigh solver

Let us prove in this section Proposition 3.1

Proof of Proposition 3.1. Note that for any function f, we have

$$Ray_\alpha(RaySolver_0(f)) = f - \alpha^2(U-c)RaySolver_0(f).$$

We therefore build the Rayleigh solver $RaySolver_\alpha(\cdot)$ by iteration, defining iteratively

$$S_0(f) := RaySolver_0(f), \qquad S_j(f) := RaySolver_0\Big(\alpha^2(U-c)S_{j-1}(f)\Big),$$

for any $f \in Y_0$. The exact Rayleigh solver of the Rayleigh equation is then defined by

$$RaySolver_\alpha(f) := \sum_{j=0}^{+\infty} S_j(f), \qquad f \in Y_0. \tag{3.10}$$

Indeed, since $f \in Y_0$, then by the estimate (3.8) and iteration, $S_j(f) \in Y_0$ and

$$\|S_j(f)\|_{Y_0} \le C\alpha^2 \|S_{j-1}(f)\|_{Y_0} \le C^j \alpha^{2j} \|f\|_{Y_0}.$$

For sufficiently small α, the series $\sum_{j=0}^{+\infty} S_j(f)$ is thus convergent in Y_0. In addition, for all $J \ge 0$,

$$Ray_\alpha\Big(\sum_{j=0}^{J} S_j(f)\Big) = f - \alpha^2 (U-c) S_J(f).$$

By taking $J \to \infty$, $\sum_{j=0}^{+\infty} S_j(f)$ defines the Rayleigh solver from Y_0 to Y_0. More generally, if $f \in Y_k$ for some $k \ge 0$, then the function $RaySolver_\alpha(f)$ lies in Y_k. Proposition 3.1 then follows by combining with Lemma 3.6. □

3.3 Case $\alpha \neq 0$: two particular solutions

Lemma 3.7. *For α small enough, there exists two functions $\phi_{j,\alpha} \in Y_4$ with $j = 1, 2$, uniformly bounded in Y_4 as α goes to 0, such that*

$$Ray_\alpha(\phi_{j,\alpha}) = 0,$$

$$\|\phi_{j,\alpha} - \phi_{j,0}\|_{Y_4} = O(\alpha^2).$$

Moreover, with the notation $U_0' = U'(0)$, we have

$$\begin{aligned} \phi_{1,\alpha}(0) &= -c + \frac{\alpha^2}{U_0'} \int_0^1 (U-c)^2\, dx + \mathcal{O}(\alpha^2 z_c \log z_c) \\ \partial_z \phi_{1,\alpha}(0) &= U_0' + \mathcal{O}(\alpha^2 \log z_c). \end{aligned}$$

Proof. We use the previous construction to build exacts solution of $Ray_\alpha(\phi) = 0$, starting from $\phi_{1,0}$ and $\phi_{2,0}$, the two solutions of $Ray_0(\phi) = 0$ that are constructed above in Lemma 3.4. Let us denote

$$\phi_{j,n}(z) := RaySolver_0\Big((U-c)\phi_{j,n-1}\Big)(z),$$

for $n \ge 1$. Clearly, we have

$$Ray_\alpha(\sum_{k=0}^{n} \alpha^{2k} \phi_{j,k}) = -\alpha^{2(n+1)} (U-c) \phi_{j,n}.$$

By the estimate obtained in Lemma 3.5, we get $\|\phi_{j,n}\|_{Y_0} \le C\|\phi_{j,n-1}\|_{Y_0}$, for all $n \ge 1$. Therefore the series in the above equation converges in Y_0 for sufficiently small α. This proves that

$$\phi_{j,\alpha}(z) := \sum_{n=0}^{\infty} \alpha^{2n} \phi_{j,n}(z) \tag{3.11}$$

are well-defined in Y_0 and are two exact solutions to $Ray_\alpha \phi = 0$.

We now detail the first terms of the asymptotic expansions of $\phi_{j,\alpha}$. First, we recall that $\phi_{1,0}(0) = U(0) - c = -c$, and $\partial_z \phi_{1,0}(0;\epsilon,c) = U_0' \neq 0$. In addition, since z_c is sufficiently small, we can write

$$\phi_{2,0}(0) = \frac{1}{U_0'} - \frac{2U_0''}{U_0'^2} z_c \log z_c + \mathcal{O}(z_c), \qquad \partial_z \phi_{2,0}(0) = \frac{2U_0''}{U_0'^2} \log z_c + \mathcal{O}(1), \tag{3.12}$$

with $U_0' = U'(0)$ and $U_0'' = U''(0)$. We also recall that z_c is a complex number with $U(z_c) = c$, and so $\mathrm{Im}\, c = U_0' \mathrm{Im}\, z_c + \mathcal{O}(z_c^2)$.

Next, by definition, $\phi_{j,1} = RaySolver_0((U-c)\phi_{j,0})$. That is, we have

$$\partial_z^k \phi_{j,1}(0;\epsilon,c) = \partial_z^k \phi_{2,0}(0;\epsilon,c) \int_0^1 \phi_{1,0}(x)\phi_{j,0}\, dx$$

for $j = 1, 2$ and $k = 0, 1$. This proves the Lemma. □

4 Airy equations

Our goal is to invert the Airy operator defined as in (2.4), and thus we wish to construct the Green function for the modified Airy equation

$$-i\varepsilon\partial_z^4\phi - (U(z) - c - 2i\varepsilon\alpha^2)\partial_z^2\phi = 0. \tag{4.1}$$

In the first paragraph we recall classical properties of the classical Airy equations. In the second one we detail Langer transformation, and then focus on the construction of the Green function.

4.1 Airy functions

The aim of this section is to recall some properties of the classical Airy functions. The classical Airy equation is

$$\partial_z^2\phi - z\phi = 0, \qquad z \in \mathbb{C}, \tag{4.2}$$

with two classical solutions named $Ai(z)$ and $Bi(z)$, which go to 0 respectively at $+\infty$ and $-\infty$. In connection with the Orr-Sommerfeld equation or precisely the critical layer equation (see (1.11) - (1.12)), we are interested in the Airy functions with a *complex* argument

$$z = e^{i\pi/6}x, \quad x \in \mathbb{R}.$$

We have therefore to introduce two independent solutions which converge to 0 respectively at $+\infty$ and $-\infty$ on this *complex* line. We will take Ai and

$$Ci = -i\pi(Ai + iBi).$$

We will need the following estimates, which proofs may be found in [11, 18]; see also [1, Appendix].

Lemma 4.1. *The classical Airy equation* (4.2) *has two independent solutions $Ai(z)$ and $Ci(z)$ so that the Wronskian determinant of Ai and Ci equals*

$$W(Ai, Ci) = Ai(z)Ci'(z) - Ai'(z)Ci(z) = 1. \tag{4.3}$$

In addition, $Ai(e^{i\pi/6}x)$ and $Ci(e^{i\pi/6}x)$ converge to 0 *as $x \to \pm\infty$ (x being real), respectively. Furthermore, there hold asymptotic bounds:*

$$\Big|Ai(k, e^{i\pi/6}x)\Big| \le C\langle x\rangle^{k/2-1/4}e^{-\sqrt{2|x|}x/3}, \qquad k \in \mathbb{Z}, \quad x \in \mathbb{R}, \tag{4.4}$$

and

$$\Big|Ci(k, e^{i\pi/6}x)\Big| \le C\langle x\rangle^{k/2-1/4}e^{\sqrt{2|x|}x/3}, \qquad k \in \mathbb{Z}, \quad x \in \mathbb{R}, \tag{4.5}$$

in which $Ai(0,z) = Ai(z)$, $Ai(k,z) = \partial_z^{-k}Ai(z)$ for $k \le 0$, and $Ai(k,z)$ is the k^{th} primitive of $Ai(z)$ for $k \ge 0$ and is defined by the inductive path integrals

$$Ai(k,z) = \int_\infty^z Ai(k-1, w)\, dw$$

so that the integration path is contained in the sector with $|\arg(z)| < \pi/3$. *The Airy functions* $Ci(k,z)$ *for* $k \neq 0$ *are defined similarly. In the above estimates, we use the usual notation* $\langle x\rangle = \sqrt{1+|x|^2}$.

Note that $Ai(x)$ is rapidly decreasing as x goes to $+\infty$, and $Ci(x)$ is rapidly increasing as x goes to $+\infty$. Note also that two derivatives lead to a multiplication by a factor x.

4.2 Green function of Airy equation

Using the properties of the Airy solutions, we can now construct the Green function for the classical Airy equation. More precisely, let us consider

$$-i\varepsilon\partial_z^2\phi - \lambda(z-z_1)\phi = f, \qquad z\in[0,1], \tag{4.6}$$

where λ is some positive constant, and z_1 is in $[0,1]$. In connection with the classical Airy equation (4.2), we let

$$\delta = \Big(\frac{\varepsilon}{i\lambda}\Big)^{1/3} = \Big(\frac{1}{i\lambda\alpha R}\Big)^{1/3} = e^{-i\pi/6}\Big(\frac{1}{\lambda\alpha R}\Big)^{1/3}$$

which can be interpreted as the critical layer size: $|\delta| \sim \epsilon^{1/3}$. Here, it is sufficient for us to take one root of $i^{1/3}$, which is $e^{i\pi/6}$. Let $G_a(x,z)$ be defined by

$$G_a(x,z) = i\delta\varepsilon^{-1}\begin{cases} Ai(X)Ci(Z), & \text{if} \quad x>z,\\ Ai(Z)Ci(X), & \text{if} \quad x<z,\end{cases} \tag{4.7}$$

with $X := \delta^{-1}(x-z_1)$ and $Z := \delta^{-1}(z-z_1)$. We note that X,Z are on the line $e^{i\pi/6}\mathbb{R}$ in the complex plane. It follows that $G_a(x,z)$ is the Green function of (4.6). Indeed, we note that by (4.3), $[G_a(x,z)]_{|x=z} = \lim_{x\to z^+} G_a(x,z) - \lim_{x\to z^-} G_a(x,z) = 0$ and $[\partial_z G_a(x,z)]_{|x=z} = -[\partial_x G_a(x,z)]_{|x=z} = i\epsilon^{-1}$.

The solution ϕ of (4.6) is given by the convolution

$$\phi(z) = \int_0^1 G_a(x,z)f(x)dx. \tag{4.8}$$

Let us detail some estimates on G_a as a warm up for the following sections. First $G_a(x,x)$ is uniformly bounded in x. From the asymptotic properties of Ai and Ci we can in fact get

$$|G_a(x,z)| \le C_0\epsilon^{-2/3} \tag{4.9}$$

for some constant C_0, uniformly in x and z. As a consequence (4.8) is well defined as soon as f is integrable and

$$\Big\|\int_0^1 G_a(x,\cdot)f(x)dx\Big\|_{L^\infty} \le C_0\epsilon^{-2/3}\|f\|_{L^1}. \tag{4.10}$$

Let us get sharper pointwise bounds on the Green function. Since $G_a(x,z)$ is symmetric in x and z, that is $G_a(x,z) = G_a(z,x)$, it suffices to give bounds on the case when $0 \le x \le z$. We first recall that Lemma 4.1 gives

$$|\partial_{\tilde z}^k Ai(e^{i\pi/6}\tilde z)\partial_{\tilde x}^\ell Ci(e^{i\pi/6}\tilde x)| \le C\langle \tilde z\rangle^{k/2-1/4}\langle \tilde x\rangle^{\ell/2-1/4} e^{\frac13\sqrt{2|\tilde x|}\tilde x} e^{-\frac13\sqrt{2|\tilde z|}\tilde z} \tag{4.11}$$

for all real numbers $\tilde x, \tilde z$. In particular, there holds the following bound, for all $\tilde x \le \tilde z$,

$$|\partial_{\tilde z}^k Ai(e^{i\pi/6}\tilde z)\partial_{\tilde x}^\ell Ci(e^{i\pi/6}\tilde x)| \le C\langle \tilde z\rangle^{k/2-1/4}\langle \tilde x\rangle^{\ell/2-1/4} e^{-\frac13\sqrt{2|\tilde z|}|\tilde x-\tilde z|}$$

We now apply the above estimate for $Z = \delta^{-1}(z-z_1)/\delta$ and $X = \delta^{-1}(x-z_1)$, which are on the line $e^{i\pi/6}\mathbb{R}$. By a view of (4.7) and the symmetry of the Green function, this yields at once

$$|\partial_z^k\partial_x^\ell G_a(x,z)| \le C\epsilon^{-\frac{k+\ell+2}{3}}\langle Z\rangle^{k/2-1/4}\langle X\rangle^{\ell/2-1/4} e^{-\frac13\sqrt{2|Z|}|X-Z|} \tag{4.12}$$

for all x and z. We stress that in the case that $x - z_1$ and $z - z_1$ have opposite sign, the estimate (4.11) in fact yields the following stronger bound on the Green function

$$|\partial_z^k\partial_x^\ell G_a(x,z)| \le C\epsilon^{-\frac{k+\ell+2}{3}}\langle Z\rangle^{k/2-1/4}\langle X\rangle^{\ell/2-1/4} e^{-\frac{\sqrt2}{3}|X|^{3/2}} e^{-\frac{\sqrt2}{3}|Z|^{3/2}}. \tag{4.13}$$

Occasionally, it is useful to derive the stronger bounds in the case where X and Z are away from each other: $|X| \le \frac12|Z|$ or $|X| \ge 2|Z|$. In this case, we have

$$|\partial_z^\ell\partial_x^k G_a(x,z)| \le C\epsilon^{-\frac{k+\ell+2}{3}} e^{-\frac16|Z|^{3/2}} e^{-\frac16|X|^{3/2}}, \tag{4.14}$$

which follows directly from the estimate (4.11). To summarize, the Green function for the Airy equation is extremely localized near the critical layer $z = z_1$. We shall see that the localization of the Green function yields a stronger estimate than that obtained in (4.10).

4.3 Green function of the primitive Airy equation

In the sequel, we will have to look for the Green function of the equation

$$-i\varepsilon\partial_z^4\phi - \lambda(z-z_1)\partial_z^2\phi = 0. \tag{4.15}$$

We will therefore have to integrate twice $G_a(x,\cdot)$, and the primitives of $Ai(z)$ and $Ci(z)$ are in use. We shall choose $G_{2,a}(x,z)$ such that $\partial_z^2 G_{2,a}(x,z)$ coincides with the Green function $G_a(x,z)$ of the Airy function constructed in the previous subsection. We construct the Green function $G_{2,a}(x,z)$ as follows:

$$G_{2,a}(x,z) = \lambda^{-1}\begin{cases} Ai(X)Ci(2,Z) + a_1(X)Z, & \text{if } x > z,\\ Ai(2,Z)Ci(X) + a_2(X) + a_1(X)X, & \text{if } x < z,\end{cases} \tag{4.16}$$

with the notation $X = \delta^{-1}(x-z_1)$ and $Z = \delta^{-1}(z-z_1)$, where $a_1(X)$ and $a_2(X)$ are chosen such that $G_{2,a}(x,z)$ and $\partial_z G_{2,a}(x,z)$ are continuous at $x = z$, namely

$$a_1(X) := Ci(X)Ai(1,X) - Ai(X)Ci(1,X), \tag{4.17}$$

and

$$a_2(X) := Ai(X)Ci(2,X) - Ci(X)Ai(2,X). \tag{4.18}$$

Lemma 4.2. *Let $G_{2,a}(x,z)$ be defined as in* (4.16)*. Then, $G_{2,a}(x,z)$ is indeed the Green function of* (4.15)*, that is, there holds*

$$-i\varepsilon\partial_z^4 G_{2,a}(x,z) - \lambda(z-z_1)\partial_z^2 G_{2,a}(x,z) = \delta_x(z)$$

for each fixed x. In addition, for each fixed z, $G_{2,a}(\cdot,z)$ solves the adjoint equation of (4.15)*, that is*

$$-i\varepsilon\partial_x^4 G_{2,a}(x,z) - \lambda\partial_x^2((x-z_1)G_{2,a}(x,z)) = \delta_z(x).$$

Proof. By construction, $\partial_z^2 G_{2,a}(x,z) = G_a(x,z)$, which solves (4.6), and so $G_{2,a}(x,z)$ solves (4.15). In addition, $\partial_z^2 G_{2,a}(x,z)$ is continuous at $x=z$ and $\partial_z^3 G_{2,a}(x,z)$ has a jump across $z=x$ which is equal to

$$[\partial_z^3 G_{2,a}(x,z)]_{|x=z} = [G_a(x,z)]_{|x=z} = i\varepsilon^{-1}.$$

Next, by direct computations we observe that $\partial_x G_{2,a}(x,z)$ and $\partial_x^2 G_{2,a}(x,z)$ are also continuous at $x=z$, and that the jump of $\partial_x^3 G_{2,a}(x,z)$ across $x=z$ is

$$[\partial_x^3 G_{2,a}(x,z)]_{|x=z} = i\varepsilon^{-1}W[Ci,Ai] = -i\varepsilon^{-1}.$$

Furthermore, direct calculations yield

$$\begin{aligned} a_1''(X) &= Xa_1(X) - 1,\\ (a_2(X) - Xa_1(X))'' &= X(a_2(X) - Xa_1(X)) + X. \end{aligned}$$

This proves

$$\begin{aligned} \epsilon\partial_x^2 G_{2,a}(x,z) &= \lambda(x-z_1)G_{2,a}(x,z) + (z-x)\chi_{\{x>z\}}\\ \epsilon\partial_x^4 G_{2,a}(x,z) &= \lambda(x-z_1)\partial_x^2 G_{2,a}(x,z) + 2\lambda\partial_x G_{2,a}(x,z), \end{aligned}$$

where $\chi_{\{x>z\}}$ equals one if $x>z$ and zero if otherwise. That is, $G_{2,a}(x,z)$ solves the adjoint equation as claimed. □

For our convenience, we denote

$$G_{2,a}(x,z) := \widetilde{G}_{2,a}(x,z) + E_{2,a}(x,z), \tag{4.19}$$

where $\widetilde{G}_{2,a}(x,z)$ denotes the localized behavior in the Green function $G_{2,a}(x,z)$, and $E_{2,a}(x,z)$ denotes the linear term in the Green function, that is

$$E_{2,a}(x,z) = \lambda^{-1}\begin{cases} a_1(X)Z, & \text{if } x>z,\\ a_2(X) + a_1(X)X, & \text{if } x<z, \end{cases} \tag{4.20}$$

By a view of the bounds (4.4) and (4.5), it follows easily that

$$|\partial_X^k a_1(X)| \le C\langle X\rangle^{(k-2)/2}, \qquad |\partial_X^k a_2(X)| \le C\langle X\rangle^{(k-3)/2} \tag{4.21}$$

for all $X \in e^{i\pi/6}\mathbb{R}$ and integers $k \ge 0$. Similarly to the estimates on the Airy's Green function $G_a(x,z)$, obtained in (4.12)-(4.14), there hold the following pointwise bounds on the localized part of the Green function:

$$|\partial_z^\ell \partial_x^k \widetilde{G}_{2,a}(x,z)| \le C\epsilon^{-\frac{k+\ell}{3}} \langle X\rangle^{k/2-1/4} \langle Z\rangle^{\ell/2-5/4} e^{-\frac{1}{3}\sqrt{2|Z|}|X-Z|} \tag{4.22}$$

for all x, z. In addition, in the case where X and Z are away from each other: $|X| \le \frac{1}{2}|Z|$ or $|X| \ge 2|Z|$, we have

$$|\partial_z^\ell \partial_x^k \widetilde{G}_{2,a}(x,z)| \le C\epsilon^{-\frac{k+\ell}{3}} e^{-\frac{1}{6}|Z|^{3/2}} e^{-\frac{1}{6}|X|^{3/2}}. \tag{4.23}$$

4.4 Langer transformation

Since the profile U depends on z in a non trivial manner, we make a change of variables and unknowns in order to go back to classical Airy equations studied in the previous section. This change has been used in the physical literature, and called the Langer's transformation; see, for instance, Langer [7], or Drazin-Reid [1, Section 27.5] and Wasow [21, Section 1.5].

Definition 4.3. *By Langer's transformation $(z,\phi) \mapsto (\eta,\Phi)$, we mean $\eta = \eta(z)$ defined by*

$$\eta(z) = \Big[\frac{3}{2}\int_{z_c}^z \Big(\frac{U-c}{U_c'}\Big)^{1/2} dz\Big]^{2/3} \tag{4.24}$$

and $\Phi = \Phi(\eta)$ defined by the relation

$$\partial_z^2\phi(z) = \dot z^{1/2}\Phi(\eta), \tag{4.25}$$

in which

$$\dot z = \frac{dz(\eta)}{d\eta}$$

and $z = z(\eta)$ is the inverse of the map $\eta = \eta(z)$.

Direct calculation gives a useful fact; $(U-c)\dot z^2 = U_c'\eta$. Next, using that $c = U(z_c)$, one observes that for z near z_c, we have

$$\begin{aligned}\eta(z) &= \Big[\frac{3}{2}\int_{z_c}^z \Big(z - z_c + \frac{U_c''}{U_c'}(z-z_c)^2 + \mathcal{O}(|z-z_c|^3)\Big)^{1/2} dz\Big]^{2/3} \\ &= z - z_c + \frac{1}{10}\frac{U_c''}{U_c'}(z-z_c)^2 + \mathcal{O}(|z-z_c|^3).\end{aligned} \tag{4.26}$$

In particular, we have

$$\eta'(z) = 1 + \mathcal{O}(|z-z_c|), \tag{4.27}$$

and thus the inverse $z = z(\eta)$ is locally well-defined and locally increasing near $z = z_c$. In addition,

$$\dot z = \frac{1}{\eta'(z)} = 1 + \mathcal{O}(|z-z_c|).$$

Next, we note that

$$\eta'(z)^2 = \frac{U-c}{U_c'\eta(z)},$$

which is nonzero away from $z = z_c$. Thus, the inverse of $\eta = \eta(z)$ exists for all $z \in [0,1]$.

The following lemma links (4.1) with the classical Airy equation.

Lemma 4.4. *Let $(z,\phi) \mapsto (\eta,\Phi)$ be the Langer's transformation defined as in Definition 4.3. There holds*

$$\partial_z^2(\dot z^{1/2}\Phi(\eta)) = \dot z^{-3/2}\partial_\eta^2\Phi(\eta) + \partial_z^2\dot z^{1/2}\Phi(\eta) \tag{4.28}$$

Next, assume that $\Phi(\eta)$ solves

$$-i\epsilon\partial_\eta^2\Phi - U_c'\eta\Phi = f(\eta).$$

Then, $\phi = \phi(z)$ solves

$$-i\varepsilon\partial_z^4\phi - (U(z)-c)\partial_z^2\phi = \dot z^{-3/2}f(\eta(z)) - i\epsilon\partial_z^2\dot z^{1/2}\Phi(\eta(z))$$

Proof. Derivatives of the identity $\partial_z^2\phi(z) = \dot z^{1/2}\Phi(\eta)$ are

$$\partial_z^3\phi(z) = \dot z^{-1/2}\partial_\eta\Phi + \partial_z\dot z^{1/2}\Phi$$

and

$$\begin{aligned}\partial_z^4\phi(z) &= \dot z^{-3/2}\partial_\eta^2\Phi + [\partial_z\dot z^{-1/2} + \dot z^{-1}\partial_z\dot z^{1/2}]\partial_\eta\Phi + \partial_z^2\dot z^{1/2}\Phi \\ &= \dot z^{-3/2}\partial_\eta^2\Phi + \partial_z^2\dot z^{1/2}\Phi.\end{aligned} \tag{4.29}$$

This proves (4.28). Putting these together and using the fact that $(U-c)\dot z^2 = U_c'\eta$, we get

$$\begin{aligned}-i\varepsilon\partial_z^4\phi - (U(z)-c)\partial_z^2\phi &= -i\epsilon\dot z^{-3/2}\partial_\eta^2\Phi - (U-c)\dot z^{1/2}\Phi - i\epsilon\partial_z^2\dot z^{1/2}\Phi \\ &= \dot z^{-3/2}f(\eta) - i\epsilon\partial_z^2\dot z^{1/2}\Phi.\end{aligned}$$

The lemma follows. □

4.5 An approximate Green function for the modified Airy equation

In this section we will construct an approximate Green function for (4.1). To do this we fulfill the Langer's transformation and first consider

$$-i\varepsilon\partial_\eta^2\Phi - U_c'\eta\Phi = 0.$$

Let us denote

$$\delta = \Big(\frac{\varepsilon}{iU_c'}\Big)^{1/3} = e^{-i\pi/6}(\alpha R U_c')^{-1/3} \tag{4.30}$$

and introduce the notation $X = \delta^{-1}\xi$ and $Z = \delta^{-1}\eta$. The Green function of the above classical Airy equation is simply

$$G_a(X,Z) = i\delta\varepsilon^{-1}\begin{cases} Ai(X)Ci(Z), & \text{if} \quad \xi > \eta, \\ Ai(Z)Ci(X), & \text{if} \quad \xi < \eta,\end{cases}$$

which satisfies the jump conditions across $X = Z$:

$$[G_a(X,Z)]_{|X=Z} = 0, \qquad [-i\epsilon\partial_Z G_a(X,Z)]_{|X=Z} = 1.$$

By definition, we have

$$\varepsilon\partial_\eta^2 G_a(X,Z) - U_c'\eta G_a(X,Z) = \delta_\xi(\eta). \tag{4.31}$$

Next, let us take $\xi = \eta(x)$ and $\eta = \eta(z)$ where $\eta(\cdot)$ is the Langer's transformation and denote $\dot{x} = 1/\eta'(x)$ and $\dot{z} = 1/\eta'(z)$. By a view of (4.25), we define the function $G(x,z)$ so that

$$\partial_z^2 G(x,z) = \dot{x}^{3/2}\dot{z}^{1/2} G_a(\delta^{-1}\eta(x), \delta^{-1}\eta(z)), \tag{4.32}$$

in which the factor $\dot{x}^{3/2}$ was added simply to normalize the jump of $G(x,z)$. It then follows from Lemma 4.4 together with $\delta_{\eta(x)}(\eta(z)) = \delta_x(z)$ that

$$-i\varepsilon\partial_z^4 G(x,z) - (U(z)-c)\partial_z^2 G(x,z) = \delta_x(z) - i\epsilon\partial_z^2 \dot{z}^{1/2}\dot{z}^{-1/2}\partial_z^2 G(x,z). \tag{4.33}$$

That is, $G(x,z)$ is indeed an approximate Green function of the modified Airy operator $-i\epsilon\partial_z^4 - (U-c)\partial_z^2$ up to a small error term of order $\epsilon\partial_z^2 G = \mathcal{O}(\epsilon^{1/3})$. It remains to solve (4.32) for $G(x,z)$, retaining the jump conditions on $G(x,z)$ across $x = z$.

In view of primitive Airy functions, let us denote

$$\widetilde{C}i(1,z) = \delta^{-1}\int_0^z \dot{y}^{1/2} Ci(\delta^{-1}\eta(y))\,dy, \qquad \widetilde{C}i(2,z) = \delta^{-1}\int_0^z \widetilde{C}i(1,y)\,dy$$

and

$$\widetilde{A}i(1,z) = \delta^{-1}\int_\infty^z \dot{y}^{1/2} Ai(\delta^{-1}\eta(y))\,dy, \qquad \widetilde{A}i(2,z) = \delta^{-1}\int_\infty^z \widetilde{A}i(1,y)\,dy.$$

Thus, we are led to introduce

$$G(x,z) = U_c'^{-1}\dot{x}^{3/2}\begin{cases} Ai(\delta^{-1}\eta(x))\widetilde{C}i(2,z) + a_1(x)(z-z_c)/\delta, & \text{if } x > z, \\ Ci(\delta^{-1}\eta(x))\widetilde{A}i(2,z) + a_2(x) + a_1(x)(x-z_c)/\delta, & \text{if } x < z, \end{cases} \tag{4.34}$$

in which $a_1(x), a_2(x)$ are chosen so that the jump conditions in (4.35) hold. Clearly, by definition, $G(x,z)$ solves (4.32). Next, by view of (4.33), we require the following jump conditions on the Green function:

$$[G(x,z)]_{|x=z} = [\partial_z G(x,z)]_{|x=z} = [\partial_z^2 G(x,z)]_{|x=z} = 0 \tag{4.35}$$

and

$$[-i\epsilon\partial_z^3 G(x,z)]_{|x=z} = 1. \tag{4.36}$$

We note that from (4.32) and the jump conditions on $G_a(X,Z)$ across $X = Z$, the above jump conditions of $\partial_z^2 G$ and $\partial_z^3 G$ follow easily. In order for the jump conditions on $G(x,z)$ and $\partial_z G(x,z)$, we take

$$\begin{aligned} a_1(x) &= Ci(\delta^{-1}\eta(x))\widetilde{A}i(1,x) - Ai(\delta^{-1}\eta(x))\widetilde{C}i(1,x), \\ a_2(x) &= Ai(\delta^{-1}\eta(x))\widetilde{C}i(2,x) - Ci(\delta^{-1}\eta(x))\widetilde{A}i(2,x). \end{aligned} \tag{4.37}$$

We obtained the following lemma.

Lemma 4.5. *Let $G(x,z)$ be defined as in* (4.34). *Then $G(x,z)$ is indeed an approximate Green function of (4.1). Precisely, there holds*

$$-i\varepsilon\partial_z^4 G(x,z) - (U(z) - c - 2i\alpha^2\epsilon)\partial_z^2 G(x,z) = \delta_x(z) + Err_A(x,z) \tag{4.38}$$

where $Err_A(x,z)$ denotes the error kernel defined by

$$Err_A(x,z) = -i\epsilon\partial_z^2 \dot{z}^{1/2}\dot{z}^{-1/2}\partial_z^2 G(x,z) + 2i\alpha^2\epsilon\partial_z^2 G(x,z). \tag{4.39}$$

It appears convenient to denote by $\widetilde{G}(x,z)$ and $E(x,z)$ the localized and non-localized parts of the Green function, respectively. Precisely, we denote

$$\widetilde{G}(x,z) = U_c'^{-1}\dot{x}^{3/2}\begin{cases} Ai(\delta^{-1}\eta(x))\widetilde{C}i(2,z), & \text{if } x > z, \\ Ci(\delta^{-1}\eta(x))\widetilde{A}i(2,z), & \text{if } x < z, \end{cases}$$

and

$$E(x,z) = U_c'^{-1}\dot{x}^{3/2}\begin{cases} a_1(x)(z-z_c)/\delta, & \text{if } x > z, \\ a_2(x) + a_1(x)(x-z_c)/\delta, & \text{if } x < z. \end{cases}$$

Let us give some bounds on the Green function, using the known bounds on $Ai(\cdot)$ and $Ci(\cdot)$. We have the following lemma.

Lemma 4.6. *Let $G(x,z) = \widetilde{G}(x,z) + E(x,z)$ be the Green function defined as in* (4.34), *and let $X = \eta(x)/\delta$ and $Z = \eta(z)/\delta$. There holds the following pointwise estimate*

$$|\partial_z^\ell \partial_x^k \widetilde{G}(x,z)| \le C\epsilon^{-\frac{k+\ell}{3}}\langle X\rangle^{k/2-1/4}\langle Z\rangle^{\ell/2-5/4}e^{-\frac{\sqrt{2}}{3}\sqrt{|Z|}|X-Z|}, \tag{4.40}$$

for all x,z. In particular, when X and Z are away from one another: $|X| \le \frac{1}{2}|Z|$ or $|X| \ge 2|Z|$, there holds the following stronger bound

$$|\partial_z^\ell \partial_x^k \widetilde{G}(x,z)| \le C\epsilon^{-\frac{k+\ell}{3}}e^{-\frac{1}{6}|Z|^{3/2}}e^{-\frac{1}{6}|X|^{3/2}}, \tag{4.41}$$

Similarly, for the non-localized term, we have

$$|\partial_x^k a_1(x)| \le C\epsilon^{-\frac{k}{3}}\langle X\rangle^{k/2-1}, \qquad |\partial_x^k a_2(x)| \le C\epsilon^{-\frac{k}{3}}\langle X\rangle^{k/2-3/2}. \tag{4.42}$$

In particular, we have $\partial_z E(x,z) = (U_c'\delta)^{-1}\dot{x}^{3/2}a_1(x)$ for $x > z$ and zero for $x < z$, $\partial_z^\ell E(x,z) = 0$, for $\ell \ge 2$, and

$$|\partial_x^k E(x,z)| \le C\epsilon^{-\frac{k}{3}}\langle X\rangle^{k/2} \times \begin{cases} \frac{|Z|}{\langle X\rangle}, & \text{if } x > z, \\ 1, & \text{if } x < z, \end{cases} \tag{4.43}$$

for $k \ge 0$.

Proof. We note that $\dot{z}(\eta(z)) = 1 + \mathcal{O}(|z - z_c|)$, which in particular yields

$$\frac{1}{2} \le \dot{z}(\eta(z)) \le \frac{3}{2}$$

for z sufficiently near z_c. We now estimate

$$\begin{aligned}
|\widetilde{A}i(1,z)| &\le |\delta|^{-1} \int_z^1 |\dot{y}^{1/2} Ai(e^{i\pi/6}Y)|\, dy \le C|\delta|^{-1} \int_z^1 \langle Y\rangle^{-1/4} \left| e^{-\sqrt{2|Y|}Y/3} \right| dy \\
&\le C|\delta|^{-1} \int_z^1 \langle Y\rangle^{-1/4} e^{-\sqrt{2|Y|}\Re Y/3}\, dy \\
&\le C \int_{\Re Z}^{\infty} \langle Y\rangle^{-1/4} e^{-\sqrt{2|Y|}y'/3}\, dy'
\end{aligned}$$

where we have made the change of variable $y' = \Re Y = \delta^{-1}\Re\eta(y)$. Now for large $\Re Z$, this expression is bounded asymptotically by

$$C\langle Z\rangle^{-3/4} e^{-\sqrt{2|Z|}\Re Z/3}.$$

This remains true for bounded Z. Similarly, we have

$$\begin{aligned}
|\widetilde{A}i(2,z)| &\le |\delta|^{-1} \int_z^1 |\widetilde{A}i(1,y)|\, dy \le C|\delta|^{-1} \int_z^1 \langle Y\rangle^{-3/4} e^{-\sqrt{2|Y|}\Re Y/3}\, dy \\
&\le C \int_Z^{\infty} \langle Y\rangle^{-3/4} e^{-\sqrt{2|Z|}\Re Y/3}\, dY \\
&\le C\langle Z\rangle^{-5/4} e^{-\sqrt{2|Z|}\Re Z/3}
\end{aligned}$$

and

$$\begin{aligned}
|\widetilde{C}i(1,z)| &\le |\delta|^{-1} \int_0^z |\dot{y}^{1/2} Ci(e^{i\pi/6}Y)|\, dy \le C|\delta|^{-1} \int_0^z \langle Y\rangle^{-1/4} e^{\sqrt{2|Y|}\Re Y/3}\, dy \\
&\le C \int_0^z \langle Y\rangle^{-1/4} e^{\sqrt{2|Y|}\Re Y/3}\, dY \\
&\le C\langle Z\rangle^{-3/4} e^{\sqrt{2|Z|}\Re Z/3}.
\end{aligned}$$

We also have

$$\begin{aligned}
|\widetilde{C}i(2,z)| &\le |\delta|^{-1} \int_0^z |\widetilde{C}i(1,y)|\, dy \le C \int_0^z \langle Y\rangle^{-3/4} e^{\sqrt{2|Y|}\Re Y/3}\, dY \\
&\le C\langle Z\rangle^{-5/4} e^{\sqrt{2|Z|}\Re Z/3}.
\end{aligned}$$

Hence, $\widetilde{A}i(k,z)$ and $\widetilde{C}i(k,z)$ satisfy the same bounds as do $Ai(k,z)$ and $Ci(k,z)$, respectively. Derivative bounds are also obtained in the same way. By combining these, together with those on $Ai(\cdot)$, $Ci(\cdot)$, the claimed bounds on $\widetilde{G}(x,z)$ follow similarly to those obtained in (4.22) and (4.23). Finally, using the above bounds on $\widetilde{Ai}(k,z)$ and $\widetilde{Ci}(k,z)$, we get

$$|\partial_x^k a_1(x)| \le C|\delta|^{-k}\langle X\rangle^{k/2-1}, \qquad |\partial_x^k a_2(x)| \le C|\delta|^{-k}\langle X\rangle^{k/2-3/2},$$

upon noting that the exponents in $Ai(\cdot)$ and $Ci(\cdot)$ are cancelled out identically. The boundedness of $E(x,z)$ thus follows easily.

This completes the proof of the lemma, recalling that $|\delta| \sim \epsilon^{1/3}$. □

Similarly, we also obtain the following simple lemma on the error kernel $Err_A(x,z)$.

Lemma 4.7. *Let $Err_A(x,z)$ be the error kernel defined as in (4.39), and let $X = \eta(x)/\delta$ and $Z = \eta(z)/\delta$. There hold*

$$|\partial_z^k \partial_x^\ell Err_A(x,z)| \le C\epsilon^{\frac{1-k-\ell}{3}}\langle X\rangle^{k/2-1/4}\langle Z\rangle^{\ell/2-1/4}e^{-\frac{\sqrt{2}}{3}\sqrt{|Z|}|X-Z|}, \tag{4.44}$$

for all x,z. In particular, when X and Z satisfy $|X| \le \frac{1}{2}|Z|$ or $|X| \ge 2|Z|$, there holds the following stronger estimate

$$|\partial_z^k \partial_x^\ell Err_A(x,z)| \le C\epsilon^{\frac{1-k-\ell}{3}}e^{-\frac{1}{6}|Z|^{3/2}}e^{-\frac{1}{6}|X|^{3/2}}. \tag{4.45}$$

Proof. We recall that

$$Err_A(x,z) = -i\epsilon\partial_z^2 \dot{z}^{1/2}\dot{z}^{-1/2}\partial_z^2 G(x,z) + 2i\alpha^2\epsilon\partial_z^2 G(x,z).$$

The lemma follows directly from the pointwise bounds (4.40)-(4.41) on $\widetilde{G}(x,z)$, respectively, and the fact that $\partial_z^2 G(x,z) = \partial_z^2\widetilde{G}(x,z)$. □

4.6 Convolution estimates

In this section, we establish the following convolution estimates.

Lemma 4.8. *Let $G(x,z)$ be the approximate Green function of the modified Airy equation constructed as in Lemma 4.5. Then there is some constant C so that*

$$\int_0^1 |\partial_z^\ell \partial_x^k \widetilde{G}(x,z)|dx \le C\epsilon^{\frac{1-k-\ell}{3}}\langle Z\rangle^{k+\ell-2}, \tag{4.46}$$

and

$$\int_0^1 |\partial_z^k E(x,z)|dx \le C\epsilon^{-\frac{k}{3}}|\log\epsilon|, \tag{4.47}$$

for all $z \in [0,1]$. Note that in (4.47), there holds $\partial_z^k E(x,z) = 0$ for $k \ge 2$.

Proof. Using the pointwise bounds obtained in Lemma 4.6 for X and Z close to one another and away from each other, respectively, we obtain

$$\begin{aligned}\int_0^1 |\partial_z^\ell \partial_x^k \widetilde{G}(x,z)|dx &\le C_0\epsilon^{-\frac{k+\ell}{3}} \int_0^1 \Big[\langle Z\rangle^{(k+\ell-3)/2} e^{-\frac{\sqrt{2}}{3}\sqrt{|Z|}|X-Z|} + e^{-\frac{1}{6}|Z|^{3/2}} e^{-\frac{1}{6}|X|^{3/2}}\Big]\,dx \\ &\le C\epsilon^{-\frac{k+\ell-1}{3}}\langle Z\rangle^{k+\ell-2},\end{aligned}$$

upon noting that $dx = \delta\dot{z}^{-1}(\eta(x))dX$ with $\dot{z}(\eta(x)) \approx 1$ and $|\delta| \approx \epsilon^{1/3}$. As for $E(x,z)$, we have

$$\int_0^1 |E(x,z)|\,dx = \int_0^z |E(x,z)|\,dx + \int_z^1 |E(x,z)|\,dx,$$

in which the first integral is bounded thanks to the boundedness of $E(x,z)$ for $x < z$; see (4.43). Whereas for the second integral, we note that $E(x,z)$ is also bounded when $x > z$ and $|X| \gtrsim |Z|$, and thus the integration over this region is also bounded. In particular, when $z \ge \Re z_c$, then if $x \ge z$, we have $|X| \ge |Z|$. It thus remains to estimate the case when $z \le \Re z_c$. We have

$$\int_0^1 |E(x,z)|\,dx \le C + C|Z|\int_z^{\Re z_c} \langle X\rangle^{-1}\,dx \le C\Big(1 + |\delta||Z|\log\langle Z\rangle\Big),$$

which is bounded by $C|\log\delta|$, recalling that $Z = \delta^{-1}\eta(z)$, with $\eta(z)$ being uniformly bounded in $z \in [0,1]$. Finally, we recall that $\partial_z E(x,z) = \delta^2\epsilon^{-1}\dot{x}^{3/2}a_1(x)$ for $x > z$ and zero for $x < z$. Hence,

$$\int_0^1 |\partial_z E(x,z)|\,dx \le C|\delta|^{-1}\int_z^1 \langle X\rangle^{-1}\,dx$$

which is bounded by $C_0|\delta|^{-1}|\log\delta|$. This proves the lemma. □

Similarly, we also obtain the following convolution estimate for the error kernel $Err_A(x,z)$.

Lemma 4.9. *Let $Err_A(x,z)$ be the error kernel of the modified Airy equation defined as in Lemma 4.5. Then there is some constant C so that*

$$\int_0^1 |\partial_x^k\partial_z^\ell Err_A(x,z)|dx \le C\epsilon^{\frac{2-k-\ell}{3}}\langle Z\rangle^{k+\ell-1} \tag{4.48}$$

for all $z \in [0,1]$.

Proof. Using the pointwise bounds obtained in Lemma 4.7 for X and Z close to one another and away from each other, respectively, we obtain

$$\begin{aligned}\int_0^1 |\partial_z^\ell\partial_x^k Err_A(x,z)|dx &\le C_0\epsilon^{\frac{1-k-\ell}{3}}\int_0^1 \Big[\langle Z\rangle^{k+\ell-1/2} e^{-\frac{\sqrt{2}}{3}\sqrt{|Z|}|X-Z|} + e^{-\frac{1}{6}|Z|^{3/2}} e^{-\frac{1}{6}|X|^{3/2}}\Big]\,dx \\ &\le C\epsilon^{\frac{2-k-\ell}{3}}\langle Z\rangle^{k+\ell-1},\end{aligned}$$

in which again the extra factor of $|\delta| \sim \epsilon^{1/3}$ was due to the change of variable $X = \eta(x)/\delta$. □

Finally, when f is localized near the critical layer $z = z_c$, we obtain a better convolution estimate as follows.

Lemma 4.10. *Let $G(x,z)$ and $Err_A(x,z)$ be the approximate Green function and the error kernel of the modified Airy equation constructed as in Lemma 4.5, and let $f = f(x)$ satisfy $|f(x)| \le C_f \langle X\rangle^k e^{-\frac{\sqrt{2}}{3}|X|^{3/2}}$, with $X = \delta^{-1}\eta(x)$ and $k \in \mathbb{Z}$. Then there is some constant C so that*

$$\begin{aligned} \int_0^1 |\partial_z^\ell \widetilde{G}(x,z) f(x)| dx &\le C C_f \epsilon^{\frac{1-\ell}{3}} \langle Z\rangle^{k+\ell-2} e^{-\frac{\sqrt{2}}{3}|Z|^{3/2}}, \\ \int_0^1 |\partial_z^\ell E(x,z) f(x)| dx &\le C C_f \epsilon^{\frac{1-\ell}{3}}. \end{aligned} \tag{4.49}$$

In addition, we also have

$$\int_0^1 |\partial_z^\ell Err_A(x,z) f(x)| dx \le C C_f \epsilon^{\frac{2-\ell}{3}} \langle Z\rangle^{k+\ell-1} e^{-\frac{\sqrt{2}}{3}|Z|^{3/2}}, \tag{4.50}$$

for all $z \in [0,1]$.

Proof. The proof is straightforward, similarly to those obtained in Lemmas 4.8 and 4.9. For instance, we have

$$\begin{aligned} &\int_0^1 |\partial_z^\ell \widetilde{G}(x,z) f(x)| dx \\ &\quad \le C_0 C_f \epsilon^{-\frac{\ell}{3}} \int_0^1 \Big[\langle Z\rangle^{k+\ell-3/2} e^{-\frac{\sqrt{2}}{3}\sqrt{|Z|}|X-Z|} + \langle X\rangle^k e^{-\frac{1}{6}|Z|^{3/2}} e^{-\frac{1}{6}|X|^{3/2}} \Big] e^{-\frac{\sqrt{2}}{3}|X|^{3/2}}\, dx \\ &\quad \le C C_f \epsilon^{\frac{1-\ell}{3}} \langle Z\rangle^{k+\ell-2} e^{-\frac{\sqrt{2}}{3}|Z|^{3/2}}. \end{aligned}$$

The estimates for $E(x,z)$ and $Err_A(x,z)$ follow similarly. □

4.7 Resolution of modified Airy equation

In this section, we shall introduce the approximate inverse of the *Airy* operator. We recall that $Airy(\phi) = -i\varepsilon\partial_z^4\phi - (U - c - 2i\epsilon\alpha^2)\partial_z^2\phi$. Let us study the inhomogeneous Airy equation

$$Airy(\phi) = f(z), \tag{4.51}$$

for some source $f(z)$. We introduce the approximate solution to this equation by defining

$$AirySolver(f) := \int_0^1 G(x,z) f(x) dx. \tag{4.52}$$

Then, since the Green function $G(x,z)$ does not solve exactly the modified Airy equation (see (4.38)), the solution $AirySolver(f)$ does not solve it exactly either. However, there holds

$$Airy(AirySolver(f)) = f + AiryErr(f) \tag{4.53}$$

where the error operator $AiryErr(\cdot)$ is defined by

$$AiryErr(f) := \int_0^1 Err_A(x,z) f(x) dx,$$

in which $Err_A(x,z)$ is the error kernel of the Airy operator, defined as in Lemma 4.5. In particular, from Lemma 4.9, we have the estimate

$$\|AiryErr(f)\|_{X_0} \le C\epsilon^{\frac{2}{3}} \|f\|_{X_0}, \tag{4.54}$$

for all $f \in X_0$. That is, $AiryErr(f)$ is indeed of order $\mathcal{O}(\epsilon^{2/3})$ in X_0.

For the above mentioned reason, we may now define by iteration an exact solver for the modified Airy operator. Let us start with a fixed $f \in X_0$. Let us define

$$\begin{aligned} \phi_n &= -AirySolver(E_{n-1}) \\ E_n &= -AiryErr(E_{n-1}) \end{aligned} \tag{4.55}$$

for all $n \ge 1$, with $E_0 = f$. Let us also denote

$$S_n = \sum_{k=1}^n \phi_k.$$

It follows by induction that

$$Airy(S_n) = f + E_n,$$

for all $n \ge 1$. Now by (4.54), we have

$$\|E_n\|_{X_0} \le C\epsilon^{\frac{2}{3}} \|E_{n-1}\|_{X_0} \le (C\epsilon^{\frac{2}{3}})^n \|f\|_{X_0}.$$

This proves that $E_n \to 0$ in X_0 as $n \to \infty$ since ϵ is small. In addition, by a view of Lemma 4.8, we have

$$|\phi_n(z)| \le C\|E_{n-1}\|_{X_0} \le C(C\epsilon^{\frac{2}{3}})^{n-1} \|f\|_{X_0}.$$

This shows that ϕ_n converges to zero in X_0, and furthermore the series

$$S_n \to S_\infty$$

in X_0 as $n \to \infty$, for some $S_\infty \in X_0$. We then denote $AirySolver_\infty(f) = S_\infty$, for each $f \in X_0$. In addition, we have $Airy(S_\infty) = f$, that is, $AirySolver_\infty(f)$ is the exact solver for the modified Airy operator.

To summarize, we have proved the following proposition.

Proposition 4.11. *Assume that ϵ is sufficiently small. There exists an exact solver $AirySolver_\infty(\cdot)$ as a well-defined operator from X_0 to X_0 so that*

$$Airy(AirySolver_\infty(f)) = f.$$

In addition, there holds

$$\|AirySolver_\infty(f)\|_{X_0} \le C\|f\|_{X_0},$$

for some positive constant C.

In addition, when f is localized near the critical layer, the bound on $AirySolver_\infty(f)$ is sharper and in particular is of order $\mathcal{O}(\epsilon^{1/3})$ as follows.

Proposition 4.12. *Assume that ϵ is sufficiently small and $f = f(x)$ satisfies $|f(x)| \le C_f \langle X\rangle^k e^{-\frac{\sqrt{2}}{3}|X|^{3/2}}$, with $X = \delta^{-1}\eta(x)$ and $k \in \mathbb{Z}$. The exact solver $AirySolver_\infty(f)$ exists and satisfies the uniform bound*

$$|\partial_z^\ell AirySolver_\infty(f)(z)| \le C C_f \epsilon^{\frac{1-\ell}{3}},$$

for $\ell \ge 0$ and for some positive constant C.

Proof. This is a direct consequence of Proposition 4.11, using the sharper bounds on the convolutions obtain in Lemma 4.10. □

5 Singularities and Airy equations

In this section, we study the smoothing effect of the modified Airy function. Precisely, let us consider the Airy equation with a singular source:

$$Airy(\phi) = -i\varepsilon\partial_z^4\phi - (U - c - 2i\epsilon\alpha^2)\partial_z^2\phi = \epsilon\partial_z^4 f(z), \tag{5.1}$$

with $f \in Y_4$; see the definition in Section 2.2. We also assume that

$$f'(1) = 0. \tag{5.2}$$

We prove the following:

Proposition 5.1. *Assume that $\alpha, \epsilon, c \ll 1$, and f satisfies the above assumptions. Then, the $AirySolver(\cdot)$ and $AiryErr(\cdot)$ operators satisfy*

$$\Big\|AirySolver(\epsilon\partial_x^4 f)\Big\|_{X_2} \le C\|f\|_{Y_4}|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}, \tag{5.3}$$

in which δ denotes the critical layer size, defined as in (4.30), with $|\delta| \sim \epsilon^{1/3}$, and

$$\Big\|AiryErr(\epsilon\partial_x^4 f)\Big\|_{X_2} \le C\|f\|_{Y_4}|\delta^2\log\delta|, \tag{5.4}$$

for some universal constant C.

The above proposition follows directly from the two following lemmas.

Lemma 5.2. *Assume that $\alpha, \epsilon, c \ll 1$. Let $G(x,z)$ be the approximated Green function to the modified Airy equation constructed as in Lemma 4.5 and let $f(z) \in Y_4$. There holds a convolution estimate:*

$$\Big|(z - z_c)^k\partial_z^k\int_0^1 G(x,z)\epsilon\partial_x^4 f(x)dx\Big| \le C\|f\|_{Y_4}|\delta\log\delta|\langle z_c/\delta\rangle^{3/2} \tag{5.5}$$

with $|\delta| \sim \epsilon^{1/3}$, for all $z \in (0,1)$, and for $k = 0, 1, 2$.

Similarly, we also have the following.

Lemma 5.3. *Assume that $\alpha, \epsilon, c \ll 1$. Let $Err_A(x,z)$ be the error defined as in Lemma 4.5 and let $f(z) \in Y_4$. There holds the convolution estimate for $Err_A(x,z)$*

$$\Big|(z - z_c)^k\partial_z^k\int_0^1 Err_A(x,z)\epsilon\partial_x^4 f(x)dx\Big| \le C\|f\|_{Y_4}|\delta^2\log\delta| \tag{5.6}$$

for all $z \in (0,1)$, and for $k = 0, 1, 2$.

Proof of Lemma 5.2 with $k = 0$. By scaling, let us assume that $\|f\|_{Y_4} = 1$. To begin our estimates, let us recall the decomposition of $G(x,z)$ into the localized and non-localized part as

$$G(x,z) = \widetilde{G}(x,z) + E(x,z),$$

where $\widetilde{G}(x,z)$ and $E(x,z)$ satisfy the pointwise bounds in Lemma 4.6. In addition, we recall that $\epsilon\partial_x^j G_{2,a}(X,Z)$ and so $\epsilon\partial_x^j G(x,z)$ are continuous across $x = z$ for $j = 0,1,2$. Using the continuity, we can integrate by parts to get

$$\begin{aligned}\phi(z) &= -\epsilon\int_0^1 \partial_x^3(\widetilde{G}+E)(x,z)\partial_x f(x)\,dx + \mathcal{B}(z) \\ &= I_\ell(z) + I_e(z) + \mathcal{B}(z)\end{aligned} \tag{5.7}$$

Here, $I_\ell(z)$ and $I_e(z)$ denote the corresponding integral that involves $\widetilde{G}(x,z)$ and $E(x,z)$ respectively, and $\mathcal{B}_0(z)$ is introduced to collect the boundary terms at $x = 0$ and is defined by

$$\mathcal{B}(z) := -\epsilon\sum_{k=0}^{2}(-1)^k\partial_x^k G(x,z)\partial_x^{3-k}(f(x))\Big|_{x=0}^{x=1}. \tag{5.8}$$

Estimate for the integral $I_\ell(z)$. Using the bounds (4.40) and (4.41) on the localized part of the Green function, we can give bounds on the integral term I_ℓ in (5.7).

Consider the case $|z - z_c| \le |\delta|$. By splitting the integral into two cases according to the estimates (4.40) and (4.41), we get

$$\begin{aligned}|I_\ell(z)| &= \Big|\epsilon\int_0^1 \partial_x^3\widetilde{G}(x,z)\partial_x f(x)\,dx\Big| \\ &\le \epsilon\int_{\{|x-z_c|\le 3|\delta|\}} |\partial_x^3\widetilde{G}(x,z)\partial_x f(x)|\,dx + \epsilon\int_{\{|x-z_c|\ge 3|\delta|\}} |\partial_x^3\widetilde{G}(x,z)\partial_x f(x)|\,dx,\end{aligned}$$

in which $\epsilon\partial_x^3\widetilde{G}(x,z)$ is bounded and so the first integral on the right is bounded by

$$C\int_{\{|x-z_c|\le 3|\delta|\}} |\partial_x f(x)|\,dx \le C\int_{\{|x-z_c|\le 3|\delta|\}} (1 + |\log(x-z_c)|)\,dx \le C|\delta\log\delta|.$$

Let us turn to the second integral on the right. Note that

$$|\delta|^{-1}|x - z_c| \ge 3 \ge 3|\delta|^{-1}|z - z_c|,$$

hence, as $X \sim |\delta|^{-1}(x - z_c)$ and similarly for z, $|X| \ge 2|Z|$. We therefore use (4.41) for x away from z to get

$$\begin{aligned}\epsilon\int_{\{|x-z_c|\ge 3|\delta|\}} |\partial_x^3\widetilde{G}(x,z)\partial_x f(x)|\,dx &\le C\int_{\{|x-z_c|\ge 3|\delta|\}} e^{-\frac{1}{6}|X|^{3/2}}(1 + |\log(x-z_c)|)\,dx \\ &\le C|\delta\log\delta|\int_{\mathbb{R}} e^{-\frac{1}{6}|X|^{3/2}}\,dX \\ &\le C|\delta\log\delta|.\end{aligned}$$

Let us now consider the case $|z - z_c| \geq |\delta|$. We again split the integral in x into two parts $|x - z_c| \leq |\delta|$ and $|x - z_c| \geq |\delta|$. Using that $\epsilon\partial_x^3 \widetilde{G}$ is bounded we get

$$\begin{aligned}\epsilon \int_{\{|x-z_c|\leq|\delta|\}} |\partial_x^3 \widetilde{G}(x,z)\partial_x f(x)|\,dx &\leq C \int_{\{|x-z_c|\leq|\delta|\}} (1 + |\log(x - z_c)|)\,dx \\ &\leq C|\delta \log \delta|.\end{aligned}$$

Next, for the integral over $\{|x - z_c| \geq |\delta|\}$, we note that for $|x - z_c| \geq |\delta|$, $|\partial_x f(x)| \leq C(1 + |\log \delta|)$. We then use the bounds (4.40) and (4.41) to get

$$\begin{aligned}&\epsilon \int_{\{|x-z_c|\geq|\delta|\}} |\partial_x^3 \widetilde{G}(x,z)\partial_x f(x)|\,dx \\ &\leq C|\log \delta|\Big[\int_{\frac{1}{2}|z|\leq|x|\leq|2|z|} e^{-\sqrt{2|Z|}|X-Z|/3}\,dx + e^{-\frac{1}{6}|Z|^{3/2}} \int_0^1 e^{-\frac{1}{6}|X|^{3/2}}\,dx\Big] \\ &\leq C|\delta \log \delta|.\end{aligned}$$

Therefore in all cases, we have $|I_\ell(z)| \leq C|\delta \log \delta|$ as claimed.

Estimate for I_e. Let us next consider the integral

$$I_e(z) = -\int_0^1 \epsilon\partial_x^3(E(x,z))\partial_x f(x)\,dx = -\Big[\int_{\{|x-z_c|\leq|\delta|\}} + \int_{\{|x-z_c|\geq|\delta|\}}\Big]\epsilon\partial_x^3(E(x,z))\partial_x f(x)\,dx,$$

with recalling that

$$E(x,z) = \delta^3\epsilon^{-1}\dot{x}^{3/2}\begin{cases} a_1(x)(z - z_c)/\delta, & \text{if } x > z, \\ a_2(x) + a_1(x)(x - z_c)/\delta, & \text{if } x < z. \end{cases}$$

Here $a_1(x), a_2(x)$ satisfy the bound

$$|\partial_x^k a_1(x)| \leq C|\delta|^{-k}(1 + |X|)^{k/2-1}, \qquad |\partial_x^k a_2(x)| \leq C|\delta|^{-k}(1 + |X|)^{k/2-3/2},$$

with $X = \eta(x)/\delta \approx (x - z_c)/\delta$.

In particular, for $|x - z_c| \leq |\delta|$, the integrand $\epsilon\partial_x^3(E(x,z))\partial_x f(x)$ is bounded by $C(1 + |\log(x - z_c)|)$ for $x < z$ and by $C(1 + |\log(x - z_c)|)|z - z_c|/|\delta|$ for $z < x$. In the latter case, we have $|z - z_c| \leq |z_c| + |\delta|$ since $z < x$ and $|x - z_c| \leq |\delta|$. Putting these together, we have

$$\begin{aligned}\Big|\int_{\{|x-z_c|\leq|\delta|\}} \epsilon\partial_x^3(E(x,z))\partial_x f(x)\,dx\Big| &\leq C(1 + |z_c/\delta|)\int_{\{|x-z_c|\leq|\delta|\}} (1 + |\log(x - z_c)|)\,dx \\ &\leq C|\delta|(1 + |z_c/\delta|)(1 + |\log \delta|).\end{aligned}$$

Next, we consider the integral over $\{|x - z_c| \geq |\delta|\}$. In this case, since $X \to \infty$ as $\delta \to 0$, $\partial_x^k a_j(x)$ is very large and therefore we have to take several integration by parts to avoid this growth. Note that if $|z_c| \leq |\delta|$ the two smaller boundary terms are not present.

$$\begin{aligned}\int_{\{|x-z_c|\geq|\delta|\}} \epsilon\partial_x^3 E(x,z)\partial_x f(x)\,dx = &-\int_{\{|x-z_c|\geq|\delta|\}} \epsilon\partial_x^2 E(x,z)\partial_x^2 f(x)\,dx \\ &+ B_0(z) + B_1(z) + B_3(z) + B_4(z),\end{aligned}$$

where $B_j(z)$ denotes the boundary terms at $x = 0, x = 1, x = z$, and at points which satisfy $|x - z_c| = |\delta|$, respectively. We have $B_1(z) = 0$ since $\partial_x f(1) = 0$, whereas

$$B_0(z) = -\epsilon\partial_x^2 E(x,z)\partial_x f(x)_{|x=0} = -\epsilon\partial_x^2\Big[a_2(x) + a_1(x)(x - z_c)/\delta\Big]\partial_x f(x)_{|x=0}.$$

From the bound on $\partial_x^k a_j(x)$ and the fact that at $x = 0$, $X \approx z_c/\delta$, we have

$$|B_0(z)| \le C|\delta|(1 + |\log\delta|)(1 + |z_c/\delta|)^{3/2}.$$

Similarly, we have

$$B_3(z) = \epsilon[\partial_x^2 E(x,z)]_{|x=z}\partial_x f(z) = \epsilon\Big[\partial_x^2 a_2(z) + 2\delta^{-1}\partial_x a_1(z)\Big]\partial_z f(z),$$

which satisfies the bound

$$|B_3(z)| \le C|\delta|(1 + |\log(z - z_c)|)(1 + |Z|)^{-1/2} \le C|\delta|(1 + |\log\delta|).$$

The last boundary term reads

$$B_4(z) = \epsilon\partial_x^2 E(x,z)\partial_x f(x)\Big|_{\{|x - z_z| = |\delta|\}}$$

This is the same as in the previous case $|x - z_c| \le |\delta|$: $B_4(z)$ is bounded by $C|\delta|(1+|\log\delta|)$ for $x < z$ and by $C|\delta|(1 + |\log\delta|)|z_c|/\delta$ for $z < x$.

To summarize, we have so far shown

$$\begin{aligned}&\int_{\{|x - z_c| \ge |\delta|\}} \epsilon\partial_x^3 E(x,z)\partial_x f(x)\,dx \\ &= -\int_{\{|x - z_c| \ge |\delta|\}} \epsilon\partial_x^2 E(x,z)\partial_x^2 f(x)\,dx + \mathcal{O}(|\delta|(1 + |\log\delta|)(1 + |z_c/\delta|)^{3/2}).\end{aligned}$$

As for the integral term on the right, $\epsilon\partial_x^2 E(x,z)$ remains large for $x > z$ due to the appearance of $|z - z_c|/|\delta|$. We again integrate by parts this term to move x-derivatives of $E(x,z)$ into $f(x)$. This will leave several boundary terms that are similar to the above and are of a smaller order since the order of derivatives that hit $E(x,z)$ decreases. Thus, we get

$$\begin{aligned}&\int_{\{|x - z_c| \ge |\delta|\}} \epsilon\partial_x^3 E(x,z)\partial_x f(x)\,dx \\ &= -\int_{\{|x - z_c| \ge |\delta|\}} \epsilon E(x,z)\partial_x^4 f(x)\,dx + \mathcal{O}(|\delta|(1 + |\log\delta|)(1 + |z_c/\delta|)^{3/2}).\end{aligned}$$

By definition of $E(x,z)$ and the assumption on $f(x)$, we have

$$\begin{aligned}\Big|\int_{\{|x - z_c| \ge |\delta|\}} \epsilon E(x,z)\partial_x^4 f(x)\,dx\Big| &\le C|\delta|^3 \int_{\{|x - z_c| \ge |\delta|\}} \Big[1 + |x - z_c|^{-3}\Big]\,dx \\ &\le C|\delta|\end{aligned}$$

To summarize, we have obtained

$$|I_e(z)| \le C|\delta|(1 + |\log \delta|)(1 + |z_c/\delta|)^{3/2},$$

for all $z \in (0,1)$.

Estimate for the boundary term $\mathcal{B}(z)$. It remains to give estimates on

$$\mathcal{B}(z) := -\epsilon \sum_{k=0}^{2} (-1)^k \partial_x^k G(x,z) \partial_x^{3-k}(f(x))\Big|_{x=0}^{x=1}.$$

We claim that

$$|\mathcal{B}(z)| \le C|\delta|(1 + |\log \delta|)(1 + |z_c/\delta|)^{3/2}.$$

In the estimate for $I_e(z)$, we have provided estimates for the boundary terms at $x = 0, 1$ that involve $E(x,z)$. It thus remains to consider the terms involving the localized part $\widetilde{G}(x,z)$ of the Green function. Using the bounds (4.40) and (4.41) for $x = 0$, we get

$$|\epsilon \partial_x^k \widetilde{G}(x,z) \partial_x^{3-k}(f(x))|_{x=0} \le C|\delta|^{3-k}(1 + |z_c|^{-2+k}(1 + |\log z_c|))e^{-\frac{2}{3}|Z|^{3/2}}$$

which is bounded by $C|\delta|(1 + |\log \delta|)(1 + |z_c/\delta|)^{k-2}$, with $k = 0, 1, 2$. At $x = 1$, $\partial_x^k(f(x))$ is bounded for $k = 0, ...4$, and $\epsilon \partial_x^k \widetilde{G}(x,z)$ is bounded by $C|\delta|^{3-k}$, for $k \le 3$, which follows directly from the bounds (4.40) and (4.41). This proves the claimed estimate for $\mathcal{B}(z)$.

□

Proof of Lemma 5.2 with $k > 0$. We now prove the lemma for the case $k = 2$; the case $k = 1$ follows similarly. We consider the integral

$$\epsilon \int_0^1 (z - z_c)^2 \partial_z^2 \widetilde{G}(x,z) \partial_x^4 f(x) dx = I_1(z) + I_2(z),$$

with $I_1(z)$ and $I_2(z)$ denoting the integration over $\{|x - z_c| \le |\delta|\}$ and $\{|x - z_c| \ge |\delta|\}$, respectively. Note that $(U(z) - c)\dot{z}^2 = U'(z_c)\eta(z)$ and recall that $Z = \eta(z)/\delta$ by definition. For the second integral $I_2(z)$, by using the bounds on the Green function for x away from z and for x near z, it follows easily that

$$\begin{aligned} |I_2(z)| \le C\Big[|\delta| \int_{\{|x-z_c|\ge|\delta|\}} (1 + |Z|)^{1/2} e^{-\frac{2}{3}\sqrt{|Z|}|X-Z|}(1 + |x - z_c|^{-1})\, dx \\ + \epsilon e^{-\frac{1}{6}|Z|^{3/2}} \int_{\{|x-z_c|\ge|\delta|\}} e^{-\frac{1}{6}|X|^{3/2}}(1 + |x - z_c|^{-3})\, dx\Big]. \end{aligned}$$

Using $|x - z_c| \ge |\delta|$ in these integrals and making a change of variable $X = \eta(x)/\delta$ to gain an extra factor of δ, we obtain

$$|I_2(z)| \le C|\delta|\Big[\int_{\mathbb{R}} (1 + |Z|)^{1/2} e^{-\frac{2}{3}\sqrt{|Z|}|X-Z|}\, dX + e^{-\frac{1}{8}|Z|^{3/2}} \int_{\mathbb{R}} e^{-\frac{1}{6}|X|^{3/2}}\, dX\Big],$$

which is clearly bounded by $C|\delta|$. It remains to give the estimate on $I_1(z)$ over the region: $|x - z_c| \le |\delta|$. In this case, we take integration by parts three times. Leaving the boundary terms untreated for a moment, let us consider the integral term

$$\epsilon \int_{\{|x-z_c|\le|\delta|\}} (z-z_c)^2 \partial_z^2 \partial_x^3 \widetilde{G}(x,z) \partial_x f(x) dx.$$

We note that the twice z-derivative causes a large factor $|\delta|^{-2}$ which combines with $(z-z_c)^2$ to give a term of order $|Z|^2$. Similarly, the small factor of ϵ cancels out with $|\delta|^{-3}$ that comes from the third x-derivative. The integral is therefore easily bounded by

$$\begin{aligned}
C\Big[&\int_{\{|x-z_c|\le|\delta|\}} e^{-\frac{2}{3}\sqrt{|Z|}|X-Z|}(1+|\log(x-z_c)|)\,dx \\
&+ e^{-\frac{1}{6}|Z|^{3/2}} \int_{\{|x-z_c|\le|\delta|\}} e^{-\frac{1}{6}|X|^{3/2}}(1+|\log(x-z_c)|)\,dx\Big] \\
&\le C \int_{\{|x-z_c|\le|\delta|\}} (1+|\log(x-z_c)|)\,dx \\
&\le C|\delta|(1+|\log\delta|).
\end{aligned}$$

Finally, the boundary terms can be treated, following the previous treatment as done in the case $k = 0$. This completes the proof of the lemma. □

Proof of Lemma 5.3 . The proof follows similarly, but more straightforwardly, from the above proof for the localized part of the Green function, upon recalling that

$$Err_A(x,z) = -i\epsilon\partial_z^2 \dot{z}^{1/2}\dot{z}^{-1/2}\partial_z^2 G(x,z) + 2i\alpha^2\epsilon\partial_z^2 G(x,z).$$

We omit further details. □

6 Construction of the slow Orr modes

In this section, we iteratively construct an exact Orr-Sommerfeld solutions. Our construction starts with the Rayleigh solutions $\phi_{j,\alpha}$, constructed in Section 3. Precisely, we obtain the following Proposition whose proof will be given at the end of the section, yielding an exact solution to the Orr-Sommerfeld equations.

Proposition 6.1. *For sufficiently small α, ϵ, c so that*

$$|\delta \log \delta| \langle z_c/\delta \rangle^{3/2} \ll 1, \tag{6.1}$$

with z_c being the critical layer $U(z_c) = c$ and $|\delta| \sim \epsilon^{1/3}$ being the critical layer size, defined as in (4.30)*, there exist exact solutions $\phi_1(z)$ and $\phi_2(z)$ in X_2 which solve the Orr-Sommerfeld equations*

$$Orr(\phi_j) = 0, \qquad j = 1, 2.$$

In addition, we can construct $\phi_1(z)$ and $\phi_2(z)$ so that ϕ_j is close to $\phi_{j,\alpha}$ in X_2, that is,

$$\|\phi_j - \phi_{j,\alpha}\|_{X_2} \le C |\delta \log \delta| \langle z_c/\delta \rangle^{3/2}$$

for some positive constant C independent of α, z_c, ϵ.

Next, we obtain the following lemma.

Lemma 6.2. *The slow Orr modes $\phi_{1,2}$ constructed in Proposition 6.1 depends analytically in c, for Im $c > 0$.*

Proof. The proof is straightforward since the only "singularities" are of the forms: $\log(U-c)$, $1/(U-c)$, $1/(U-c)^2$, and $1/(U-c)^3$, which are of course analytic in c when Im $c > 0$. ☐

Remark 6.3. In fact, it can be shown that the solutions $\phi_{1,2}$ can be extended C^γ-Hölder continuously on the axis $\{\text{Im } c = 0\}$, for $0 \le \gamma < 1$.

6.1 Principle of the construction

We now present the idea of the iterative construction. We start from the Rayleigh solutions $\phi_{j,\alpha}(z)$ constructed in Section 3. Since they solve the Rayleigh equation exactly, we have

$$Orr(\phi_{j,\alpha}) = i\epsilon(\partial_z^2 - \alpha^2)^2 \phi_{j,\alpha} = Diff(\phi_{j,\alpha}). \tag{6.2}$$

Here we observe that the right hand side, denoted $O_1(z)$, is of order $\mathcal{O}(\epsilon)$. However, it contains a singularity near the critical layer $z = z_c$ since $\phi_{j,\alpha}(z)$ has a singularity of order $(z - z_c)\log(z - z_c)$. We then apply the Airy solver to smooth out the singularity. Precisely, we denote

$$A_{j,0} := AirySolver(Diff(\phi_{j,\alpha})).$$

By Proposition 5.1,

$$\|A_{j,0}\|_{X_2} \le C \|\phi_{j,\alpha}\|_{Y_4} |\delta \log \delta| \langle z_c/\delta \rangle^{3/2}.$$

We then modify $\phi_{j,\alpha}$ by adding this corrector $A_{j,0}$. We introduce

$$\phi_{j,1} := \phi_{j,\alpha} + A_{j,0}.$$

We then have

$$Orr(\phi_{j,1}) = O_{j,1} := -AiryErr(Diff(\phi_{j,\alpha})) + Reg(AirySolver(Diff(\phi_{j,\alpha}))), \tag{6.3}$$

which has no singularity other than $(z - z_c)\log(z - z_c)$ and of smaller order as compared to the right hand side of (6.2). By Proposition 5.1, and as Reg is a simple multiplication by a bounded function,

$$\|Orr(\phi_{j,1})\|_{X_2} \le C\|\phi_{j,\alpha}\|_{Y_4}|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}.$$

We then approximately solve (6.3) by

$$\psi_{j,2} = -RaySolver_\alpha\Big(O_{j,1}\Big).$$

We thereby gain the control of two derivatives and an extra $z-z_c$ term thanks to Proposition 3.1 and get

$$\|\psi_{j,2}\|_{Y_4} \le C\|O_{j,1}\|_{X_2}.$$

We inductively proceed the above construction. Let us assume that we have constructed approximate solutions $\phi_{j,N}(z) \in X_2$, $j = 1, 2$ and $N \ge 1$, so that

$$Orr(\phi_{1,N}) = O_{j,N},$$

with an error $O_{j,N}$ which is sufficiently small. We then introduce

$$\psi_{j,N} := -RaySolver_\alpha\Big(O_{j,N}\Big)$$

in order to solve approximately the equation "in the interior of the domain". Observe that by a view of (2.6) and (3.2)

$$Orr(\phi_{j,N} + \psi_{j,N}) = S_{j,N} = -Diff(RaySolver_\alpha(O_{j,N})). \tag{6.4}$$

We expect $Diff(RaySolver_\alpha(O_{j,N}))$ to have a better error estimate, precisely due to the extra ε present in the $Diff$ operator. However, the Rayleigh equation contains a singular solution of the form $(z - z_c)\log(z - z_c)$, and therefore $\psi_{1,N}$ admits the same singularity at $z = z_c$. As a consequence, $Diff(\psi_{1,N})$ consists of singularities of orders $\log(z - z_c)$ and $(z - z_c)^{-k}$, for $k = 1, 2, 3$, and is large in the critical layer. To remove these singularities, we then use the *Airy* operator. More precisely, the *Airy* operator smoothes out the singularity inside the critical layer. To do so, we introduce

$$A_{j,N} = AirySolver(S_{j,N})$$

which take care of the singularities in the critical layer. We then define

$$\phi_{1,N+1} := \phi_{1,N} + \psi_{j,N} + A_{j,N}$$

which solves

$$Orr(\phi_{1,N+1}) = AiryErr(S_{s,N}) - Reg(AirySolver(S_{s,N})).$$

The point here is that although $S_{s,N}$ contains the mentioned singularity, $AirySolver(S_{s,N})$ and so $Orr(\phi_{1,N+1})$ consist of no singularity, and furthermore the right hand side term $O_{j,N+1}$ has a better error as compared to $O_{j,N}$. To ensure the convergence, let us introduce the iterating operator $Iter(f)$ defined by

$$\begin{aligned} Iter(f) :=& AiryErr(Diff(RaySolver_\alpha(f))) \\ &- Reg(AirySolver(Diff(RaySolver_\alpha(f)))). \end{aligned} \tag{6.5}$$

Then

$$O_{j,N+1} = Iter(O_{j,N}).$$

We shall prove the following key lemma which gives sufficient estimates on the $Iter(\cdot)$ operator.

Lemma 6.4. *For $f \in X_2$, the $Iter(\cdot)$ operator defined as in* (6.5) *is a well-defined map from X_2 to X_2. Furthermore, there holds*

$$\|Iter(f)\|_{X_2} \le C|\delta \log \delta| \langle z_c/\delta \rangle^{3/2} \|f\|_{X_2}, \tag{6.6}$$

for some universal constant C.

Proof. Take $f \in X_2$. By Proposition 3.1, $F(z) := RaySolver_\alpha(f)(z)$ is well-defined for all $z \in [0,1]$, and satisfies

$$\|F\|_{Y_4} \le C\|f\|_{X_2}.$$

Furthermore, $\partial_z F(1) = 0$. Next, Proposition 5.1 can be applied to get

$$\|AirySolver(Diff(F))\|_{X_2} \le C\|F\|_{Y_4} |\delta \log \delta| \langle z_c/\delta \rangle^{3/2}, \tag{6.7}$$

and

$$\|AiryErr(Diff(F))\|_{X_2} \le C\|F\|_{Y_4} |\delta^2 \log \delta|. \tag{6.8}$$

Combining these estimates together and recalling that $Reg(\phi) := (i\varepsilon\alpha^4 - U'' - \alpha^2(U-c))\phi$ is simply a multiplication by a bounded function, the Lemma follows at once. □

Proof of Proposition 6.1. Using the previous Lemma we construct by iteration functions $\phi_{j,N}$ such that

$$Orr(\phi_{j,N})(z) = O_{j,N}(z), \tag{6.9}$$

where the error $O_{j,N}(z)$ satisfies

$$\|O_{j,N}\|_{X_2} \le C\Big[C|\delta \log \delta| \langle z_c/\delta\rangle^{3/2}\Big]^N,$$

and where $\phi_{j,N}$ satisfy the same bound in Y_4. Thanks to the smallness assumption (6.1), we can now define ϕ_j, which satisfy $Orr(\phi_j) = 0$, by the following convergent series

$$\phi_j(z) = \phi_{j,\alpha}(z) + AirySolver(Diff(\phi_{j,\alpha})) + \sum_{n\ge 1}^{+\infty} \Big[\psi_{j,n} + AirySolver\Big(Diff(\psi_{j,n})\Big)\Big] \tag{6.10}$$

with $\psi_{j,n} = -RaySolver_\alpha(O_{j,n})$ and $O_{j,n+1}(z) := Iter(O_{j,n})(z)$, in which $O_{j,1}$ is defined as in (6.3) and the iterated operator $Iter(\cdot)$ is defined as in (6.5), the exact Rayleigh solver $RaySolver_\alpha(\cdot)$ is constructed as in Proposition 3.1.

□

6.2 First order expansion of ϕ_1 at $z = 0$

In this paragraph we explicitly compute the boundary contribution of the first terms in the expansion of $\phi_1(0)$. We shall use the estimates obtained in Lemma 3.7. In study of the dispersion relation, we are interested in various ratios between these solutions. For convenience, let us define

$$K_1 := \frac{\phi_1(0)}{\partial_z\phi_1(0)}. \tag{6.11}$$

In this section, we will prove the following Lemma.

Lemma 6.5. *Let ϕ_1 be defined as in* (6.10)*, and let K_1 be defined as in* (6.11)*. For sufficiently small α, ϵ, c, there hold*

$$\begin{aligned} K_1 &= \frac{-c}{U_0'} + \frac{\alpha^2}{|U_0'|^2}\int_0^1 U^2\,dx + \mathcal{O}\Big(\alpha^2 z_c \log z_c + \delta^2\log\delta + \alpha^2\delta\log\delta\langle z_c/\delta\rangle^{3/2}\Big),\\ Im\ K_1 &= -\frac{Im\ c}{U_0'}\Big(1 + \mathcal{O}(\alpha^2\log\alpha)\Big) + \mathcal{O}\Big(\delta^2\log\delta + \alpha^2\delta\log\delta\langle z_c/\delta\rangle^{3/2}\Big). \end{aligned} \tag{6.12}$$

In particular, $K_1 = \mathcal{O}(\alpha^2 + |z_c| + |\delta|)$.

The proof of Lemma 6.5 follows directly from several lemmas, obtained below in this section, together with Lemma 3.7. We first give the boundary estimates on $A_{j,0}(z)$

Lemma 6.6. *Let $A_{1,0}(z) = AirySolver(Diff(\phi_{1,\alpha}))$. There hold*

$$\begin{aligned} |A_{1,0}(z)| &\le C|\delta|^3 + C\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}\\ |\partial_z A_{1,0}(z)| &\le C|\delta|^2 + C|z-z_c|^{-1}\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2} \end{aligned} \tag{6.13}$$

Proof. We recall that $\phi_{1,\alpha} = U - c + \mathcal{O}(\alpha^2)$ in Y_4 by Lemma 3.7. Hence its leading order $U - c$ is sufficiently smooth, and in particular, we have

$$Diff(U-c) = \mathcal{O}(\delta^3).$$

Proposition 4.11 then yields

$$\partial_z^k AirySolver(Diff(U-c))(z) = \mathcal{O}(\delta^{3-k}),$$

for $k = 0, 1$. This proves that $AirySolver(Diff(U-c))$ is of order δ^3 in X_2. Next, since the $\mathcal{O}(\alpha^2)$ term in $\phi_{1,\alpha}$ is of order α^2 in Y_4 (Lemma 3.7), Proposition 5.1 (see also (6.7)) yields

$$\|AirySolver(Diff(\mathcal{O}(\alpha^2)))\|_{X_2} \le C\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}.$$

Combining, we obtain

$$\|A_{1,0}\|_{X_2} \le C|\delta|^3 + C\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}. \tag{6.14}$$

By a view of the definition of the function space X_2, this proves the lemma. □

Proof of Lemma 6.5. Let us recall from (6.10) that ϕ_1 is defined by

$$\phi_1(z) = \phi_{1,\alpha}(z) + AirySolver(Diff(\phi_{1,\alpha})) + \sum_{n\ge 1}^{+\infty}\Big[\psi_{1,n} + AirySolver\Big(Diff(\psi_{1,n})\Big)\Big].$$

It remains to give estimates on the terms in the summation. We recall that $\psi_{1,1} = -RaySolver_\alpha(O_{1,1})$, with

$$\begin{aligned} O_{1,1}(z) &= -AiryErr(Diff(\phi_{1,\alpha})) + Reg(AirySolver(Diff(\phi_{1,\alpha}))) \\ &= -AiryErr(Diff(\phi_{1,\alpha})) + Reg(A_{1,0}). \end{aligned}$$

By definition of $Reg(\phi) = (i\varepsilon\alpha^4 - U'' - \alpha^2(U-c))\phi$, we have $|Reg(A_{1,0})| \le C|A_{1,0}|$ and hence has the same bound as that of $A_{1,0}$, obtained as in Lemma 6.6. Next, using the estimate (6.8), we have

$$\|AiryErr(Diff(\phi_{1,\alpha}))\|_{X_2} \le C\|\phi_{1,\alpha}\|_{Y_4}|\delta^2\log\delta| \le C|\delta^2\log\delta|.$$

Next, from Proposition 3.1 and Lemma 6.6, we have obtained

$$\|\psi_{1,1}\|_{Y_4} \le C\|O_{1,1}\|_{X_2} \le C|\delta^2\log\delta| + C\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2},$$

in which we have absorbed the bound $\mathcal{O}(\delta^3)$ into $\delta^2\log\delta$. Next order terms are smaller, and so in particular satisfy the same bound.

Combining the above estimates, together with the expansions on $\phi_{1,\alpha}(0)$ and $\partial_z\phi_{1,\alpha}(0)$ obtained in Lemma 3.7, we have obtained

$$\begin{aligned} \phi_1(0) &= -c + \frac{\alpha^2}{U_0'}\int_0^1 (U-c)^2\,dx \\ &\quad + \mathcal{O}\Big(\alpha^2 z_c\log z_c + |\delta^2\log\delta| + \alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}\Big) \end{aligned} \tag{6.15}$$

for small z_c, α, ϵ. As for the derivative, from the definition of norm $\|\cdot\|_{X_2}$, we also have

$$\partial_z\phi_1(0) = U_0' + \mathcal{O}\Big(\alpha^2 \log z_c + |z_c|^{-1}|\delta^2 \log\delta| + |z_c|^{-1}\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2}\Big) \tag{6.16}$$

The claimed estimate of $\phi_1/\partial_z\phi_1$ now follows at once, upon recalling that $c \sim z_c$ is sufficiently small.

Finally, let us study the imaginary part of $\phi_1/\partial_z\phi_1$. It is clear from the above expansions that

$$\mathrm{Im}\,\Big(\frac{\phi_1}{\partial_z\phi_1}\Big)(0) = -\frac{\mathrm{Im}\,c}{U_0'}\Big(1 + \mathcal{O}(\alpha^2\log\alpha)\Big) + \mathcal{O}(\alpha^2|\delta\log\delta|\langle z_c/\delta\rangle^{3/2})$$

in which again we have used the fact that $c \sim z_c$. This proves the lemma. □

6.3 First order expansion of $\phi_{1,2}$ at $z = 1$

Similarly to the previous section, we are interested in the ratio:

$$K_2 := \frac{\partial_z\phi_1(1)}{\partial_z\phi_2(1)}. \tag{6.17}$$

We shall prove the following lemma.

Lemma 6.7. *Let ϕ_j be defined as in* (6.10)*, and let K_2 be defined as in* (6.17)*. For sufficiently small α, ϵ, c, there hold*

$$K_2 = \alpha^2(1 + \mathcal{O}(\alpha^2))\int_0^1 U^2\,dx. \tag{6.18}$$

Proof. We recall that

$$\phi_j(z) = \phi_{j,\alpha} + AirySolver(Diff(\phi_{j,\alpha})) + \sum_{n\ge 1}\Big[\psi_{j,n} + AirySolver\Big(Diff(\psi_{j,n})\Big)\Big]$$

in which $\psi_{j,n} = -RaySolver_\alpha(O_{j,n})$. By definition of the $RaySolver_\alpha(\cdot)$ operator, together with the assumption that $\partial_z\phi_{1,0}(1) = U'(1) = 0$, it follows directly that

$$\partial_z\phi_{j,\alpha}(1) = 0, \qquad \partial_z\psi_{j,n}(1) = 0,$$

for $j = 1, 2$ and $n \ge 1$. In addition, the term $AirySolver(Diff(\psi_{j,n})$ is of a higher order. It thus suffices to give estimates on the derivative of

$$A_{j,0}(z) = AirySolver(Diff(\phi_{j,\alpha}))(z).$$

Next, we recall from Lemma 3.7 that

$$\begin{aligned}\phi_{1,\alpha} &= (U-c)\Big[1 + \alpha^2\int_0^z \phi_{1,0}\phi_{2,0}\,dx - \alpha^2\frac{\phi_{2,0}}{U-c}\int_0^z (U-c)^2\,dx\Big]\\ &\quad + \alpha^2\phi_{2,0}\int_0^1 (U-c)^2\,dx + \mathcal{O}(\alpha^4)\phi_{2,0}\\ \phi_{2,\alpha} &= \phi_{2,0} + \mathcal{O}(\alpha^2),\end{aligned}$$

in which $\phi_{2,0}(z)$ contains a $(z - z_c)\log(z - z_c)$ singularity near the critical layer $z = z_c$. We note that the first bracket term in $\phi_{1,\alpha}$ is regular near $z = z_c$, and thus can be neglected when convoluted with $AirySolver(Diff(\cdot))$ due to the extra factor of ϵ in the $Diff(\cdot)$ operator. We are only concerned with the singular terms, which occur at order $\mathcal{O}(1)$ in $\phi_{2,\alpha}$, whereas at order $\mathcal{O}(\alpha^2)$ in $\phi_{1,\alpha}$. Precisely, we have the following expansions:

$$
\begin{aligned}
|\partial_z A_{1,0}(1)| &= \alpha^2(1 + \mathcal{O}(\alpha^2))K_0 \int_0^1 (U - c)^2 \, dx \\
|\partial_z A_{2,0}(1)| &= K_0(1 + \mathcal{O}(\alpha^2)),
\end{aligned}
$$

in which $K_0 := \partial_z AirySolver(Diff(\phi_{2,0}))_{|z=1}$. Putting these together proves the lemma. $\square$

7 Construction of the fast Orr modes

In this section we provide a similar construction to that obtained in Proposition 6.1 for the slow Orr modes. The construction will begin with the Airy solutions. Precisely, let us introduce

$$\phi_{3,0}(z) := \gamma_3 Ai(2, \delta^{-1}\eta(z)), \qquad \phi_{4,0}(z) := \gamma_4 Ci(2, \delta^{-1}\eta(z)), \tag{7.1}$$

in which $\gamma_3 = 1/Ai(2, \delta^{-1}\eta(0))$ and $\gamma_4 = 1/Ci(2, \delta^{-1}\eta(1))$ are normalizing constants. Here, $Ai(2, \cdot)$ and $Ci(2, \cdot)$ are the second primitive of the Airy solutions $Ai(\cdot)$ and $Ci(\cdot)$, respectively, and $\eta(z)$ denotes the Langer's variable

$$\delta = \Big(\frac{\varepsilon}{iU_c'}\Big)^{1/3}, \qquad \eta(z) = \Big[\frac{3}{2}\int_{z_c}^{z}\Big(\frac{U-c}{U_c'}\Big)^{1/2}\,dz\Big]^{2/3}. \tag{7.2}$$

We recall that as $Z = \eta(z)/\delta$ which tends to the infinity along the line $e^{i\pi/6}\mathbb{R}$, the Airy solution $Ai(2, e^{i\pi/6}Z)$ behaves as $e^{\mp\frac{\sqrt{2}}{3}|Z|^{3/2}}$, whereas $Ci(2, e^{i\pi/6}Z)$ is of order $e^{\pm\frac{\sqrt{2}}{3}|Z|^{3/2}}$. In particular, since $\eta(0) \sim -z_c$, $\delta^{-1}\eta(0) \to \infty$ with an angle approximately $\frac{7}{6}\pi$. Hence, the normalizing constant γ_3 is approximately of order $\langle z_c/\delta\rangle^{5/4}e^{-\frac{\sqrt{2}}{3}|z_c/\delta|^{3/2}}$. Later on in the study of the dispersion relation, we shall consider the case when $|z_c/\delta|$ is sufficiently large. Since $A(2, \delta^{-1}\eta(z))$ decays exponentially fast as $z \gg \text{Re } z_c$, the function $\phi_{3,0}(z)$ is uniformly bounded. Let us also recall that the critical layer is centered at $z = z_c$ and has a typical size of $|\delta| \sim \epsilon^{1/3}$. Inside the critical layer, the Airy functions play a crucial role.

Proposition 7.1. *For sufficiently small α, ϵ, c so that*

$$|\delta \log \delta|\langle z_c/\delta\rangle^{3/2} \ll 1, \tag{7.3}$$

with z_c being the critical layer $U(z_c) = c$ and $|\delta| \sim \epsilon^{1/3}$ being the critical layer size, defined as in (4.30)*, there are exact solutions $\phi_j(z)$, $j = 3, 4$, solving the Orr-Sommerfeld equation*

$$Orr(\phi_j) = 0, \qquad j = 3, 4.$$

In addition, we can construct $\phi_j(z)$ so that $\phi_j(z)$ is approximately close to $\phi_{j,0}(z)$ in the sense that

$$|\partial_z^k(\phi_j(z) - \phi_{j,0}(z))| \le C|\delta|^{1-k}, \qquad j = 3, 4, \qquad k = 0, 1, \tag{7.4}$$

for some fixed constants η, C.

From the construction, we also obtain the following lemma.

Lemma 7.2. *The fast Orr mode $\phi_{3,4}(z)$ constructed in Proposition 7.1 depends analytically in c with $Im\ c \neq 0$.*

Proof. This is simply due to the fact that both Airy function and the Langer transformation (7.2) are analytic in their arguments. $\square$

7.1 Iterative construction

Let us prove Proposition 7.1 in this section.

Proof of Proposition 7.1. We start with $\phi_{3,0}(z) = \gamma_3 Ai(2, \delta^{-1}\eta(z))$. By the estimates on the Airy functions in Lemma 4.1, when the z-derivative hits the Airy functions, it gives an extra $\delta^{-1}\langle Z\rangle^{1/2}$, and hence, $\phi_{3,0}(z)$ satisfies

$$|\partial_z^k \phi_{3,0}(z)| \le C_0\gamma_3|\delta|^{-k}\langle Z\rangle^{-5/4+k/2}e^{-\sqrt{2|Z|}Z/3}, \qquad k \ge 0,$$

uniformly for all $z \in [0,1]$. We note that thanks to the normalizing constant γ_3, the above estimate in particular yields that

$$|\partial_z^k \phi_{3,0}(0)| \le C_0|\delta|^{-k}\langle z_c/\delta\rangle^{k/2}, \qquad k \ge 0,$$

which could be large in the limit $\epsilon, z_c \to 0$. When z is away from zero, the exponent $e^{-\sqrt{2|Z|}Z/3}$ is sufficiently small, of order $e^{-1/|\delta|^{3/2}}$ in the limit $\delta \to 0$. This controls any polynomial growth in $1/\delta$. Next, direct calculations yield

$$\begin{aligned}
Airy(\phi_{3,0}) := {}& \varepsilon\delta^{-1}\gamma_3\eta^{(4)}Ai(1,Z) + 4\varepsilon\delta^{-2}\gamma_3\eta'\eta^{(3)}Ai(Z) + 3\varepsilon\delta^{-2}\gamma_3(\eta'')^2 Ai(Z)\\
&+ \varepsilon\delta^{-4}\gamma_3(\eta')^4 Ai''(Z) + 6\varepsilon\delta^{-3}\gamma_3\eta''(\eta')^2 Ai'(Z)\\
&- \gamma_3(U-c)\Big[\eta''\delta^{-1}Ai(1,Z) + \delta^{-2}(\eta')^2 Ai(Z)\Big],
\end{aligned}$$

with $Z = \delta^{-1}\eta(z)$. Let us first look at the leading terms with a factor of $\varepsilon\delta^{-4}$ and of $(U-c)\delta^{-2}$. Using the facts that $\eta' = 1/\dot z$, $\delta^3 = \varepsilon/U_c'$, and $(U-c)\dot z^2 = U_c'\eta(z)$, we have

$$\begin{aligned}
\varepsilon\delta^{-4}&(\eta')^4 Ai''(Z) - \delta^{-2}(\eta')^2(U-c)Ai(Z)\\
&= \varepsilon\delta^{-4}(\eta')^4\Big[Ai''(Z) - \delta^2\varepsilon^{-2}(U-c)\dot z^2 Ai(Z)\Big]\\
&= \varepsilon\delta^{-4}(\eta')^4\Big[Ai''(Z) - ZAi(Z)\Big] = 0.
\end{aligned}$$

The next terms in $Airy(\phi_{3,0})$ are

$$\begin{aligned}
6\varepsilon\delta^{-3}&\gamma_3\eta''(\eta')^2 Ai'(Z) - \gamma_3(U-c)\eta''\delta^{-1}Ai(1,Z)\\
&= \gamma_3\Big[6\eta''(\eta')^2 U_c' Ai'(Z) - ZU_c'\eta''(\eta'^2)Ai(1,Z)\Big]\\
&= \gamma_3\eta''(\eta')^2 U_c'\Big[6Ai'(Z) - ZAi(1,Z)\Big],
\end{aligned}$$

which is of order $\mathcal{O}(1)$. The rest is of order $\mathcal{O}(\varepsilon^{1/3})$ or smaller. Precisely, we obtain

$$\begin{aligned}
Airy(\phi_{3,0}) = {}& \gamma_3\eta''(\eta')^2 U_c'\Big[6Ai'(Z) - ZAi(1,Z)\Big]\\
&+ \varepsilon\delta^{-2}\gamma_3\Big[\delta\eta^{(4)}Ai(1,Z) + (4\eta'\eta^{(3)} + 3(\eta'')^2)Ai(Z)\Big],
\end{aligned}$$

with noting that the last term is of order $\varepsilon^{1/3}$, since $\delta \sim \varepsilon^{1/3}$. This shows that $Airy(\phi_{3,0})(z)$ is very localized and depends primarily on the fast variable Z as does $Ai(\cdot)$. Furthermore, from the estimates on $Ai(\cdot)$, we have

$$|Airy(\phi_{3,0})(z)| \le C\gamma_3 \langle Z\rangle^{1/4} e^{-\sqrt{2|Z|}Z/3}$$

for some constant C, uniformly in $z \in [0,1]$. By the identity (2.6), it follows that

$$Orr(\phi_{3,0}) = I_0(z) := Reg(\phi_{3,0}) + Airy(\phi_{3,0}),$$

in which $Reg(\phi) := (i\varepsilon\alpha^4 - U'' - \alpha^2(U-c))\phi$. Hence, the bound on $\phi_{3,0}$ and on $Airy(\phi_{3,0})$ immediately yields

$$|\partial_z^k I_0(z)| \le C_0\gamma_3|\delta|^{-k}\langle Z\rangle^{1/4+k/2} e^{-\sqrt{2|Z|}Z/3} \tag{7.5}$$

uniformly in $z \in [0,1]$. Again as $z \to 0$, we obtain the following bound, using the normalizing constant γ_3,

$$|\partial_z^k I_0(0)| \le C_0|\delta|^{-k}\langle z_c/\delta\rangle^{3/2+k/2}.$$

To obtain a better error estimate, let us introduce $\phi_{3,1}(z) := \phi_{3,0}(z) - AirySolver_\infty(I_0)(z)$. We then get

$$Orr(\phi_{3,1}) = I_1(z) := -Reg(AirySolver_\infty(I_0))(z),$$

in which by a view of Lemma 4.10 and Proposition 4.12, $I_1(z)$ is of order $\mathcal{O}(\delta)$ smaller than that of $I_0(z)$. Precisely, we have

$$|\partial_z^k I_1(z)| \le C|\delta|^{1-k}\gamma_3\langle Z\rangle^{-7/4+k/2} e^{-\sqrt{2|Z|}Z/3} + C_k|\delta|^{1-k}, \tag{7.6}$$

for $k \ge 0$, in which C_k vanishes for $k \ge 2$. Indeed, in the above estimate, the terms on the right are due to the convolution with the localized and non-localized part of the Green function of the $Airy$ operator, respectively. Hence, when z-derivative hits the non-localized part $E(x,z)$, we have by definition, $\partial_z^k E(x,z) = 0$ for $k \ge 1$ and $x < z$, and $|\partial_z E(x,z)| \le C_0|\delta|^{-1}(1+|X|)^{-1}$ and $\partial_z^k E(x,z) = 0$ for $x > z$ and for $k \ge 2$. In particular, there is no linear growth in Z in the last term on the right of (7.6). In addition, we also note that as $z \to 0$, there holds the uniform estimate:

$$|\partial_z^k I_1(0)| \le C|\delta|^{1-k}\langle z_c/\delta\rangle^{-1/2+k/2} + C|\delta|^{1-k}.$$

In particular, $I_1 = \mathcal{O}(\delta)$ and $\partial_z I_1 = \mathcal{O}(1)$. We then inductively introduce

$$I_{n+1} := Iter(I_n)$$

where $Iter(\cdot)$ is defined as in (6.5). Proposition 6.1 ensures the convergence of the series

$$\phi_{3,N}(z) = \phi_{3,0}(z) - AirySolver_\infty(I_0)(z) + \sum_{n=1}^{N}\Big[\psi_n + AirySolver_\infty\Big(Diff(\psi_n)\Big)\Big] \tag{7.7}$$

in which $\psi_n := -RaySolver_\alpha(I_n)$. The limit of $\phi_{3,N}$ as $N \to \infty$ yields the third Orr modes as claimed.

A similar construction applies for $\phi_{4,0} = \gamma_4 Ci(2, \delta^{-1}\eta(z))$, since both $Ai(2,\cdot)$ and $Ci(2,\cdot)$ solve the same primitive Airy equation.

This proves the proposition. □

7.2 First order expansion of ϕ_3 at $z = 0$

By the construction in Proposition 7.1, we obtain the following first order expansion of ϕ_3 at the boundary $z = 0$, directly from (7.4):

$$\phi_3(0) = \phi_{3,0}(0) + \mathcal{O}(\delta), \qquad \partial_z\phi_3(0) = \partial_z\phi_{3,0}(0) + \mathcal{O}(1).$$

By definition, we have $\phi_{3,0}(0) = 1$ and

$$\partial_z\phi_{3,0}(0) = \delta^{-1}\frac{Ai(1, \delta^{-1}\eta(0))}{Ai(2, \delta^{-1}\eta(0))}.$$

In the study of the linear dispersion relation, we are interested in the ratio $\partial_z\phi_3/\phi_3$. Again, for convenience, let us introduce

$$K_3 := \frac{\phi_3(0)}{\partial_z\phi_3(0)}, \qquad \text{and} \qquad C_{Ai}(Y) := \frac{Ai(2, Y)}{Ai(1, Y)}. \tag{7.8}$$

From the above estimates on $\phi_3(0)$ and $\partial_z\phi_3(0)$, it follows at once that

$$K_3 = \frac{\delta C_{Ai}(\delta^{-1}\eta(0))}{1 + \mathcal{O}(\delta)C_{Ai}(\delta^{-1}\eta(0))}(1 + \mathcal{O}(\delta)).$$

As will be calculated below, $\delta C_{Ai}(\delta^{-1}\eta(0)) \sim \delta\langle\eta(0)/\delta\rangle^{-1/2} \ll 1$. Hence, K_3 is estimated by

$$K_3 = \delta C_{Ai}(\delta^{-1}\eta(0))(1 + \mathcal{O}(\delta)C_{Ai}(\delta^{-1}\eta(0)))(1 + \mathcal{O}(\delta)). \tag{7.9}$$

The following lemma is crucial later on to determine instability.

Lemma 7.3. *Let ϕ_3 be the Orr-Sommerfeld solution constructed in Proposition 7.1, and let K_3 be defined as in* (7.8)*. There holds*

$$K_3 = -e^{\pi i/4}|\delta||z_c/\delta|^{-1/2}(1 + \mathcal{O}(|z_c/\delta|^{-3/2})) \tag{7.10}$$

as long as z_c/δ is sufficiently large. In particular, the imaginary part of $\phi_3/\partial_z\phi_3$ becomes negative when z_c/δ is large. In addition, when $z_c/\delta = 0$,

$$K_3 = 3^{1/3}\Gamma(4/3)|\delta|e^{5i\pi/6}, \tag{7.11}$$

for $\Gamma(\cdot)$ the usual Gamma function.

Here, we recall that $\delta = e^{-i\pi/6}(\alpha R U_c')^{-1/3}$, and from the estimate (4.26), $\eta(0) = -z_c + \mathcal{O}(z_c^2)$. Therefore, we are interested in the ratio $C_{Ai}(Y)$ for complex $Y = -e^{i\pi/6}y$, for y being in a small neighborhood of $\mathbb{R}^+$. Without loss of generality, in what follows, we consider $y \in \mathbb{R}^+$. Lemma 7.3 follows directly from the following lemma.

Lemma 7.4. *Let $C_{Ai}(\cdot)$ be defined as above. Then, $C_{Ai}(\cdot)$ is uniformly bounded on the ray $Y = e^{7i\pi/6}y$ for $y \in \mathbb{R}^+$. In addition, there holds*

$$C_{Ai}(-e^{i\pi/6}y) = -e^{5i\pi/12}y^{-1/2}(1 + \mathcal{O}(y^{-3/2}))$$

for all large $y \in \mathbb{R}^+$. At $y = 0$, we have

$$C_{Ai}(0) = -3^{1/3}\Gamma(4/3).$$

Proof. Thus, using asymptotic behavior of Ai, yields

$$C_{Ai}(Y) = -Y^{-1/2}(1 + \mathcal{O}(|Y|^{-3/2}))$$

for large Y. This proves the estimate for large y. The value at $y = 0$ is easily obtained from those of $Ai(k, 0)$. This completes the proof of the lemma. □

7.3 First order expansion of $\phi_{3,4}$ at $z = 1$

As will be clear in the next section, we shall need to estimate the values of derivatives of $\phi_{3,4}$ at $z = 1$ as well as the ratio

$$K_4 := \frac{\phi_4'(1)}{\phi_4'''(1)}. \tag{7.12}$$

Lemma 7.5. *Let $\phi_{3,4}$ be the Orr-Sommerfeld solution constructed in Proposition 7.1. There hold*

$$\begin{aligned} \partial_z^k \phi_3(1) &\lesssim |\delta^2 \log \delta| \langle z_c/\delta \rangle^{3/2} \\ \partial_z^k \phi_4(1) &\gtrsim \mathcal{O}(e^{1/\sqrt{|\epsilon|}}) \end{aligned} \tag{7.13}$$

in L^∞, for $k = 1$ and $k = 3$.

Proof. We recall that the construction in Proposition 7.1 gives

$$\phi_j(z) = \phi_{j,0}(z) - AirySolver_\infty(I_0)(z) + \sum_{n\ge 1} \Big[\psi_n + AirySolver\Big(Diff(\psi_n)\Big)\Big] \tag{7.14}$$

with $\psi_n := -RaySolver_\alpha(I_n)$. Let us give estimates for ϕ_3. We recall that $\phi_{3,0}(z) = \gamma_3 Ai(2, \delta^{-1}\eta(z))$, and $I_0(z)$ satisfies (7.5). Thanks to (4.4), the claimed estimate for $\phi_{3,0}(1)$ and its derivatives follows easily, upon noting that $\eta(1) \approx 1$ and $\delta \approx \epsilon^{1/3}$. Next, let $k \ge 1$. By definition (see (4.52) and (4.34)), we have

$$\partial_z^k AirySolver(I_0)(1) = \int_0^1 \partial_z^k \widetilde{G}(x, 1) I_0(x)\, dx.$$

Now, since $I_0(x)$ is very localized, a very similar calculation as done in Lemma 4.10 yields

$$|\partial_z^k AirySolver(I_0)(1)| \lesssim |\delta| e^{-1/\sqrt{\epsilon}}, \qquad \forall\ k \ge 1.$$

As for the next term ψ_n in (7.7), we observe that $\partial_z^k \psi_n(1) = -\partial_z^k RaySolver_\alpha(I_n)(1) = 0$, by definition of the $RaySolver_\alpha$ operator. In addition, we recall that $I_1 = \mathcal{O}(\delta)$ and so $\psi_n = \mathcal{O}(\delta)$, for $n \geq 1$. Finally, as in the above estimate, we obtain

$$\partial_z^k AirySolver\Big(Diff(\psi_n)\Big)(1) = \int_0^1 \partial_z^k \widetilde{G}(x,1) Diff(\psi_n)\, dx, \qquad k \geq 1.$$

Lemma 5.2 then yields

$$|\partial_z^k AirySolver\Big(Diff(\psi_n)\Big)(1)| \lesssim |\delta^2 \log \delta| \langle z_c/\delta \rangle^{3/2}$$

for $k \geq 1$ and $n \geq 1$. This proves the claimed estimate for ϕ_3 at the boundary $z = 1$.

Similarly, as for $\partial_z^k \phi_4(1)$, we have started the expansion with $\phi_{4,0}(z) = \gamma_4 Ci(2, \delta^{-1}\eta(z))$, which is of order $e^{1/\sqrt{\epsilon}}$ at the boundary $z = 1$. This yields the claimed lower bound for the derivatives of ϕ_4 at the boundary. □

Lemma 7.6. *Let ϕ_4 be the Orr-Sommerfeld solution constructed in Proposition 7.1, and let K_4 be defined as in* (7.12)*. There holds*

$$K_4 = \mathcal{O}(\delta^2). \tag{7.15}$$

Proof. Indeed, up to an error of order $\mathcal{O}(\delta^2)$, the fast mode $\phi_4(z)$ primarily depends on the fast variable $Z = \eta(z)/\delta$. This shows that taking the z-derivative yields a large factor of order $1/\delta$, and the estimate for the ratio thus follows. □

8 Study of the dispersion relation

8.1 Linear dispersion relation

A solution of (1.7)–(1.9) is a linear combination of the slow solutions $\phi_{1,2}$ that link with the Rayleigh solutions and the localized solutions $\phi_{3,4}$ that link with the Airy functions. Let us then introduce an exact Orr-Sommerfeld solution of the form

$$\phi := A_1\phi_1 + A_2\phi_2 + A_3\phi_3 + A_4\phi_4, \tag{8.1}$$

for some parameters $A_j = A_j(\epsilon, c)$, where $\phi_{1,2} = \phi_{1,2}(z;\varepsilon,c)$ and $\phi_{3,4} = \phi_{3,4}(z;\varepsilon,c)$ are constructed in Propositions 6.1 and 7.1, respectively. It is clear that $\phi(z)$ is an exact solution to the Orr-Sommerfeld equation. The boundary conditions (1.8)-(1.9) at $z = 0, 1$ yield that the determinant

$$W_0(\epsilon,c) := \det \begin{pmatrix} \phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) \\ \phi_1'(0) & \phi_2'(0) & \phi_3'(0) & \phi_4'(0) \\ \phi_1'(1) & \phi_2'(1) & \phi_3'(1) & \phi_4'(1) \\ \phi_1'''(1) & \phi_2'''(1) & \phi_3'''(1) & \phi_4'''(1) \end{pmatrix} = 0. \tag{8.2}$$

This identity represents an eigenvalue dispersion relation, from which we shall obtain the existence of unstable eigenvalue c with $\mathrm{Im}\, c > 0$ for a certain range of parameter $\alpha = \alpha(\epsilon)$.

We first relate this dispersion relation to those ratios K_j, $j = 1, ..., 4$, defined previously in (6.11), (6.17), (7.8), and (7.12), respectively. Indeed, by dividing the last column in the above matrix by $\phi_4'''(1)$ and recalling from Lemma 7.5 that at $z = 1$, the derivatives of $\phi_4(z)$ are of order $e^{1/\sqrt{\epsilon}}$, the last column in the determinant can be replaced by

$$\begin{pmatrix} 0 \\ 0 \\ K_4 \\ 1 \end{pmatrix} + \mathcal{O}(e^{-1/\sqrt{\epsilon}}), \qquad \text{with} \quad K_4 = \frac{\phi_4'(1)}{\phi_4'''(1)}.$$

Similarly, derivatives of $\phi_3(z)$ are of order $e^{-1/\sqrt{\epsilon}}$ at $z = 1$. Thus, again Lemma 7.5 shows that the third column in the above determinant can be replaced by

$$\begin{pmatrix} K_3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathcal{O}(\delta^2 \log\delta \langle z_c/\delta\rangle^{3/2}), \qquad \text{with} \quad K_3 = \frac{\phi_3(0)}{\phi_3'(0)}.$$

In addition, by a view of Lemma 7.6, we have $K_4 = \mathcal{O}(\delta^2)$. This proves that the relation $W_0(\epsilon, c) = 0$ reduces to

$$W_1(\epsilon,c) := \det \begin{pmatrix} \phi_1(0) & \phi_2(0) & K_3 \\ \phi_1'(0) & \phi_2'(0) & 1 \\ \phi_1'(1) & \phi_2'(1) & 0 \end{pmatrix} = \mathcal{O}(\delta^2 \log\delta \langle z_c/\delta\rangle^{3/2}).$$

Define $K_1 = \phi_1(0)/\phi_1'(0)$ and $K_2 = \phi_1'(1)/\phi_2'(1)$ as in (6.11) and (6.17), respectively. The above leads to a new dispersion relation:

$$K_3 = \frac{\phi_1(0) - K_2\phi_2(0)}{\phi_1'(0) - K_2\phi_2'(0)} + \mathcal{O}(\delta^2 \log \delta \langle z_c/\delta\rangle^{3/2})$$
$$= \frac{K_1 - K_2\frac{\phi_2(0)}{\phi_1'(0)}}{1 - K_2\frac{\phi_2'(0)}{\phi_1'(0)}} + \mathcal{O}(\delta^2 \log \delta \langle z_c/\delta\rangle^{3/2})$$

Now, recall from Lemmas 6.5 and 6.7 that $K_1 = \mathcal{O}(\alpha^2 + |z_c| + |\delta|)$ and $K_2 = \mathcal{O}(\alpha^2)$. In addition, we have $\phi_1'(0) \approx U_0' \neq 0$, $\phi_{2,0}(0) \approx 1/U_0'$, and $\phi_2'(0) \approx \log z_c$. The above dispersion relation is further reduced to

$$K_3 = K_1 - \frac{K_2}{|U_0'|^2} + \mathcal{O}(\alpha^2 \log z_c)(\alpha^2 + |z_c| + |\delta|) + \mathcal{O}(\delta^2 \log \delta \langle z_c/\delta\rangle^{3/2}). \tag{8.3}$$

Up to now, the three parameters α, ϵ, c are taken sufficiently small, without a priori knowledge of their relative size. We next show the existence of $c = c(\alpha, \epsilon)$ for small α, ϵ, satisfying the dispersion relation (8.3).

Lemma 8.1. *For small α, ϵ, there is a unique $c = c(\alpha, \epsilon)$ near zero so that the linear dispersion* (8.3) *(and thus* (8.2)*) holds. In addition, as long as $\delta \log \delta \langle z_c/\delta\rangle^{3/2}$ remains bounded, there holds*

$$|c(\alpha, \epsilon)| \le C(\alpha^2 + |\delta|), \tag{8.4}$$

in the limit $\alpha, \epsilon \to 0$. In particular, $|z_c| \le C(\alpha^2 + |\delta|)$.

Proof. The proof is straightforward. Indeed, we set

$$F(c, \alpha, \epsilon) := K_3 - K_1 - \frac{K_2}{|U_0'|^2} + \mathcal{O}(\alpha^2 \log z_c)(\alpha^2 + |z_c| + |\delta|) + \mathcal{O}(\delta^2 \log \delta \langle z_c/\delta\rangle^{3/2})$$

so that $F(c, \alpha, \epsilon) = 0$ is equivalently to the dispersion relation (8.3). It follows that $F(c, \alpha, \epsilon)$ is analytic in c, with $\Im c > 0$, and $F(0, 0, 0) = 0$. Next, we recall from Lemmas 6.5, 6.7, and 7.3 that $K_1 = -c/U_0' + \mathcal{O}(\alpha^2 + |\delta|)$, $K_2 = \mathcal{O}(\alpha^2)$, and $K_3 = \mathcal{O}(\delta\langle z_c/\delta|\rangle^{-1/2})$, respectively. Thus, there holds

$$\partial_c F(0, 0, 0) = 1.$$

The existence of $c = c(\alpha, \epsilon)$ so that $F(c(\alpha, \epsilon), \alpha, \epsilon) = 0$ follows directly from the Implicit Function Theorem.

Finally, since $K_1 = -c/U_0' + \mathcal{O}(\alpha^2 + |\delta|)$, the dispersion relation (8.3) reads

$$c = \mathcal{O}(\alpha^2 + |\delta|) + \mathcal{O}(\alpha^2 \log z_c)(\alpha^2 + |z_c|) + \mathcal{O}(\delta^2 \log \delta \langle z_c/\delta\rangle^{3/2}).$$

Using the boundedness assumption of $\delta \log \delta \langle z_c/\delta\rangle^{3/2}$ and the fact that $z_c \sim c$, the claimed estimate on $c(\alpha, \epsilon)$ follows at once. $\square$

8.2 Ranges of α

By Lemma 8.1, $(c(\alpha,\varepsilon), z_c(\alpha,\varepsilon)) \to 0$ as $(\alpha,\varepsilon)\to 0$. In addition, as suggested by physical results (see, e.g., [1, 15]), and as will be proved below, for instability, we would search for α between $(\alpha_{\text{low}}(R), \alpha_{\text{up}}(R))$ with

$$\alpha_{\text{low}}(R) \approx R^{-1/7}, \qquad \alpha_{\text{up}}(R) \approx R^{-1/11},$$

for sufficiently large R. These values of $\alpha_j(R)$ form lower and upper branches of the marginal (in)stability curve for the shear profile U. More precisely, we will show that there is a critical constant A_{1c} so that with $\alpha_{\text{low}}(R) = A_1 R^{-1/7}$, the imaginary part of c turns from negative (stability) to positive (instability) when the parameter A_1 increases across $A_1 = A_{1c}$. Similarly, there exists an A_{2c} so that with $\alpha = A_2 R^{-1/11}$, Im c turns from positive to negative as A_2 increases across $A_2 = A_{2c}$. In particular, we obtain instability in the intermediate zone: $\alpha \approx R^{-\beta}$ for $1/11 < \beta < 1/7$.

We note that the ranges of α restrict the absolute value of $\delta = (\varepsilon/iU_c')^{1/3}$, with $\epsilon = \frac{1}{\alpha R}$, to lie between δ_2 and δ_1, with $\delta_1 \approx \alpha^2$ and $\delta_2 \approx \alpha^{10/3}$, respectively. In particular, $|\delta| \lesssim \alpha^2$. Therefore, in the case $\alpha \approx \alpha_{\text{low}}(R)$, the critical layer is accumulated on the boundary, and thus the fast-decaying mode in the critical layer plays a role of a boundary sublayer; in this case, the mentioned Langer transformation plays a crucial role. In the latter case when $\alpha \approx \alpha_{\text{up}}(R)$, the critical layer is well-separated from the boundary at $z = 0$.

In what follows, we shall restrict the three small parameters to satisfy the following bounds:

$$\alpha^{10/3} \lesssim |\delta| \lesssim \alpha^2, \qquad |z_c| \lesssim \alpha^2. \tag{8.5}$$

We note in particular that within the above choices, there hold

$$\delta \log\delta \langle z_c/\delta\rangle^{3/2} \lesssim \alpha^{4/3} \ll 1.$$

That is, the smallness assumption made in the construction of the Orr-Sommerfeld solutions in Propositions 6.1 and 7.1, and the boundedness assumption made in Lemma 8.1 are satisfied. In particular, the dispersion relation (8.3) can now be simplified to

$$K_3 = K_1 - \frac{K_2}{|U_0'|^2} + \mathcal{O}(\alpha^4 \log\alpha). \tag{8.6}$$

with $K_1 = -c + \mathcal{O}(\alpha^2)$, $K_2 = \mathcal{O}(\alpha^2)$, and $K_3 \approx \delta\langle z_c/\delta|\rangle^{-1/2}$.

In the next subsections, we shall prove the following proposition, confirming the physical results.

Proposition 8.2. *For R sufficiently large, we show that $\alpha_{\text{low}}(R) = A_{1c}R^{-1/7}$ and $\alpha_{\text{up}}(R) = A_{2c}R^{-1/11}$, for some critical constants A_{1c}, A_{2c}, are indeed the lower and upper marginal branch for stability, respectively. In all cases of instability: $\alpha = AR^{-\beta} \in (\alpha_{\text{low}}(R), \alpha_{\text{up}}(R))$, there holds*

$$Im\ c \quad \approx \quad A^{-3/2} R^{(3\beta-1)/2}, \tag{8.7}$$

and in particular, we obtain the growth rate

$$\alpha Im\ c \quad \approx \quad R^{-(1-\beta)/2}, \tag{8.8}$$

with $\beta \in [\frac{1}{11}, \frac{1}{7}]$.

8.3 Lower stability branch: $\alpha \approx R^{-1/7}$

Let us consider the case $\alpha = AR^{-1/7}$, for some constant A. We recall that $\delta \approx (\alpha R)^{-1/3} = A^{-1/3}R^{-2/7}$. That is, $\delta \approx \alpha^2$ for the fixed constant A. By a view of (8.4), we then have $|z_c| \approx C|\delta|$. More precisely, we have

$$z_c/\delta \quad \approx \quad A^{4/3}. \tag{8.9}$$

Thus, we are in the case that the critical layer goes up to the boundary with z_c/δ staying bounded in the limit $\alpha, \epsilon \to 0$.

We prove in this section the following lemma.

Lemma 8.3. *Let* $\alpha = AR^{-1/7}$. *For* R *sufficiently large, there exists a critical constant* A_c *so that the eigenvalue* $c = c(\alpha, \epsilon)$ *has its imaginary part changing from negative (stability) to positive (instability) as* A *increases past* $A = A_c$. *In particular,*

$$Im\ c \quad \approx \quad A^{-1/3}R^{-2/7}.$$

Proof. By taking the imaginary part of the dispersion relation (8.6) and using the bounds from Lemmas 6.5 and 7.3, we obtain

$$(-1 + \mathcal{O}(\alpha^2))\text{Im } c + \mathcal{O}(\alpha^4 \log \alpha) = \text{Im } \Big(\frac{\phi_3(0)}{\partial_z \phi_3(0)}\Big) = \mathcal{O}(\delta \langle z_c/\delta \rangle^{-1/2}). \tag{8.10}$$

which clearly yields Im $c = \mathcal{O}(\delta\langle z_c/\delta\rangle^{-1/2})$ and so Im $c \approx A^{-1/3}R^{-2/7}$. Next, also from Lemma 7.3, the right-hand side is positive when z_c/δ is small, and becomes negative when $z_c/\delta \to \infty$. Consequently, together with (8.9), there must be a critical number A_c so that for all $A > A_c$, the right-hand side is positive, yielding the lemma as claimed. □

8.4 Intermediate zone: $R^{-1/7} \ll \alpha \ll R^{-1/11}$

Let us now turn to the intermediate case when

$$\alpha = AR^{-\beta}$$

with $1/11 < \beta < 1/7$. In this case $\delta \approx \alpha^{-1/3}R^{-1/3} \approx A^{-1/3}R^{\beta/3-1/3}$ and hence $\delta \ll \alpha^2$. That is, the critical layer is away from the boundary: $\delta \ll z_c$ by a view of (8.5). We prove the following lemma.

Lemma 8.4. *Let $\alpha = AR^{-\beta}$ with $1/11 < \beta < 1/7$. For arbitrary fixed positive A, the eigenvalue $c = c(\alpha, \epsilon)$ always has positive imaginary part (instability) with*

$$Im\, c \quad \approx \quad A^{-3/2} R^{(3\beta-1)/2}.$$

Proof. As mentioned above, z_c/δ is unbounded in this case. Since $z_c \approx \alpha^2$ and $\delta \ll \alpha^2$, we indeed have

$$z_c/\delta \quad \approx \quad A^{7/3} R^{(1-7\beta)/3} \to \infty,$$

as $R \to \infty$ since $\beta < 1/7$. By Lemma 7.3, we then have

$$\text{Im } (K_3) = \mathcal{O}(\delta \langle z_c/\delta \rangle^{-1/2}) \approx A^{-3/2} R^{(3\beta-1)/2}, \tag{8.11}$$

and furthermore the imaginary of K_3 is positive since $z_c/\delta \to \infty$. It is crucial to note that in this case

$$\alpha^4 \log \alpha \approx R^{-4\beta} \log R,$$

which remains neglected in the dispersion relation (8.10) as compared to the size of the imaginary part of K_3, since $\beta > 1/11$.

This yields the lemma at once. □

8.5 Upper branch instability: $\alpha \approx R^{-1/11}$

Finally, let us study the upper branch case: $\alpha = AR^{-1/11}$. In this case, the term of order $\alpha^4 \log \alpha$ is no longer neglected as compared to K_3 in the dispersion relation (8.10). Precisely, we have

$$K_3 \approx A^{-3/2} R^{-4/11}, \qquad \alpha^4 \log \alpha \approx A^4 R^{-4/11} \log R.$$

By a view of the linear dispersion relation just above the equation (8.3), the new dispersion relation now reads

$$\frac{\phi_1(0) - K_2 \phi_2(0)}{\phi_1'(0) - K_2 \phi_2'(0)} = K_3 + \mathcal{O}(\epsilon) = \mathcal{O}(A^{-11/2} \alpha^4).$$

The left-hand side of (8.3) consists precisely of the Rayleigh modes $\phi_{1,2}$, whereas the right-hand side can be neglected as compared to the $\alpha^4 \log \alpha$ terms. Since we have started with the case of the stable Rayleigh profiles, the corresponding eigenvalue is in the stable half-plane as $R \to \infty$. This shows that there must be a critical value A_{2c} from instability (due to the previous case: $\alpha \ll R^{-1/11}$, or equivalently, $A \ll 1$) to stability, when A increases past A_{2c}.

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