Stability of shear flows near a boundary

Emmanuel Grenier[1] Toan T. Nguyen[2]

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*Equipe Projet Inria NUMED, INRIA Rhône Alpes, Unité de Mathématiques Pures et Appliquées., UMR 5669, CNRS et École Normale Supérieure de Lyon, 46, allée d'Italie, 69364 Lyon Cedex 07, France. Email: egrenier@umpa.ens-lyon.fr

†Department of Mathematics, Penn State University, State College, PA 16803. Email: nguyen@math.psu.edu. TN is partly supported by the NSF under grant DMS-1764119 and an AMS Centennial fellowship.

Contents

Chapter 1

Introduction

1.1 Introduction

This book is devoted to the study of the linear and nonlinear stability of shear flows for Navier Stokes equations for incompressible fluids with Dirichlet boundary conditions in the case of small viscosity.

More precisely, we shall consider the classical Navier-Stokes equations for an incompressible fluid in a spatial domain $\Omega \subset \mathbb{R}^d$, with $d \geq 2$,

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

together with the initial condition

$$u(0, \cdot) = u_0, \tag{1.3}$$

and the classical no-slip boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.4}$$

In these equations, $u(t,x)$ is the velocity of the fluid, $p(t,x)$ the pressure, $\nu > 0$ the viscosity and u_0 an initial velocity field. The Dirichlet boundary condition (1.4) expresses the fact that the fluid "sticks" to the boundary, and hence its velocity vanishes on $\partial\Omega$.

In this book we will focus on the study of the stability of particular solutions of Navier Stokes, known as "shear flows" solutions. These solutions are of the form

$$U_{shear}(t,x,y,z) = \begin{pmatrix} U_s(t,z) \\ 0 \end{pmatrix}, \tag{1.5}$$

where U_s satisfies the classical heat equation

$$\partial_t U_s - \nu \partial_z^2 U_s = 0. \tag{1.6}$$

Note that the solution of this equation is well defined and smooth for any positive time. The shear flow changes of time scales of order $O(\nu^{-1})$.

We will also consider the case of a time independent shear flow $U_{shear}(z)$, up to an additional forcing term in the right hand side of (1.1), of the form

$$F_{shear}(t, x, y, z) = \begin{pmatrix} -\nu \partial_z^2 U_s \\ 0 \end{pmatrix}. \tag{1.7}$$

Note that the forcing term is small, of order $O(\nu)$.

We are interested in the stability of these shear flows for small viscosity ν. As $\nu \to 0$, formally, the limit is the classical Euler equations for an incompressible ideal fluid, namely

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \tag{1.8}$$

$$\nabla \cdot u = 0, \tag{1.9}$$

together with the initial condition

$$u(0, \cdot) = u_0, \tag{1.10}$$

and the boundary condition

$$u \cdot n = 0 \quad \text{on} \quad \partial\Omega, \tag{1.11}$$

where $n(x)$ denotes the unit vector normal to the boundary $\partial\Omega$.

Let us now describe the general landscape in an informal way. Two cases arise.

- If the shear flow $U_{shear}(0, x, y, z)$ is spectrally unstable for Euler equation, then we can expect that it is also unstable for Navier Stokes equations provided ν is small enough. We will prove in this book that it is indeed the case: if ν is small enough, the shear flow is linearly and nonlinearly unstable for Navier Stokes equations.

- If the shear flow $U_{shear}(0, x, y, z)$ is spectrally stable for Euler equation, then we could expect that it would be also stable for Navier Stokes equations. This turns out not to be the case, which is somehow counter intuitive. We will prove in this book that such profiles are linearly unstable for Navier Stokes equations, and give preliminary nonlinear instability results.

1.2 Main results

Let us now state the two main results of the book, in an informal way.

Result 1

If the shear layer profile U_s is spectrally unstable for Euler equations, then it is also spectrally unstable for Navier Stokes equations provided the viscosity is small enough. It is also nonlinearly unstable in the following sense. For arbitrarily large N and s we can find a perturbation of order ν^N in Sobolev space H^s, such that at a later time T_ν of order $\log \nu^{-1}$, the perturbation reaches a size $O(1)$ in L^∞ and L^2.

Result 2

If the shear layer profile U_s is spectrally stable for Euler equations, then it is spectrally *unstable* for Navier Stokes equations provided the viscosity is small enough. The corresponding eigenvalue has a very small real part of order $O(\nu^{1/2})$. It is also "weakly" nonlinearly unstable in the following sense. For arbitrarily large N and s we can find a perturbation of order ν^N in Sobolev space H^s, such that at a later time T_ν of order $\nu^{-1/2} \log \nu^{-1}$, the perturbation reaches a size $O(\nu^{1/4})$ in L^∞ and L^2.

1.3 Physical introduction

The question of the linear and nonlinear instability of shear flows is one of the most classical questions of fluid mechanics. Its study goes back to Rayleigh at the end of the nineteenth century, and expanded through the beginning of the twentieth century, thanks to Orr, Sommerfeld, C.C. Lin, Tollmien, Schlichting,...

Physically, the inviscid limit is linked to the high Reynolds number limit. The Reynolds number is a non-dimensional number, defined by

$$Re = \frac{UL}{\nu} \tag{1.12}$$

where U is a typical velocity of the flow, L a typical length and ν the viscosity. Clearly, for each fixed U and L, $\nu \to 0$ if and only if $Re \to \infty$. Throughout out the book, small viscosity or high Reynolds number is used interchangeably.

In his seminal experiments in 1883, Reynolds first pointed out that flows at a high Reynolds number experience turbulence. In other words, well-organized flows can become chaotic under infinitesimal disturbances when the Reynolds number exceeds a critical number.

Let us give a simple example, and consider a time independent solution u_0 of Navier Stokes equations with a forcing term f_0. Let us prove that u_0 is stable provided the Reynolds number is small enough. For sake of simplicity, let us assume that Ω is a bounded domain. Let u be any time-dependent solution of Navier Stokes equations with the same forcing term f. Then, the perturbation $v = u - u_0$ satisfies

$$\partial_t v + (u_0 + v) \cdot \nabla v + v \cdot \nabla u_0 - \nu \Delta v + \nabla q = 0,$$

$$\nabla \cdot v = 0$$

with the zero boundary condition $v = 0$ on $\partial\Omega$. The energy induced by the perturbation satisfies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2\, dx + \nu \int_\Omega |\nabla v|^2\, dx = -\int_\Omega (v \cdot \nabla) u_0 \cdot v\, dx.$$

Clearly, the term on the right is responsible for any possible energy production and is bounded by $\|v\|_{L^2}^2 \|\nabla u_0\|_{L^\infty}$. The Poincaré inequality gives

$$\|v\|_{L^2}^2 \leq C(\Omega) \|\nabla v\|_{L^2}^2,$$

for some constant $C(\Omega)$ that depends on Ω. This yields

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \leq \Big(C(\Omega)\|\nabla u_0\|_{L^\infty} - \nu\Big)\|v\|_{L^2}^2.$$

Hence, if

$$\nu \geq C(\Omega)\|\nabla u_0\|_{L^\infty}$$

then

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq 0.$$

There is no energy production for any perturbation of the steady solution. This implies that the steady solution is nonlinearly stable. By the dimensional analysis, it follows that $C(\Omega)$ behaves like CL^2 and $\|\nabla u_0\|_{L^\infty}$ like $\|u_0\|_{L^\infty}/L$, where L is a typical length of Ω. Thus, the stationary solution u_0 is stable, provided that the ratio

$$Re = \frac{\|u_0\|_{L^\infty} L}{\nu}$$

is sufficiently small, or equivalently, the corresponding Reynolds number is small.

On the contrary, if the Reynolds number is large enough, the steady solution u_0 would become unstable. This second assertion is much more delicate to prove, and has been the subject of many studies since the 19^{th} century, from both a mathematical and a physical point of view. Many works focus on the particular case of a shear layer profile, with the fluid domain Ω being the half plane $\mathbb{R}^2_+$ or half space $\mathbb{R}^3_+$, and the flow of the form

$$u_0 = \begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix}, \qquad z \geq 0,$$

where $U(0) = 0$ and $\lim_{z\to+\infty} U(z)$ exists and is finite. Let us formalize our claim.

Claim: Any shear layer profile is linearly and nonlinearly unstable if the Reynolds number is large enough.

The rigorous study of this claim turns out to be very delicate. The natural idea is to study the stability of a shear profile for the limiting system, namely Euler equations ($Re = \infty$), and then to make a perturbative argument to deduce the stability or instability in the high Reynolds number regime. This perturbation however turns out to be very singular and subtle. In fact, two cases arise:

Case 1: Instability for Euler equations: there are shear layer profiles that are unstable to Euler equations. When viscosity is added, but remains sufficiently small, it is then conceivable that the shear layer remains unstable for Navier Stokes equations. This is indeed the case, but requires a delicate analysis to prove it rigorously. In this book, we shall provide a complete nonlinear proof of this instability, arising from that of Euler equations.

Case 2: Stability for Euler equations: there are shear layer profiles that are stable for Euler equations. For instance, all the profiles that do not have an inflection point are stable, thanks to the classical Rayleigh's stability condition. In this case, a naive idea is that viscosity should have an overall stabilizing effect: for a small viscosity, the shear layer would be stable for Navier Stokes equations. Strikingly, this appears to be false, which is somehow paradoxical. A small viscosity does destabilize the flow.

That is, *all shear profiles are unstable for large Reynolds numbers.* Physics textbooks claim that there are lower and upper marginal stability branches

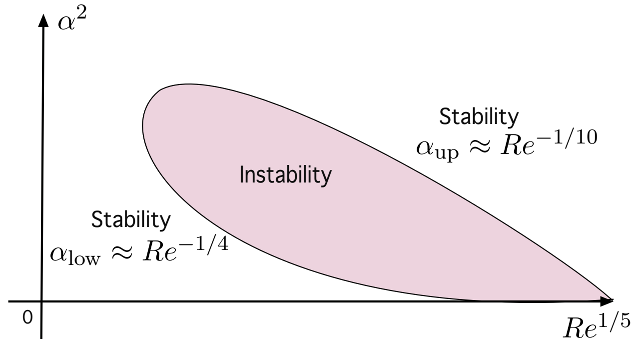

Figure 1.1: *Instability in the stable case.*

$\alpha_{\rm low}(Re), \alpha_{\rm up}(Re)$, depending on the Reynolds number, so that whenever the horizontal wave number α of a perturbation belongs to $[\alpha_{\rm low}(Re), \alpha_{\rm up}(Re)]$, the linearized Navier-Stokes equations about the shear profile is spectrally and linearly unstable, that is, admits an exponentially growing solution.

The asymptotic behavior of these branches $\alpha_{\rm low}$ and $\alpha_{\rm up}$ depends on the profile:

- for plane Poiseuille flow in a channel: $U(z) = 1 - z^2$ for $-1 < z < 1$,

$$\alpha_{\rm low}(R) = A_{1c}R^{-1/7} \qquad \text{and} \qquad \alpha_{\rm up}(R) = A_{2c}R^{-1/11} \tag{1.13}$$

- for boundary layer profiles,

$$\alpha_{\rm low}(R) = A_{1c}R^{-1/4} \qquad \text{and} \qquad \alpha_{\rm up}(R) = A_{2c}R^{-1/6} \tag{1.14}$$

- for Blasius (a particular boundary layer) profile (see figure 1.1),

$$\alpha_{\rm low}(R) = A_{1c}R^{-1/4} \qquad \text{and} \qquad \alpha_{\rm up}(R) = A_{2c}R^{-1/10}. \tag{1.15}$$

These expressions have been compared with modern numerical computations and also with real experiments, showing a very good agreement.

Heisenberg, then Tollmien and C. C. Lin [33] were among the first physicists to use asymptotic expansions to study the instability (see also Drazin and Reid [11] for a complete account of the physical literature on the subject). Their formal analysis has been compared with modern numerical computations and also with experiments, showing a very good agreement; see [11, Figure 5.5] or Figure 1.1 for a sketch of the marginal stability curves. Not until recently, the complete mathematical proof of the linear stability theory was given [17, 18, 19].

1.4 Rayleigh and Orr Sommerfeld equations

This section is devoted to the introduction of Rayleigh and Orr Sommerfeld equations, which are reformulations of linearized Euler and Navier Stokes equations near a steady shear flow

$$U_{shear} := \begin{pmatrix} U_s(z) \\ 0 \end{pmatrix}.$$

The first observation is that if a shear profile $U(z)$ is unstable in three dimensions, then it is also unstable in two dimension. Physically this is known as Squire's theorem. We therefore just need to focus on the two dimensional case.

The 2D Euler equations linearized around a shear flow read

$$\begin{aligned} v_t + U_{shear} \cdot \nabla v + v \cdot \nabla U_{shear} + \nabla q = 0 \\ \nabla \cdot v = 0, \end{aligned} \tag{1.16}$$

with $v = 0$ on the boundary $z = 0$. Let

$$\omega = \nabla \times v = \partial_x v - \partial_z u$$

be the vorticity. Then, ω solves

$$(\partial_t + U_s \partial_x)\omega + vU'' = 0.$$

We write this equation in term of the stream function defined through

$$u = \partial_z \psi \quad \text{and} \quad v = -\partial_x \psi.$$

We note that

$$\omega = \partial_z u - \partial_x v = \Delta \psi.$$

The vorticity equation then reads

$$(\partial_t + U_s \partial_x)\Delta\psi - U_s'' \partial_x \psi = 0.$$

We then take the Fourier transform in the tangential variable x and the Fourier-Laplace transform in time and introduce

$$\psi = e^{i\alpha(x-ct)}\phi(z),$$

in which α is a real-valued positive wave number (Fourier variable in x),

$$\lambda = -i\alpha c$$

is the Laplace variable in time (and so, c is a complex number), and $\phi(z)$ is a complex-valued function. The link between λ and c is traditional.

Putting $\psi = e^{i\alpha(x-ct)}\phi(z)$ into the vorticity equation yields the well-known Rayleigh equations, which are just linearized Euler equations in the stream function formulation, and after an horizontal Fourier transform and a time Fourier Laplace transform

$$\begin{cases} (U_s - c)(\partial_z^2 - \alpha^2)\phi - U_s''\phi = 0 \\ \phi_{|z=0} = 0, \qquad \lim_{z\to\infty} \phi(z) = 0. \end{cases} \tag{1.17}$$

Here, the boundary conditions is exactly: $v = 0$ on the boundary.

Note that in three dimensional space, we have to take Fourier transform in two horizontal variables instead of only one. Up to a change in horizontal variables, the analysis is similar. This remark is known as Squire's theorem in the physical literature.

Its important to note that, up to a multiplication by $-i\alpha$, the spectrum of linearized Euler equations and Rayleigh equations are identical. Rayleigh is much easier to deal with than Euler linearized equation since the divergence free condition is built in.

Now if we consider (1.16) with a source term

$$f(t, x, y) = e^{i\alpha(x-ct)}F(z),$$

where $F(z)$ is a two components vector, we are led to

$$(U_s - c)(\partial_z^2 - \alpha^2)\phi - U_s''\phi = \frac{S}{i\alpha} \tag{1.18}$$

where

$$S = \partial_z F_1 - i\alpha F_2.$$

Note that (1.18) is the resolvent equation of linearized Euler equations with forcing term f.

Rayleigh equation is a second order differential equation, with boundary conditions at $z = 0$ and at infinity. It is singular when $U_s - c$ vanishes, or at least is very small. This remark will play a central part in the study of Euler stable profiles.

If we consider Navier Stokes equations instead of Euler equations, we get the so called Orr Sommerfeld equations, namely

$$\begin{cases} (U - c)(\partial_z^2 - \alpha^2)\phi - U''\phi = \epsilon(\partial_z^2 - \alpha^2)^2\phi, \qquad \epsilon = \dfrac{\nu}{i\alpha} \\ \phi_{|z=0} = \phi'_{|z=0} = 0, \qquad \lim_{z\to\infty} \phi(z) = 0. \end{cases} \tag{1.19}$$

These are fourth order ordinary differential equations. As ν goes to 0, the "viscous" term $\epsilon(\partial_z^2 - \alpha^2)^2\phi$ disappears, and this fourth order equation degenerates in a second order equation, namely Rayleigh equations. This singular limit is classical if Rayleigh is not singular, namely if $U_s - c$ is bounded away from 0. It Rayleigh is itself singular, namely if $U_s - c$ vanishes at some point z_c, then near z_c the situation is dramatic, since both the fourth order and the second order terms vanish at z_c. The Orr Sommerfeld equation then reduces to $-U''\phi = 0$, which is a very severe degeneracy. Such a z_c is called a critical layer. The study of Orr Sommerfeld in the critical layer involves Airy functions.

1.5 Link with Prandtl equations

The question of the stability of shear flows is highly connected to the so called Prandtl equation. More precisely, the study of the inviscid limit with the Dirichlet boundary condition is a long standing problem, with a rich physical and mathematical history. The leading question is the convergence of Navier-Stokes solutions in energy norm in the inviscid limit:

For $\nu > 0$, let u^ν be a sequence of (smooth) solutions of the Navier-Stokes problem (1.1)-(1.4) on a given interval $[0,T]$. Does u^ν converge in $L^\infty([0,T], L^2(\Omega))$, as $\nu \to 0$, to a vector field u, which is a solution of the Euler problem (1.8)-(1.11) ?

The main problem in this inviscid limit is the change in the boundary condition. As the viscosity vanishes, the Laplacian term $-\nu\Delta u$ disappears, leading to a change in the number of boundary conditions. Instead of $u = 0$, in the limit we only have $u \cdot n = 0$ on the boundary. In particular, the tangential velocity may be non zero in the limit. This change in the number of boundary conditions leads to a boundary layer type behavior near $\partial\Omega$: the velocity u will rapidly change near the boundary in order to"recover", or rather "correct", the missing boundary conditions. It turns out that the Dirichlet boundary condition (1.4), which is the main focus of this book, is the most difficult boundary condition to study the inviscid limit problem. There are other interesting physical boundary conditions such as Navier slip boundary conditions. These latter conditions are in fact easier to handle in the inviscid limit, for the reason that they yield certain control on the vorticity near the boundary, see for instance [27, 36].

Up to now, this question appears to be out of reach in the case when Ω has a boundary. Its answer is probably linked to some extent to the under-standing of wall turbulence, a mathematically widely open subject. There is a beautiful criterium by Kato [29] which asserts that the convergence fails in the inviscid limit if and only if the dissipation of energy in a strip of size ν near the boundary does not go to zero. For similar conditional results, see [2, 7, 31], and the references therein. Instead of L^2, we will restrict ourself to L^∞ norms and focus on the following question:

Can we describe the limiting behavior of u^ν in $L^\infty([0,T],\Omega)$ in the inviscid limit ?

This question is stronger, since it requires to describe the behavior of u^ν very close to the wall, which may lead to a cascade of boundary layers of thinner and thinner sizes starting from the classical Prandtl layer of size $\sqrt{\nu}$. In this book we will give preliminary results in that direction.

We end the introduction with a brief mathematical bibliography on the study of Prandtl boundary layers. The existence and uniqueness of solutions to the Prandtl equations have been constructed for monotonic data by Oleinik [39] in the sixties. There are also recent reconstructions [1, 37] of Oleinik's solutions via a more direct energy method. For data with analytic or Gevrey regularity, the well-posedness of the Prandtl equations is established in [42, 15, 10], among others. In the case of non-monotonic data with Sobolev regularity, the Prandtl boundary layer equations are known to be ill-posed ([13, 12, 25]).

On the other hand, the justification of Prandtl's boundary layer Ansatz for the behavior of solutions to the Navier Stokes equations has been justified for analytic data in a pioneered work by Caflisch and Sammartino [42, 43]. See also [34, 38]. The stability of shear flows under perturbations with Gevrey regularity is recently proved in [14].

However these positive results hide a strong instability occurring at high spatial frequencies. For some profiles, instabilities with horizontal wave numbers of order $\nu^{-1/2}$ grow like $\exp(Ct/\sqrt{\nu})$. Within an analytic framework, these instabilities are initially of order $\exp(-D/\sqrt{\nu})$ and grow like $\exp((Ct-D)/\sqrt{\nu})$. They remain negligible in bounded time (as long as $t < D/2C$ for instance).

Within Sobolev spaces, these instabilities are predominant. Indeed, the first author proved in [16] that Prandtl's asymptotic expansion for Sobolev data near unstable profiles is false, up to a remainder of order $\nu^{1/4}$ in L^∞ norm. In this book, we shall present our recent instability results for data with Sobolev regularity, which in particular prove that the Prandtl's boundary layer Ansatz is false and the boundary layer asymptotic expansions for stable monotone profiles are invalid.

1.6 Structure of the book

The aim of this book is to provide a comprehensive presentation to recent advances on boundary layers stability. It targets graduate students and researchers in mathematical fluid dynamics and only assumes that the readers have a basic knowledge on ordinary differential equations and complex analysis. No prerequisites are required in fluid mechanics, excepted a basic knowledge on Navier Stokes and Euler equations, including Leray's theorem.

Part I is devoted to the presentation of classical results and methods: Green functions techniques, resolvent techniques, analytic functions. Part II focuses on the linear analysis, first of Rayleigh equations, then of Orr Sommerfeld equations. This enables the construction of Green functions for Orr Sommerfeld, and then the construction of the resolvent of linearized Navier Stokes equations. Part III details the construction of approximate solutions for the complete nonlinear problem and nonlinear instability results.

Part I

Preliminaries

Chapter 2

Estimates using resolvent

The aim of this chapter is to give a short introduction to the use of the classical Laplace transform to construct solutions for the following simple equations: linear systems in finite dimension space, heat equation and second order linear parabolic equations. These techniques will be used for Rayleigh and Orr Sommerfeld equations in forthcoming chapters.

2.1 Finite dimension case

Let us consider in this section the following simple linear ordinary differential system of N equations

$$\dot{x} = Ax, \tag{2.1}$$

with initial condition

$$x(0) = x_0. \tag{2.2}$$

Here, A is a given fixed $N \times N$ constant matrix, $x(t)$ is a vector function in $\mathbb{R}^N$, and x_0 a given vector. The solution to (2.1)-(2.2) is simply given by

$$x(t) = \exp(tA)x_0.$$

In the case when A is diagonalizable, we can write

$$A = PDP^{-1}$$

where D is diagonal, with coefficients $(\lambda_i)_{1\le i\le N}$, and P is a change of basis. Then, there holds

$$x(t) = P\exp(tD)P^{-1}x_0, \tag{2.3}$$

in which $\exp(tD)$ is diagonal with coefficients $(e^{\lambda_i t})_{1\le i\le N}$.

In the case when A is no longer diagonalizable, then polynomials in time will appear in front of the exponential term. We will not detail this case here. Of course, this approach is limited to matrices and finite dimensional systems. In order to deal with partial differential equations, we will have to develop another approach, based on the Laplace / Fourier transform.

To be more specific, let us define

$$y(t) = x(t) 1_{t \geq 0}.$$

Then $y(t)$ satisfies in the sense of distributions

$$\dot{y} = Ay + x_0 \delta_0$$

where δ_0 is the Dirac mass centered at $t = 0$. Note that y may have an exponential growth in large times. To turn this potential growth into a decay we multiply y by e^{-Mt}, for some sufficiently large constant $M > 0$, and introduce

$$z(t) = y(t) e^{-Mt} = x(t) e^{-Mt} 1_{t \geq 0}.$$

Then

$$\dot{z}(t) = (A - M\mathrm{Id}) z(t) + x_0 \delta_0, \tag{2.4}$$

where Id denotes the identity matrix. Occasionally, we simply drop the notation Id. Now $z(t)$ has an exponential decay as t goes to $+\infty$, and vanishes for negative t.

To solve (2.4), we may introduce its Fourier transform $\hat{z}(\xi)$, yielding

$$i\xi \hat{z}(\xi) = (A - M\mathrm{Id}) \hat{z}(\xi) + x_0.$$

Therefore,

$$\hat{z}(\xi) = -(A - M\mathrm{Id} - i\xi)^{-1} x_0.$$

Now inverting the Fourier transform gives

$$z(t) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi t} (A - M\mathrm{Id} - i\xi)^{-1} x_0 d\xi,$$

which solves (2.4). Using analyticity in ξ, we can shift the contour integration, yielding

$$x(t) = -\frac{1}{2\pi} \int_{\Im \xi = iM} e^{it\xi} (A - i\xi)^{-1} x_0 d\xi.$$

Following the traditional notation, we introduce the Laplace transform variable $\lambda = i\xi$, and thus

$$x(t) = \frac{1}{2i\pi} \int_{\Gamma} e^{\lambda t} (\lambda - A)^{-1} x_0 d\lambda \tag{2.5}$$

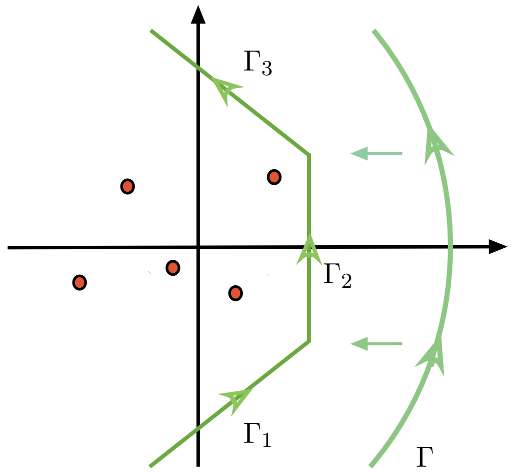

Figure 2.1: *Illustrated are possible point spectrum of A and a contour decomposition of Γ lying on the right of the spectrum.*

where Γ is a contour "on the right" of the spectrum of A, as depicted in Figure 2.1. This defines a solution to the ODE problem (2.1)-(2.2).

Let us introduce the resolvent operator

$$R(\lambda) = (\lambda - A)^{-1}. \tag{2.6}$$

Then, the solution $x(t)$ is represented by

$$x(t) = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} R(\lambda) x_0 d\lambda, \tag{2.7}$$

in which the function $R(\lambda)x_0$ is simply the solution ϕ of the resolvent equation

$$\lambda\phi = A\phi + x_0$$

(which is of course the Laplace transform of (2.1) with source term x_0).

Now, in order to obtain bounds on the solution $x(t)$ in term of time growth, we may decompose the contour of integration Γ to the left, as long as Γ does not cross the spectrum of A; see Figure 2.1. For instance, we can deform Γ as follows:

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

where $\Gamma_2 = P + i[-B, B]$, $\Gamma_1 = (1+i)\mathbb{R}_- + P - iB$, and $\Gamma_3 = (-1+i)\mathbb{R}_+ + P + iB$, for some constant P and sufficiently large constant B so that the spectrum of A remains on the left.

The integral on Γ_2 is simply bounded by Ce^{Pt}. The integrals on Γ_1 and Γ_3 are bounded by

$$Ce^{Pt}\int_{\mathbb{R}_+}\frac{e^{-st}}{1+s}ds \le C'e^{Pt}.$$

Hence, for any constant P that is greater than the maximum of real part of the spectrum of A, there exists a constant C_P so that

$$\|x(t)\| \le C_P e^{Pt}, \qquad \forall\ t \ge 0.$$

To get optimal bounds, and even an explicit expression for the solution, we need to detail the behavior of the resolvent operator $R(\lambda)$

$$R(\lambda) := \frac{1}{\det(\lambda - A)}\mathrm{com}(\lambda - A)^{tr}$$

in which $\mathrm{com}(M)$ denotes the comatrix of a matrix M. Note that $R(\lambda)$ is an holomorphic function, singular on $Sp(A)$, and which has a Laurent's asymptotic expansion at $\lambda \in Sp(A)$. By the residue theorem, the integral (2.7) is the sum of the residues at the poles of $R(\lambda)$. These residues are sum of products of projectors with polynomials in time (of degree the multiplicity of the eigenvalue minus one). We thereby recover (2.3) in the case of simple eigenvalues.

2.2 Heat equation

Let us now detail how to solve the classical heat equation

$$\partial_t u - \nu\partial_z^2 u = 0, \tag{2.8}$$

with initial condition $u_0(z)$ on the whole line $z \in \mathbb{R}$, using a similar approach. Using the Laplace transform approach, we have

$$u(t) = \frac{1}{2i\pi}\int_\Gamma e^{\lambda t}u_\lambda d\lambda \tag{2.9}$$

in which Γ is a contour of integration lying on the right of the spectrum of $\nu\partial_z^2$, say in L^2, and u_λ is the solution of the resolvent equation

$$\lambda u_\lambda - \nu\partial_z^2 u_\lambda = u_0. \tag{2.10}$$

Next, we solve (2.10) via the Green function approach. To this end, we first construct increasing and decreasing solutions $\psi_\pm$ of the homogenous equation

$$\lambda\psi_\pm = \nu\partial_z^2\psi_\pm, \tag{2.11}$$

which are

$$\psi_\pm(z) = e^{\mp z\sqrt{\lambda/\nu}}. \tag{2.12}$$

Note that $\psi_\pm$ is holomorphic in $z \in \mathbb{C}$, and locally in $\lambda \neq 0$, with a "branching point" at $\lambda = 0$. However if the contour Γ does not cross $\mathbb{R}_-$, we can choose $\sqrt{\lambda/\nu}$ (and hence $\psi_\pm$) in an holomorphic way such that its real part is strictly positive.

On $\mathbb{R}$, the Green function of (2.10) is simply

$$G_\lambda(x,z) = -\frac{1}{2\sqrt{\lambda\nu}} e^{-|x-z|\sqrt{\lambda/\nu}},$$

and hence, the solution of (2.10) is given, for real values of z, by

$$u_\lambda(z) = \int_{\mathbb{R}} G_\lambda(x,z) u_0(x) dx. \tag{2.13}$$

Now to explicitly compute the value of $u(t)$ via (2.9), we choose a contour Γ which passes on the right of 0 and is asymptotic to the half lines $L_\pm = \mathbb{R}_+ e^{\pm 2i\theta}$ for some $\pi/2 < 2\theta < \pi$. In this case, the integral of (2.9) converges and gives the solution of (2.8).

Alternatively, we introduce the temporal Green function $G(t,x,z)$, defined by

$$G(t,x,z) = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} G_\lambda(x,z)\, d\lambda, \tag{2.14}$$

and derive its pointwise bounds. The solution to the heat equation is then the convolution of $G(t,x,z)$ with the initial data $u_0(z)$. To estimate the integral (2.14), we bound the integrand by

$$|e^{\lambda t} e^{-|x-z|\sqrt{\lambda/\nu}}| = e^{\Re\lambda t - |x-z| \Re\sqrt{\lambda/\nu}}.$$

As the imaginary part of $\sqrt{\lambda/\nu}$ plays no role in the modulus estimate, we parametrize Γ through the imaginary part of $\sqrt{\lambda/\nu}$, introduce

$$\sqrt{\lambda/\nu} = a + ik,$$

and parametrize Γ through k. Note that $e^{\Re\lambda t - |x-z|\Re\sqrt{\lambda/\nu}}$ is minimal (for each fixed k) when

$$a = \frac{|x-z|}{2\nu t}.$$

We therefore fix a to this value. We then have

$$\frac{\lambda}{\nu} = (a+ik)^2,$$

which leads to the following choice of Γ

$$\Gamma := \Big\{ \lambda = a^2\nu - k^2\nu + 2iak\nu, \quad k \in \mathbb{R} \Big\}.$$

Then, we compute

$$\begin{aligned} |G(t,x,z)| &\le \frac{1}{4\pi\sqrt{\nu}} \int_\Gamma e^{\Re\lambda t - |x-z|\Re\sqrt{\lambda/\nu}} \frac{|d\lambda|}{|\sqrt{\lambda}|} \\ &\le \frac{1}{2\pi} \int_{\mathbb{R}} e^{a^2\nu t - |x-z|a} e^{-k^2\nu t}\, dk \\ &\le \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{|x-z|^2}{4\nu t}}. \end{aligned}$$

The temporal Green function of the heat problem (2.8) behaves exactly as the classical Gaussian kernel.

2.3 Second order linear parabolic equations

Let us turn to systems of the form

$$\partial_t u - \nu\Delta u = A(x)u \tag{2.15}$$

where $A(x)$ is a given smooth matrix, with initial data u_0. Formula (2.7) can be extended to this case and gives

$$u(t) = \int_\Gamma e^{\tau t} R(\tau) u_0 \, d\tau \tag{2.16}$$

where

$$R(\tau) = (i\tau \mathrm{Id} - \nu\Delta - A)^{-1} \tag{2.17}$$

is the resolvent, and where Γ is a contour on the right of the singularities of R. To bound $u(t)$ we need to bound $R(\tau)$, namely to solve

$$i\tau v - \nu\Delta v - A(x)v = u_0(x), \tag{2.18}$$

which is a system of linear ordinary differential equations.

Green function

To solve (2.18) we introduce the Green function $G(x,y)u$, solution of

$$i\tau G - \nu\Delta G - A(x)G = \delta_y u, \tag{2.19}$$

where u is a fixed vector. Note that we skipped τ is the notation of the Green function. The solution of (2.18) is then given by

$$v(x) = \int_{\mathbb{R}} G(x,y)u_0(y)dy. \tag{2.20}$$

Next to define the Green function G we introduce ψ_j^- and ψ_j^+ with $1 \le j \le N$, independent solutions of

$$i\tau v - \nu \Delta v - A(x)v = 0$$

which go to 0 as $x \to -\infty$ (for ψ_j^-) or $x \to +\infty$ (for ψ_j^+). Then

$$G(x,y)u = \sum_{1 \le j \le N} \alpha_j(y)\psi_j^-(x)$$

for $x < y$ and for some constants $\alpha_j(y)$ depending on u and

$$G(x,y)u = \sum_{1 \le j \le N} \beta_j(y)\psi_j^+(x)$$

for $x > y$ and for some constant $\beta_j(y)$. Note that $G(x,y)u$ is continuous at y and that its derivative has a jump u at y. Hence

$$\sum_{1 \le j \le N} \alpha_j \psi_j^+ = \sum_{1 \le j \le N} \beta_j \psi_j^-$$

and

$$\sum_{1 \le j \le N} \alpha_j \partial_x \psi_j^+ = \sum_{1 \le j \le N} \beta_j \partial_x \psi_j^- - \nu^{-1} u.$$

Let

$$M = \begin{pmatrix} \psi_j^+ & \psi_j^- \\ \partial_x \psi_j^+ & \partial_x \psi_j^- \end{pmatrix}.$$

Note that M is simply the jacobian matrix of the independent solutions ψ_j^+, ψ_j^-, which depends on y and τ. Then

$$\begin{pmatrix} \alpha_j \\ -\beta_j \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ -\nu^{-1} u \end{pmatrix}. \tag{2.21}$$

We introduce the Evans function

$$\mathcal{E} = \|\nu^{-1} M^{-1}\|. \tag{2.22}$$

Then if $|\tau| \le \alpha$ we get

$$\sum_{1 \le j \le N} |\alpha_j| + |\beta_j| \le \mathcal{E}^{-1}|u| \tag{2.23}$$

for some constant C.

Analyticity

Let us study $\psi_j^\pm$. Let us assume that $A(x)$ is holomorphic in the strip $S = \{|\Im x| \le \sigma_0\}$ for some positive σ_0. For bounded τ, the functions $\psi_j^\pm$ are solutions of a holomorphic equation on S. As a consequence they are defined and holomorphic on S. It is also possible to show that $\psi_j^\pm(z)$ go to zero as $\Re z$ go to $\pm\infty$.

For large τ, $\psi_j^\pm$ behave like the solutions ψ of

$$\tau\psi - \nu\partial_x^2\psi = 0,$$

namely like

$$\psi_\pm(z) \sim \frac{1}{2}\sqrt{\frac{\nu}{\tau}}e^{-\pm z\sqrt{\tau/\nu}},$$

which is similar to the heat equation case. The bounds on the Green function for large τ are therefore the same as the bounds for the heat equation.

Bounds on the resolvant

To bound the resolvant $R(\tau)$ we need

- a bound for bounded τ, given by (2.23)
- another bound for unbounded τ, of the form $\tau = A + e^{i\theta}t$ with $t > 0$, for t large. In this case A can be treated as a small perturbation of $i\tau\mathrm{Id} - \nu\Delta$, which lead to bounds similar to the case of the heat equation.

Combining these two estimates we can then bound $R(\tau)$ and thus $u(t)$. We will not detail this point here.

In (2.16) the contour Γ can be moved to another contour Γ' on its left, leaving some eigenvalues λ_i with $1 \le i \le p$ on its right. Then, if all the eigenvalues are simple,

$$u(t) = \int_{\Gamma'} e^{\tau t}R(\tau)u_0 d\tau + \sum_i P_i u_0 e^{\lambda_i t}, \tag{2.24}$$

where P_i is the projector on the eigenspace E_i associated with the eigenvalue λ_i.

By carefully choosing the contour Γ' we can get an asymptotic expansion on $u(t)$. In particular if there exists some unstable eigenvalue λ_i then $u(t)$ generically behaves like $\exp(\lambda_i t)$. Note that the eigenvalues are simply the singularities of $\mathcal{E}$, namely the zeros of the Evans function $\det(M)$, which is explicit in the functions $\psi_j^\pm$.

Chapter 3

From linear to nonlinear instability

This chapter is devoted to simple remarks on ordinary differential equations in order to clarify basic ideas on the link between linear and nonlinear instabilities.

3.1 From linear to nonlinear instability

Let us recall in this section how we can prove nonlinear instability, assuming linear instability in the case of ordinary differential equations. Let us consider the following system

$$\partial_t \phi = A\phi + Q(\phi, \phi) \tag{3.1}$$

where ϕ is a vector, A a matrix and Q a quadratic term. Let us assume that A is spectrally unstable, namely that there exists an eigenvalue λ and a corresponding eigenvector v_0 with $\Re\lambda > 0$. For simplicity we assume that λ is a single eigenvalue. We want to prove nonlinear instability, and more precisely

Theorem 3.1.1. *There exists a constant σ_0 such that for any arbitrarly small $\varepsilon > 0$, there exists a solution ϕ to (3.1) satisfying the following properties:*

$$|\phi(0)| \le \varepsilon,$$

$$|\phi(T_\varepsilon)| \ge \sigma_0$$

for some positive time T_ε.

The proof of this nonlinear instability result relies on two steps: first the construction of an accurate approximate solution, then an estimate between the approximate and the true solution.

3.1.1 Construction of an approximate solution

We look for an approximate solution of the form

$$\phi_{app} = \sum_{i=1}^{N} \phi_i$$

where N will be chosen later. We start with

$$\phi_1 = \varepsilon v_0 e^{\lambda t},$$

where ε is a small parameter. Note that ϕ_0 is bounded by

$$|\phi_1| \le C\varepsilon e^{\Re\lambda t}.$$

For $i \ge 2$, ϕ_i is constructed by iteration through

$$\partial_t \phi_i = A\phi_i + \sum_{j+k=i} Q(\phi_j, \phi_k),$$

with $\phi_i(0) = 0$ (where $\phi_j = 0$ for $j \le 0$). We will prove by iteration that

$$|\phi_j| \le C_j \varepsilon^j e^{j\Re\lambda t}. \tag{3.2}$$

This bound is true for $j = 1$. If it is true for $j < i$ then

$$\phi_i(t) = \sum_{j+k=i} \int_0^t e^{A(t-\tau)} Q(\phi_j, \phi_k) d\tau,$$

which leads to

$$|\phi_i(t)| \le C_i \varepsilon^i \sum_{j+k=i} \int_0^t e^{\Re\lambda(t-\tau)} e^{i\Re\lambda\tau} d\tau$$

which gives (3.2).

Now, ϕ_{app} is an approximate solution in the sense that

$$\partial_t \phi_{app} = A\phi_{app} + Q(\phi_{app}, \phi_{app}) + R_{app},$$

where R_{app} is an error term. A short computation shows that

$$|R_{app}| \le C_{N+1} \varepsilon^{N+1} e^{(N+1)\Re\lambda t}.$$

3.1.2 Stability

The next step is to use a stability estimate. Let ϕ be the solution of (3.1) with initial data $\phi_0(0)$. Let

$$\theta = \phi - \phi_{app}$$

be the difference between the "true" solution and the approximate one. We have

$$\partial_t \theta = A\theta + Q(\phi_{app}, \theta) + Q(\theta, \phi_{app}) + Q(\theta, \theta) - R_{app}.$$

Let us work on the largest time T_0 such that on $0 \le t \le T_0$, $|\phi_{app}(t)| \le 1$ and $|\theta(t)| \le 1$. Let $0 \le t \le T_0$. We take the scalar product of the previous equation with θ. This leads to

$$\partial_t |\theta|^2 \le C_0 |\theta|^2 + |R_{app}|^2$$

since $(A\theta, \theta)$, $(Q(\phi_{app}, \theta), \theta)$, $(Q(\theta, \phi_{app}), \theta)$, $(Q(\theta, \theta), \theta)$ are all bounded by $C_0|\theta|^2$ for some constant C_0. Therefore

$$|\theta|^2(t) \le \varepsilon^{2(N+1)} \int_0^t e^{C_0(t-\tau)} e^{2(N+1)\Re\lambda\tau} d\tau \le C\varepsilon^{2(N+1)} e^{2(N+1)\Re\lambda t},$$

provided

$$2(N+1)\Re\lambda > C_0,$$

namely, provided N is large enough.

We now define

$$T_1 = -\frac{\log\varepsilon}{\Re\lambda} - \sigma$$

where σ will be chosen later. Before T_1, all the ϕ_i are bounded A direct computation shows that provided σ is large enough, $T_1 < T_0$. Then at T_1,

$$|\phi(T_1)| \ge |\phi_{app}(T_1)| - |\theta(T_1)| \ge e^{-\sigma} - \sum_{i=1}^{N+1} C_i e^{-i\sigma} \ge \frac{e^{-\sigma}}{2}$$

provided σ is small enough. This ends the proof of the Theorem, chosing $\sigma_0 = e^{-\sigma}/2$.

3.1.3 Link with Navier Stokes equations

The previous theorem is a toy model for the stability of viscous boundary layers. In the case of Navier Stokes equations, Q is the nonlinear transport term and A the linearized Navier Stokes equations near a boundary layer profile. The first difficulty is to prove the existence of an unstable mode for linearized Navier Stokes equation, which is the equivalent of the construction of λ and v_0. This is obtained through a detailed spectral analysis of Orr Sommerfeld equations. The second difficulty is to have good estimates on e^{At}, which is done through the construction of the corresponding Green function. The last step, namely the energy estimate, is the standard L^2 estimate on Navier Stokes.

However, because of large gradients in the sublayer of the instability, the construction ends before the instability reaches $O(1)$. This approach stops when the instability reaches a size $O(\nu^{1/4})$, as is detailed in [16]. To go up to $O(1)$ we will have to design a new method, developed in the next chapter.

3.2 Small unstable eigenvalues

Let us recall that we face two different cases. A first case arises when the shear layer profile is unstable for Rayleigh equations. In this case the largest unstable eigenvalue for Navier Stokes equation is of order $O(1)$. In the second case, when the shear layer is stable for Rayleigh equation, $\Re\lambda$ is small, of order $\nu^{1/4}$. This latest case can not be handled by the techniques developed in the previous section. We will discuss it, always on a toy model case, in this section.

Let us focus in this section on systems of the form

$$\partial_t \phi = A_\varepsilon \phi + Q(\phi, \phi) \tag{3.3}$$

where ϕ is a vector, A_ε is a diagonal matrix and depends on a small parameter $\varepsilon > 0$, and Q is quadratic. We assume that A_ε has only one eigenvalue with positive real part, denoted by $\lambda(\varepsilon)$. We assume that

$$\Re\lambda(\varepsilon) = \varepsilon,$$

which is always possible, up to a renumbering of ε. Let $v_0(\varepsilon)$ be the corresponding eigenvalue. Then

$$\phi_1(t) = \varepsilon^N v_0(\varepsilon) e^{\lambda(\varepsilon)t}$$

is an exponential growing solution to the linearized equation, which initially is of order $O(\varepsilon^N)$. Let us construct an approximate solution of (3.3). The second term in the expansion is defined by

$$\partial_t \phi_2 = A_\varepsilon \phi_2 + Q(\varepsilon^N v_0 e^{\lambda(\varepsilon)t}, \varepsilon^N v_0 e^{\lambda(\varepsilon)t}),$$

which leads to

$$\phi_2(t) = \int_0^t e^{A_\varepsilon(t-\tau)} Q\Big(\varepsilon^N v_0 e^{\lambda(\varepsilon)\tau}, \varepsilon^N v_0 e^{\lambda(\varepsilon)\tau}\Big) d\tau,$$

assuming $\phi_2(0) = 0$. Note that

$$|Q\Big(\varepsilon^N v_0 e^{\lambda(\varepsilon)\tau}, \varepsilon^N v_0 e^{\lambda(\varepsilon)t}\Big)| \le C\varepsilon^{2N} e^{2\Re\lambda\tau},$$

hence

$$|\phi_2(t)| \le C\varepsilon^{2N} \int_0^t e^{\Re\lambda(t-\tau)} e^{2\Re\lambda\tau} d\tau \le C \frac{C\varepsilon^{2N}}{\Re\lambda} e^{2\Re\lambda t}.$$

Therefore $\phi_2(t)$ is of size $\varepsilon^{2N-1} e^{2\Re\lambda t}$, namely much larger than in the previous section, where it was of order $\varepsilon^{2N} e^{2\Re\lambda t}$. Iterating the process we can construct an approximate solution of the form

$$\phi \sim \sum_n \phi_n,$$

where

$$|\phi_n| \le C_n \varepsilon^{n(N-1)+1} e^{n\Re\lambda t}.$$

The first two terms of this series are of the same order when $e^{\Re\lambda t}$ is of order ε^{1-N}. Then all the terms of the series are of the same order, namely of order $O(\varepsilon)$. It is therefore not possible to construct a solution of (3.3) larger than $O(\varepsilon)$ with this method.

To understand this point, let us take the simplest possible example, namely the scalar example where $A_\varepsilon = \varepsilon$ and $Q(u) = \alpha u^2$. We assume that $\phi(0) > 0$. Then (3.3) is simply

$$\partial_t \phi = \varepsilon\phi + \alpha\phi^2 \tag{3.4}$$

for some constant α. The qualitative evolution of ϕ depends on the sign of α. If $\alpha < 0$, then $\phi(t)$ converges to $\alpha^{-1}\varepsilon$ as t goes to infinity. If on the contrary $\alpha > 0$ then $\phi(t)$ blows up in finite time. Qualitatively, $\phi(t)$ increases exponentially until it reaches $O(\varepsilon)$ where the quadratic term becomes important and speeds up the growth.

Therefore in the case of small eigenvalues, the nonlinear term plays a crucial role. The situation is in fact very close to a bifurcation.

Let us go back to Navier Stokes equations. When the shear flow is linearly stable for Euler equation, it will be shown that it is linearly unstable for Navier Stokes equations, but with a very slow instability, of order $O(\nu^{1/4})$, which plays the role of the ε of this paragraph. We then suffer from similar limitations: the instability can only be proven up to a size $O(\nu^{1/4})$ in L^∞ norm.

Chapter 4

Analyticity and generator functions

This chapter is devoted to the definition of various spaces of analytic functions.

4.1 Analyticity

As a warm up we first describe some very classical analytic spaces.

4.1.1 On the real line

Let us first define various spaces of analytic functions of the real line $\mathbb{R}$. The first idea is to consider the extension of analytic functions to the complex plane. Namely, for $\rho > 0$, we define the complex strip

$$\Gamma_\rho = \Big\{ z \in \mathbb{C}, \quad |\Im z| \le \rho \Big\},$$

and consider the space of functions f which are holomorphic on this strip. We then define the analytic spaces $\mathcal{A}^\rho$ by: $f \in \mathcal{A}^\rho$ if and only if there exists a bounded holomorphic function g defined on Γ_ρ such that $f(z) = g(z)$ for all real $z \in \mathbb{R}$. The norm on $\mathcal{A}^\rho$ is defined by

$$\|f\|_\rho = \sup_{z \in \Gamma_\rho} |g(z)| e^{\beta |\Re z|}$$

for some fixed nonnegative number β. The exponential weight allows to control the behavior of f at infinity. By a slight abuse of notation, we denote by f the extension of $f(z)$ to Γ_ρ.

Proposition 4.1.1. *For every functions f and g in $\mathcal{A}^\rho$ there holds*

$$\|fg\|_\rho \le \|f\|_\rho \|g\|_\rho.$$

Proof. Let f and g be the extensions to the complex strip Γ_ρ. Then

$$|f(z)g(z)|e^{\beta|\Re z|} \le |f(z)|e^{\beta|\Re z|}|g(z)|e^{\beta|\Re z|}$$

since $\beta \ge 0$. We then can the supremum norm of the right hand side, and then of the left hand side. □

One of the main advantages of analytic functions is that the L^∞ norm of their derivatives can be controlled by the L^∞ norm of the function. Namely we have

Proposition 4.1.2. *There exists a constant $C > 0$ such that for any function $f \in \mathcal{A}^\rho$ and for any $0 < \rho' < \rho$, we have*

$$\|\partial_z f\|_{\rho'} \le \frac{C}{\rho - \rho'} \|f\|_\rho.$$

Proof. The proof is based on the use of Cauchy's formula

$$\partial_z f(z) = \frac{1}{2i\pi} \int_{C(z,R)} \frac{f(z')}{(z'-z)^2} dz'$$

where $C(z, R)$ is the circle, centered at z and of radius R. We can take R sufficiently small so that $C(z, R) \in \Gamma_\rho$, for $z \in \Gamma_\rho$. Hence

$$|\partial_z f(z)| \le \frac{\|f\|_\rho e^{-\beta|\Re z|}}{d(z, \partial\Gamma_\rho)},$$

where $d(z, \partial\Gamma_\rho)$ is the distance from z to the boundary of Γ_ρ. Now if $z \in \Gamma_{\rho'}$,

$$d(z, \partial\Gamma_{\rho,r}) \ge C(\rho - \rho'),$$

which ends the proof. □

4.1.2 On the half line

We next study the case of the half real line $\mathbb{R}_+$. Let $\sigma > 0$ and let $r > 0$. Let us introduce the "pencil" type domain

$$\Gamma_{\sigma,r} = \Big\{ z, |\Im z| \le \min(\sigma \Re z, \sigma r), \Re z \ge 0 \Big\}.$$

In the sequel, r will be fixed and we will omit it in the various notations. Let f be a smooth function of real $z \geq 0$. As in the previous case, we define the analytic spaces $\mathcal{B}^\sigma$ by: $f \in \mathcal{B}^\sigma$ if and only if there exists a bounded holomorphic function g defined on $\Gamma_{\sigma,r}$ such that $f(z) = g(z)$ for all real $z \geq 0$. The norm on $\mathcal{B}^\sigma$ is defined by

$$\|f\|_\sigma = \sup_{z \in \Gamma_{\sigma,r}} |g(z)| e^{\beta \Re z}$$

for some fixed nonnegative constant β. By a slight abuse of notation, we denote by f the extension of $f(x)$ to $\Gamma_{\sigma,r}$. As in Proposition 4.1.1, we have

$$\|fg\|_\sigma \leq \|f\|_\sigma \|g\|_\sigma,$$

For every functions f and g in $\mathcal{B}^\sigma$. In addition, we have

Proposition 4.1.3. *Let*

$$\varphi(z) = \frac{z}{1+z}.$$

There exists a constant $C > 0$ such that for any function $f \in \mathcal{B}^\sigma$, and for any $0 < \sigma' < \sigma$, we have $\varphi(z)\partial_z f \in \mathcal{B}^{\sigma'}$ and

$$\|\varphi(z)\partial_z f\|_{\sigma'} \leq \frac{C}{\sigma - \sigma'} \|f\|_\sigma. \tag{4.1}$$

Proof. The proof is similar to that of Proposition 4.1.2, upon noting that for $z \in \Gamma_{\sigma',r}$, we have

$$d(z, \partial\Gamma_{\sigma,r}) \geq C(\sigma - \sigma')|\varphi(z)|,$$

which ends the proof. □

4.1.3 In two space dimensions

Let us now define analytic spaces for functions which are defined on $\mathbb{R} \times \mathbb{R}_+$. We define the domain $\Gamma_{\rho,\sigma,r}$ by

$$\Gamma_{\rho,\sigma,r} = \Big\{(x,y), \quad |\Im x| \leq \rho, \quad |\Im y| \leq \min(\sigma \Re y, \sigma r), \Re y \geq 0\Big\}.$$

We then define $\mathcal{A}^{\rho,\sigma}$ as the space of holomorphic functions, defined on $\Gamma_{\rho,\sigma,r}$, together with the norm

$$\|f\|_{\rho,\sigma} = \sup_{z \in \Gamma_{\rho,\sigma,r}} |f(z)| e^{\beta \Re y}$$

where $\beta > 0$ is a fixed constant. Note that ρ measures the regularity in the x variable, and σ in the y variable. Propositions similar to 4.1.1 and 4.1.2 hold true, namely

$$\|fg\|_{\rho,\sigma} \le \|f\|_{\rho,\sigma}\|g\|_{\rho,\sigma},$$

$$\|\partial_x f\|_{\rho',\sigma} \le \frac{C}{\rho - \rho'}\|f\|_{\rho,\sigma}, \qquad \|\varphi(z)\partial_z f\|_{\rho,\sigma'} \le \frac{C}{\sigma - \sigma'}\|f\|_{\rho,\sigma}.$$

4.2 Generator functions

We now introduce other classes of analytic functions, which turn be more adapted to Navier Stokes equations.

4.2.1 In one space dimension

Another way to say that a function f is holomorphic, is to say that its successive derivatives $\partial_z^n f$ grow at most like $n!M^n$ for some constant M. This leads to the following definitions.

Let $\|.\|_n$ be a family of norms, which satisfies, for any $0 \le p \le n$

$$\|fg\|_n \le C_0\|f\|_p\|g\|_{n-p}$$

for a given constant C_0 independent on f, g, n and p. In the sequel we will take the supremum norm or a weighted norm, specially designed for boundary layers.

Let $f(z)$ be a given function of a real variable $z \in \mathbb{R}$ or $z \in \mathbb{R}_+$. We define its generator function $Gen(f)$ by

$$Gen(f)(z) = \sum_{n\ge 0} \|\partial_z^n f\|_n \frac{z^n}{n!}.$$

Note that $Gen(f)$ is defined on an interval of the form $(-R, R)$, with $R > 0$ if f is analytic.

Note that $Gen(f)$ is a non negative, nondecreasing, convex function. All its derivatives, at any order, are non negative, nondecreasing and convex.

Proposition 4.2.1. *Let f and g be two given functions, and let $N > 0$. Then*

$$Gen(fg) \le C_0 Gen(f)Gen(g). \tag{4.2}$$

Proof. We have

$$\partial_z^n(fg) = \sum_{0\le k\le n} \frac{n!}{k!(n-k)!}\partial_z^k f \partial_z^{n-k} g,$$

hence

$$\|\partial_z^n(fg)\|_n \le C_0 \sum_{0\le k\le n} \frac{n!}{k!(n-k)!}\|\partial_z^k f\|_k \|\partial_z^{n-k} g\|_{n-k}.$$

Therefore

$$\|\partial_z^n(fg)\|_n \frac{z^n}{n!} \le C_0 \sum_{0\le k\le n} \|\partial_z^k f\|_k \frac{z^k}{k!} \|\partial_z^{n-k} g\|_{n-k} \frac{z^{n-k}}{(n-k)!}.$$

Summing with respect to n we get (4.2). □

4.2.2 On the half plane

In two space dimensions, we will use the Fourier transform in the x variable. Let α be the dual variable. For any smooth function $f(x,y)$ we introduce $f_\alpha(y)$, its Fourier transform in x, which satisfies

$$f(x,y) = \sum_\alpha e^{i\alpha x} f_\alpha(y).$$

We then introduce the generator function

$$Gen(f)(z_1,z_2) = \sum_{k,\alpha} e^{|\alpha| z_1} \|\partial_y^k f_\alpha\|_k \frac{z_2^k}{k!}.$$

Generator functions have very nice properties, since the generator of a product is dominated by the product of the generators, and the generator of the x derivative is bounded by the x derivative of the generator, two strong and very useful properties!

Proposition 4.2.2. *For any functions f and g, there hold*

$$Gen(fg) \le C_0 Gen(f) Gen(g), \qquad Gen(\partial_x f) \le \partial_{z_1} Gen(f).$$

Proof. We have

$$\begin{aligned} Gen(fg) = &\sum_{k,\alpha,k'\le k,\alpha'} e^{|\alpha| z_1} \|\partial_y^{k'} f'_\alpha \partial_y^{k-k'} f_{\alpha-\alpha'}\|_k \frac{z_2^k}{k'!(k-k')!} \\ &\le C_0 \sum_{k,\alpha,k'\le k,\alpha'} e^{|\alpha'| z_1} \|\partial_y^{k'} f'_\alpha\|_{k'} e^{|\alpha-\alpha'| z_1} \|\partial_y^{k-k'} f_{\alpha-\alpha'}\|_{k-k'} \frac{z_2^k}{k'!(k-k')!} \\ &\le C_0 Gen(f) Gen(g). \end{aligned}$$

The equality involving $Gen(\partial_x f)$ is straightforward. Note that we do not have a similar expression for $Gen(\partial_y f)$ since $\partial_y f$ may become large near 0. □

4.2.3 Boundary layer norms

We will now focus on the case where there is a boundary layer in the second variable y. More precisely, if we assume that f has a boundary layer of size δ near $y = 0$ then we expect, for any k and ℓ, $\partial_x^k f$ and $y^\ell \partial_y^\ell f$ are bounded. For this reason, we introduce analytic boundary layer norms: for $z_1, z_2 \geq 0$ we define

$$\begin{aligned} Gen_0(f)(z_1, z_2) &= \sum_{\alpha\in\mathbb{Z}} \sum_{\ell\geq 0} e^{z_1|\alpha|} \|\partial_y^\ell f_\alpha\|_{\ell,0} \frac{z_2^\ell}{\ell!}, \\ Gen_\delta(f)(z_1, z_2) &= \sum_{\alpha\in\mathbb{Z}} \sum_{\ell\geq 0} e^{z_1|\alpha|} \|\partial_y^\ell f_\alpha\|_{\ell,\delta} \frac{z_2^\ell}{\ell!}, \end{aligned} \tag{4.3}$$

in which $f_\alpha(y)$ denotes the Fourier transform of $f(x, y)$ with respect to the x variable. In these sums,

$$\begin{aligned} \|f_\alpha\|_{\ell,0} &= \sup_y \varphi(y)^\ell |f_\alpha(y)|, \\ \|f_\alpha\|_{\ell,\delta} &= \sup_y \varphi(y)^\ell |f_\alpha(y)| \Big(\delta^{-1} e^{-y/\delta} + 1\Big)^{-1}, \end{aligned}$$

where

$$\varphi(y) = \frac{y}{1+y}$$

and where the boundary layer thickness δ is equal to

$$\delta = \gamma_0 \nu^{1/4}$$

for some sufficiently large $\gamma_0 > 0$.

In practice, these generator functions $Gen_0(\cdot)$ and $Gen_\delta(\cdot)$ will respectively control the velocity and the vorticity of the solutions of Navier Stokes equations, and the constant γ_0 will be chosen so that $\gamma_0^{-1} \leq \sqrt{\Re\lambda_0/2}$, where λ_0 is the maximal unstable eigenvalue of the linearized Euler equations around U.

For convenience, we introduce the following generator functions of one-dimensional functions $f = f(y)$:

$$\begin{aligned} Gen_{0,\alpha}(f)(z_2) &= \sum_{\ell\ge 0} \|\partial_y^\ell f\|_{\ell,0} \frac{z_2^\ell}{\ell!}, \\ Gen_{\delta,\alpha}(f)(z_2) &= \sum_{\ell\ge 0} \|\partial_y^\ell f\|_{\ell,\delta} \frac{z_2^\ell}{\ell!}. \end{aligned} \tag{4.4}$$

Of course, it follows that

$$Gen_0(f) = \sum_{\alpha\in\mathbb{Z}} e^{z_1|\alpha|} Gen_{0,\alpha}(f_\alpha)$$

for functions of two variables $f = f(x, y)$, and similarly for Gen_δ.

We note that Gen_0, Gen_δ and all their derivatives are non negative for positive z_1 and z_2. It follows easily that for any $\ell, \ell' \ge 0$, we have

$$\begin{gathered} \|f\|_{\ell,\delta} \le \|f\|_{\ell,0}, \qquad \|f\|_{\ell+1,\delta} \le \|f\|_{\ell,\delta}, \\ \|fg\|_{\ell,\delta} \le \|f\|_{\ell',0}\|g\|_{\ell-\ell',\delta}. \end{gathered} \tag{4.5}$$

Next, we have the following Proposition

Proposition 4.2.3. *Let f and g be two functions. For non negative z_1 and z_2, there hold*

$$Gen_\delta(fg) \le Gen_0(f) Gen_\delta(g),$$

$$Gen_\delta(\partial_x f) = \partial_{z_1} Gen_\delta(f), \qquad Gen_\delta(\partial_x^2 f) = \partial_{z_1}^2 Gen_\delta(f),$$

$$Gen_\delta(\varphi\partial_y f) \le C_0 \partial_{z_2} Gen_\delta(f),$$

for some universal constant C_0, provided $|z_2|$ is small enough.

Proof. First, note that

$$(fg)_\alpha = \sum_{\alpha'\in\mathbb{Z}} f_{\alpha'} g_{\alpha-\alpha'},$$

and

$$\partial_y^\beta (fg)_\alpha = \sum_{\alpha'\in\mathbb{Z}} \sum_{0\le\beta'\le\beta} \frac{\beta!}{\beta'!(\beta-\beta')!} \partial_y^{\beta'} f_{\alpha'} \partial_y^{\beta-\beta'} g_{\alpha-\alpha'}.$$

Thus,

$$
\begin{aligned}
&Gen_\delta(fg)(z_1,z_2)\\
&= \sum_{\alpha\in\mathbb{Z}}\sum_{\beta\geq 0} e^{z_1|\alpha|}\|\partial_y^\beta (fg)_\alpha\|_{\beta,\delta}\frac{z_2^\beta}{\beta!}\\
&\leq \sum_{\alpha\in\mathbb{Z}}\sum_{\beta\geq 0}\sum_{\alpha'\in\mathbb{Z}}\sum_{0\leq\beta'\leq\beta} e^{z_1|\alpha|}\|\partial_y^{\beta'} f_{\alpha'}\|_{\beta',0}\|\partial_y^{\beta-\beta'} g_{\alpha-\alpha'}\|_{\beta-\beta',\delta}\frac{z_2^\beta}{\beta'!(\beta-\beta')!}\\
&\leq \sum_{\alpha,\alpha'\in\mathbb{Z}}\sum_{\beta\geq 0}\sum_{\beta\geq\beta'} e^{z_1|\alpha'|}e^{z_1|\alpha-\alpha'|}\|\partial_y^{\beta'} f_{\alpha'}\|_{\beta',0}\|\partial_y^{\beta-\beta'} g_{\alpha-\alpha'}\|_{\beta-\beta',\delta}\frac{z_2^{\beta'}z_2^{\beta-\beta'}}{\beta'!(\beta-\beta')!}\\
&\leq Gen_0(f)(z_1,z_2)Gen_\delta(g)(z_1,z_2).
\end{aligned}
$$

Next, we write

$$
\begin{aligned}
Gen_\delta(\partial_x f) &= \sum_{\alpha\in\mathbb{Z}}\sum_{\ell\geq 0} e^{z_1|\alpha|}\|\alpha\partial_y^\ell f_\alpha\|_{\ell,\delta}\frac{z_2^\ell}{\ell!}\\
&= \partial_{z_1}\sum_{\alpha\in\mathbb{Z}}\sum_{\ell\geq 0} e^{z_1|\alpha|}\|\partial_y^\ell f_\alpha\|_{\ell,\delta}\frac{z_2^\ell}{\ell!} = \partial_{z_1} Gen_\delta(f),
\end{aligned}
$$

and similarly for $Gen_\delta(\partial_x^2 f)$. Finally, we compute

$$
\begin{aligned}
Gen_\delta(\varphi\partial_y f) &= \sum_{\alpha\in\mathbb{Z}}\sum_{\ell\geq 0} e^{z_1|\alpha|}\|\partial_y^\ell(\varphi\partial_y f_\alpha)\|_{\ell,\delta}\frac{z_2^\ell}{\ell!}\\
&\leq \sum_{\alpha\in\mathbb{Z}}\sum_{\ell\geq 0}\sum_{0\leq\ell'\leq\ell} e^{z_1|\alpha|}\|\partial_y^{\ell'}\varphi\partial_y^{\ell-\ell'+1} f_\alpha\|_{\ell,\delta}\frac{z_2^\ell}{\ell'!(\ell-\ell')!}\\
&\leq \Big(1+\sum_{\ell'\geq 0}\|\partial_y^{\ell'}\varphi\|_{0,0}\frac{z_2^{\ell'}}{\ell'!}\Big)\sum_{\alpha\in\mathbb{Z}}\sum_{\ell-\ell'\geq 0} e^{z_1|\alpha|}\|\partial_y^{\ell-\ell'+1} f_\alpha\|_{\ell-\ell'+1,\delta}\frac{z_2^{\ell-\ell'}}{(\ell-\ell')!}\\
&\leq C_0\partial_{z_2} Gen_\delta(f),
\end{aligned}
$$

where we distinguished the cases $\ell' = 0$ and $\ell' > 0$. As φ is analytic, $\sum_{\ell'\geq 0}\|\partial_y^{\ell'}\varphi\|_{0,0} z_2^{\ell'}/\ell'!$ converges provided z_2 is small enough. The Proposition follows. □

4.3 Laplace equation

In this section, we study the Laplace equation and the generator functions of solutions to the Laplace equation. In the latter chapters, we shall apply a similar analysis to the more complex Orr Sommerfeld equations, which is the resolvent equation for linearized Navier-Stokes equations around a boundary layer profile.

4.3.1 In one space dimension

As an exercice we now investigate the analytic regularity of the solution of the classical Laplace equation with damping. Let $\alpha > 0$ be fixed and let us study the following equation

$$\partial_y^2 \phi - \alpha^2 \phi = f \tag{4.6}$$

on the half line $z \geq 0$, with boundary condition

$$\phi(0) = 0. \tag{4.7}$$

The Green function of $\partial_z^2 - \alpha^2$ is

$$G(x,z) = -\frac{1}{2\alpha}\Big(e^{-\alpha|z-x|} - e^{-\alpha|z+x|}\Big) = \begin{cases} G_-(x,z) & 0 \leq x \leq z \\ G_+(x,z), & 0 \leq z \leq x. \end{cases}$$

Or, equivalently, using

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}),$$

we have

$$G_-(x,z) = -\frac{1}{\alpha} e^{-\alpha z} \sinh(\alpha x), \qquad G_+(x,z) = -\frac{1}{\alpha} e^{-\alpha x} \sinh(\alpha z).$$

For real values of z, the solution ϕ of (4.6) is explicitly given by

$$\phi(z) = \int_0^z G_-(x,z) f(x)\, dx + \int_z^\infty G_+(x,z) f(x)\, dx. \tag{4.8}$$

This equality defines the solution for real values of z. It can be extended to any complex value of z provided we replace the integral from 0 to z by the integral over the segment $[0,z]$ and the integral from z to $+\infty$ by the integral on the half line $z + \mathbb{R}_+$.

We will prove the following classical proposition, which asserts that the inversion of the Laplace operator leads to a gain of two derivatives. We recall that

$$\|f\|_{0,0} = \sup_{y \geq 0} |f(y)|.$$

Let us first recall the following classical result:

Proposition 4.3.1. *(L^∞ bounds).*
Let ϕ solve the one-dimensional Laplace problem (4.6)*, with Dirichlet boundary condition. There holds*

$$\alpha^2\|\phi\|_{0,0} + |\alpha|\,\|\partial_y\phi\|_{0,0} + \|\partial_y^2\phi\|_{0,0} \le C\|f\|_{0,0}, \tag{4.9}$$

where the constant C is independent of the integer $\alpha \neq 0$.

Proof. We will only consider the case $\alpha > 0$, the opposite case being similar. The Green function of $\partial_y^2 - \alpha^2$ is

$$G(x,y) = -\frac{1}{2\alpha}\Big(e^{-\alpha|x-y|} - e^{-\alpha|x+y|}\Big)$$

and its absolute value is bounded by $\alpha^{-1}e^{-\alpha|x-y|}$. The solution ϕ of (4.6) is explicitly given by

$$\phi(y) = \int_0^\infty G(x,y)f(x)dx. \tag{4.10}$$

A direct bound leads to

$$|\phi(y)| \le \alpha^{-1}\|f\|_{0,0}\int_0^\infty e^{-\alpha|x-y|}\,dx \le C\alpha^{-2}\|f\|_{0,0}$$

in which the extra α^{-1} factor is due to the x-integration. Splitting the integral formula (4.8) in $x < y$ and $x > y$ and differentiating it, we get

$$\|\partial_y\phi\|_{0,0} \le C\alpha^{-1}\|f\|_{0,0}.$$

We then use the equation to bound $\partial_y^2\phi$, which ends the proof of (4.9). □

Next, in the case when f has a boundary layer behavior, we obtain the following result:

Proposition 4.3.2. *(Boundary layers norms)*
Let ϕ solve the one-dimensional Laplacian problem (4.6) *with Dirichlet boundary condition. Provided*

$$|\delta\alpha^2| \le 1 \tag{4.11}$$

there holds

$$\|\nabla_\alpha\phi\|_{0,0} \le C\|f\|_{0,\delta} \tag{4.12}$$

and

$$|\alpha|^2\,\|\phi\|_{0,0} + \|\partial_y^2\phi\|_{0,\delta} \le C\|f\|_{0,\delta} \tag{4.13}$$

where the constant C is independent of the integer α.

Note that in the case of boundary layer norms, we only gain "one" derivative in supremum norm, but the usual two derivatives in boundary layer norm.

Proof. Using (4.8), we estimate

$$|\phi(y)| \le \alpha^{-1}\|f\|_{0,\delta}\int_0^\infty e^{-\alpha|y-x|}\Big(1+\delta^{-1}e^{-x/\delta}\Big)dx$$
$$\le \alpha^{-1}\|f\|_{0,\delta}\Big(\alpha^{-1}+\delta^{-1}\int_0^\infty e^{-x/\delta}dx\Big)$$

which yields the claimed bound for $\alpha\phi$. The bound on $\partial_y\phi$ is obtained by differentiating (4.8).

Let us turn to (4.13). Note that $|\partial_x G(x,y)| \le 1$. As $G(0,y) = 0$ this gives $|G(x,y)| \le |x|$. Therefore

$$|G(x,y)| \le \min(\alpha^{-1}e^{-\alpha|x-y|}, |x|),$$

and hence

$$|\phi(y)| \le \|f\|_{0,\delta}\int_0^\infty \min(|x|, \alpha^{-1}e^{-\alpha|x-y|})\Big(\delta^{-1}e^{-x/\delta}+1\Big)dx$$
$$\le C\|f\|_{0,\delta}\Big(\delta+\alpha^{-2}\Big)$$

which gives the desired bound when $|\delta\alpha^2| \le 1$. We then use the equation to get the bound on $\|\partial_y^2\phi\|_{0,\delta}$. □

4.3.2 Laplace equation and generator functions

In this section, we will study the generator functions of solutions to the Laplace equation $\Delta\phi = \omega$. In the sequel, it is important to keep in mind that, in the application to Prandtl boundary layer stability, ω will have a boundary layer behavior, namely will behave like $\delta^{-1}e^{-Cy/\delta}$, whereas the stream function ϕ will be bounded in the limit.

Proposition 4.3.3. *Let*

$$\Delta_\alpha\phi_\alpha = \omega_\alpha$$

on $\mathbb{R}_+$ *with the Dirichlet boundary condition* $\phi_{\alpha|_{y=0}} = 0$. *For* $|\delta\alpha^2| \le 1$, *there are positive constants* C_0, θ_0 *so that*

$$Gen_{\delta,\alpha}(\nabla_\alpha^2\phi_\alpha) + Gen_{0,\alpha}(\nabla\phi_\alpha) \le C_0 Gen_{\delta,\alpha}(\omega_\alpha), \tag{4.14}$$

for all z_2 so that $|z_2| \le \theta_0$.
Moreover if $\phi = \Delta^{-1}\omega$ and if $\omega_\alpha = 0$ for all α such that $|\delta\alpha^2| \ge 1$, then

$$Gen_0(\nabla\phi) \le C Gen_\delta(\omega), \tag{4.15}$$

$$\partial_{z_1} Gen_0(\nabla\phi) \le C\partial_{z_1} Gen_\delta(\omega), \tag{4.16}$$

$$\partial_{z_2} Gen_0(\nabla\phi) \le C\partial_{z_2} Gen_\delta(\omega) + Gen_\delta(\omega). \tag{4.17}$$

Proof. For $n \ge 1$, from the elliptic equation $\Delta_\alpha\phi_\alpha = \omega_\alpha$, we compute

$$\Delta_\alpha(\varphi^n\partial_y^n\phi_\alpha) = \varphi^n\partial_y^n\omega_\alpha + 2\partial_y(\varphi^n)\partial_y^{n+1}\phi_\alpha + \partial_y^2(\varphi^n)\partial_y^n\phi_\alpha.$$

Note that

$$\partial_y(\varphi^n)\partial_y^{n+1}\phi_\alpha = n\varphi'\varphi^{n-1}\partial_y^{n+1}\phi_\alpha,$$

and hence the $\|.\|_{0,\delta}$ norm of this term is bounded by $n\|\varphi^{n-1}\partial_y^{n+1}\phi_\alpha\|_{0,\delta}$. Moreover,

$$\partial_y^2(\varphi^n)\partial_y^n\phi_\alpha = \Big(n(n-1)\varphi'^2\varphi^{n-2} + n\varphi''\varphi^{n-1}\Big)\partial_y^n\phi_\alpha$$

whose $\|.\|_{0,\delta}$ norm is bounded by $n(n-1)\|\varphi^{n-2}\partial_y^n\phi_\alpha\|_{0,\delta}$. Using Proposition 4.3.2, we get

$$\begin{aligned}&|\alpha|^2\|\varphi^n\partial_y^n\phi_\alpha\|_{0,0} + \|\partial_y^2(\varphi^n\partial_y^n\phi_\alpha)\|_{0,\delta} + \|\nabla_\alpha(\varphi^n\partial_y^n\phi_\alpha)\|_{0,0}\\ &\le C\|\varphi^n\partial_y^n\omega_\alpha\|_{0,\delta} + Cn\|\varphi^{n-1}\partial_y^{n+1}\phi_\alpha\|_{0,\delta} + Cn(n-1)\|\varphi^{n-2}\partial_y^n\phi_\alpha\|_{0,\delta}.\end{aligned}$$

Expanding the left hand side, we get

$$\begin{aligned}&|\alpha|^2\|\varphi^n\partial_y^n\phi_\alpha\|_{0,0} + \|\varphi^n\partial_y^{n+2}\phi_\alpha\|_{0,\delta} + \|\varphi^n\partial_y^n\nabla_\alpha\phi_\alpha\|_{0,0}\\ &\le C_0\|\varphi^n\partial_y^n\omega_\alpha\|_{0,\delta} + C_0 n\|\varphi^{n-1}\partial_y^{n+1}\phi_\alpha\|_{0,\delta}\\ &+ C_0 n(n-1)\|\varphi^{n-2}\partial_y^n\phi_\alpha\|_{0,\delta} + C_0 n\|\varphi^{n-1}\partial_y^n\phi\|_{0,0}.\end{aligned} \tag{4.18}$$

Let

$$A_n = |\alpha|^2\|\varphi^n\partial_y^n\phi_\alpha\|_{0,0} + \|\varphi^n\partial_y^{n+2}\phi_\alpha\|_{0,\delta} + \|\varphi^n\partial_y^n\nabla_\alpha\phi_\alpha\|_{0,0}.$$

Multiplying by $z_2^n/n!$ and summing over n, we get

$$\begin{aligned}\sum_{n\ge0} A_n\frac{z_2^n}{n!} &\le C_0\sum_{n\ge0}\|\varphi^n\partial_y^n\omega_\alpha\|_{0,\delta}\frac{z_2^n}{n!} + C_0\sum_{n\ge1}A_{n-1}\frac{z_2^n}{(n-1)!}\\ &\quad + C_0\sum_{n\ge2}A_{n-2}\frac{z_2^n}{(n-2)!}\\ &\le C_0\sum_{n\ge0}\|\varphi^n\partial_y^n\omega_\alpha\|_{0,\delta}\frac{z_2^n}{n!} + C_0(z+z^2)\sum_{n\ge0}A_n\frac{z_2^n}{n!},\end{aligned}$$

hence

$$\sum_{n\geq 0} A_n \frac{z_2^n}{n!} \leq C_0' \sum_{n\geq 0} \|\varphi^n \partial_y^n \omega_\alpha\|_{0,\delta} \frac{z_2^n}{n!}$$

provided $|z_2|$ is small enough, which ends the proof of (4.14).

Next (4.15) is a direct consequence of (4.14), just summing in α. If we multiply (4.14) by $|\alpha|$ before summing it, this gives (4.16). Now we multiply (4.18) by $z_2^{n-1}/(n-1)!$ instead of $z_2^n/n!$. This gives

$$\sum_{n\geq 1} A_n \frac{z_2^{n-1}}{(n-1)!} \leq C_0 \sum_{n\geq 1} \|\varphi^n \partial_y^n \omega_\alpha\|_{0,\delta} \frac{z_2^{n-1}}{(n-1)!} + C_0 \sum_{n\geq 1} A_{n-1} n \frac{z_2^{n-1}}{(n-1)!}$$
$$+ C_0 \sum_{n\geq 2} n(n-1) A_{n-2} \frac{z_2^{n-1}}{(n-1)!}.$$

The terms in the right hand side may be absorbed by the left hand side provided z_2 is small enough, except $C_0 A_0$, which is bounded by $Gen_{\delta,\alpha}(\omega_\alpha)$. This ends the proof of the Proposition. □

4.4 Divergence free vector fields

4.4.1 Generator function and divergence free condition

Note that for any functions u and g, Proposition 4.2.3 yields

$$Gen_\delta(u\partial_x g) \leq Gen_0(u)\partial_{z_1} Gen_\delta(g). \tag{4.19}$$

This is not true for $Gen_\delta(v\partial_y g)$, due to the boundary layer weight. We will investigate $Gen_\delta(v\partial_y g)$ when (u, v) satisfies the divergence free condition, namely

$$\partial_x u + \partial_y v = 0.$$

Precisely, we will prove the following Proposition.

Proposition 4.4.1. *For $|z_2| \leq 1$, there holds*

$$Gen_\delta(v\partial_y g) \leq C\Big(Gen_0(v) + \partial_{z_1} Gen_0(u)\Big)\partial_{z_2} Gen_\delta(g).$$

Note that we "loose" one derivative: our bound involves $\partial_x u$.

Proof. We compute

$$Gen_\delta(v\partial_y g) = \sum_{\alpha\in\mathbb{Z}} \sum_{\beta\geq 0} e^{z_1|\alpha|} \|\partial_y^\beta (v\partial_y g)_\alpha\|_{\beta,\delta} \frac{z_2^\beta}{\beta!},$$

in which

$$\partial_y^\beta (v\partial_y g)_\alpha = \sum_{\alpha'\in\mathbb{Z}} \sum_{0\le\beta'\le\beta} \frac{\beta!}{\beta'!(\beta-\beta')!} \partial_y^{\beta'} v_{\alpha'} \partial_y^{\beta-\beta'+1} g_{\alpha-\alpha'}.$$

For $\beta' > 0$, using the divergence-free condition $\partial_y v_\alpha = -i\alpha u_\alpha$, we estimate

$$\|\partial_y^{\beta'} v_{\alpha'} \partial_y^{\beta-\beta'+1} g_{\alpha-\alpha'}\|_{\beta,\delta} \le \|\alpha' \partial_y^{\beta'-1} u_{\alpha'}\|_{\beta'-1,0} \|\partial_y^{\beta-\beta'+1} g_{\alpha-\alpha'}\|_{\beta-\beta'+1,\delta}.$$

On the other hand, for $\beta' = 0$, we estimate

$$\|v_{\alpha'} \partial_y^{\beta+1} g_{\alpha-\alpha'}\|_{\beta,\delta} \le \|\varphi^{-1} v_{\alpha'}\|_{0,0} \|\partial_y^{\beta+1} g_{\alpha-\alpha'}\|_{\beta+1,\delta}.$$

We note that for $y \ge 1$, $\varphi(y) \ge 1/2$ and hence

$$\|\chi_{\{y\ge1\}} \varphi^{-1} v_{\alpha'}\|_{0,0} \le 2\|v_{\alpha'}\|_{0,0}.$$

When $y \le 1$, using again the divergence-free condition, we write

$$v_{\alpha'}(y) = -i\alpha' \int_0^y u_{\alpha'}(y')dy' = -i\alpha' y \int_0^1 u_{\alpha'}(x,\theta y)d\theta.$$

Therefore,

$$\varphi(y)^{-1} |v_{\alpha'}(y)| \le \sup_y |\alpha' u_{\alpha'}(y)|$$

for $y \le 1$. This proves that

$$\|\varphi^{-1} v_{\alpha'}\|_{0,0} \le 2\|v_{\alpha'}\|_{0,0} + \|\alpha' u_{\alpha'}\|_{0,0}.$$

Combining these inequalities for any $\alpha \in \mathbb{Z}$ and $\beta \ge 0$, we obtain

$$\|\partial_y^\beta (v\partial_y g)_\alpha\|_{\beta,\delta} \le \sum_{\alpha'\in\mathbb{Z}} (2\|v_{\alpha'}\|_{0,0} + \|\alpha' u_{\alpha'}\|_{0,0}) \|\partial_y^{\beta+1} g_{\alpha-\alpha'}\|_{\beta+1,\delta}$$

$$+ \sum_{\alpha'\in\mathbb{Z}} \sum_{1\le\beta'\le\beta} \|\alpha' \partial_y^{\beta'-1} u_{\alpha'}\|_{\beta'-1,0} \|\partial_y^{\beta-\beta'+1} g_{\alpha-\alpha'}\|_{\beta-\beta'+1,\delta} \frac{\beta!}{\beta'!(\beta-\beta')!}.$$

It remains to multiply by $e^{z_1|\alpha|} z_2^\beta/\beta!$ and to sum all the terms over α, α', β and β'. The second term in the right hand side is bounded by the product of

$$\sum_\alpha \sum_\beta e^{|\alpha|z_1} \|\alpha \partial_y^\beta u_\alpha\|_{\beta,0} \frac{z_2^{\beta+1}}{(\beta+1)!},$$

which is bounded by $Gen_0(\partial_x u)$ provided $|z_2| \le 1$ and of

$$\sum_\alpha \sum_\beta e^{|\alpha|z_1} \|\partial_y^{\beta+1} g_\alpha\|_{\beta+1} \frac{z_2^\beta}{\beta!},$$

which equals $\partial_{z_2} Gen_\delta(g)$. The first term is similar, which ends the proof. □

4.4.2 Bilinear estimates

Let us now bound derivatives of the transport term $u\partial_x g + v\partial_y g$.

Proposition 4.4.2. *Let*

$$\mathcal{A} = \Big(Id + \partial_{z_1} + \partial_{z_2}\Big) Gen_\delta$$

and

$$\mathcal{B} = Gen_0(u) + Gen_0(v) + \partial_{z_1} Gen_0(u) + \mathcal{A}(g).$$

Then

$$\mathcal{A}(u\partial_x g + v\partial_y g) \le C\mathcal{B}\partial_{z_1}\mathcal{B} + C\mathcal{B}\partial_{z_2}\mathcal{B}.$$

Note that all the terms in $\mathcal{A}$ are non negative, since all the derivatives of generator functions are non negative.

Proof. Let us successively bound all the terms appearing in $\mathcal{A}(u\partial_x g + v\partial_y g)$. First, $Gen_\delta(u\partial_x g + v\partial_y g)$ has been bounded in (4.19) and in the previous proposition. Next we compute

$$\begin{aligned} \partial_{z_1} Gen_\delta(u\partial_x g) &= Gen_\delta(\partial_x(u\partial_x g)) = Gen_\delta(\partial_x u \partial_x g + u\partial_x^2 g) \\ &\le \partial_{z_1} Gen_0(u)\partial_{z_1} Gen_\delta(g) + Gen_0(u)\partial_{z_1}^2 Gen_\delta(g). \end{aligned} \tag{4.20}$$

Moreover, using Proposition 4.4.2,

$$\begin{aligned} \partial_{z_1} Gen_\delta(v\partial_y g) &= Gen_\delta(\partial_x(v\partial_y g)) = Gen_\delta(\partial_x v\partial_y g + v\partial_y\partial_x g) \\ &\le C(\partial_{z_1} Gen_0(v) + \partial_{z_1}^2 Gen_0(u))\partial_{z_2} Gen_\delta(g) \\ &\quad + C(Gen_0(v) + \partial_{z_1} Gen_0(u))\partial_{z_2}\partial_{z_1} Gen_\delta(g). \end{aligned}$$

Let us now bound the term $\partial_{z_2} Gen_\delta(v\partial_y g)$. Precisely, we have to bound

$$\begin{aligned} &\frac{z_2^n}{n!}\|\varphi^{n+1}\partial_y^{n+1}(v_{\alpha'}\partial_y g_{\alpha-\alpha'})\|_{0,\delta} = \frac{z_2^n}{n!}\|\varphi^{n+1}\partial_y^n(\partial_y v_{\alpha'}\partial_y g_{\alpha-\alpha'} + v_{\alpha'}\partial_y^2 g_{\alpha-\alpha'})\|_{0,\delta} \\ &\le \sum_{0\le k\le n}\frac{z_2^n}{k!(n-k)!}\|\varphi^{n+1}\partial_y^{k+1} v_{\alpha'}\partial_y^{n+1-k} g_{\alpha-\alpha'} + \varphi^{n+1}\partial_y^k v_{\alpha'}\partial_y^{n+2-k} g_{\alpha-\alpha'}\|_{0,\delta}. \end{aligned}$$

Let us split this sum in two. The first sum equals, using the divergence free condition,

$$\begin{aligned} &\sum_{0\le k\le n}\frac{z_2^n}{k!(n-k)!}\|\varphi^k\partial_y^k\partial_x u_{\alpha'}\ \varphi^{n+1-k}\partial_y^{n+1-k} g_{\alpha-\alpha'}\|_{0,\delta} \\ &\le \sum_{0\le k\le n}\frac{z_2^k}{k!}\|\varphi^k\partial_y^k\partial_x u_{\alpha'}\|_{0,0}\frac{z_2^{n-k}}{(n-k)!}\|\varphi^{n+1-k}\partial_y^{n+1-k} g_{\alpha-\alpha'}\|_{0,\delta}. \end{aligned}$$

Multiplying by $e^{|\alpha| z_1}$ and summing over α and α', the sum is bounded by

$$Gen_0(\partial_x u)\partial_{z_2} Gen_\delta(g) = \partial_{z_1} Gen_0(u)\partial_{z_2} Gen_\delta(g).$$

On the other hand, the second sum equals to

$$\sum_{0\le k\le n} \frac{z_2^n}{k!(n-k)!} \|\varphi^{n+1}\partial_y^k v_{\alpha'}\ \partial_y^{n+2-k} g_{\alpha-\alpha'}\|_{0,\delta}. \tag{4.21}$$

We follow the proof of the previous Proposition. First, for $k > 0$, this sum equals to

$$\sum_{1\le k\le n} \frac{z_2^n}{k!(n-k)!} \|\varphi^{k-1}\partial_y^{k-1}\partial_x u_{\alpha'}\ \varphi^{n+2-k}\partial_y^{n+2-k} g_{\alpha-\alpha'}\|_{0,\delta}.$$

Multiplying by $e^{|\alpha| z_1}$, the corresponding sum is bounded by

$$\partial_{z_1} Gen_0(u)\partial_{z_2}^2 Gen_\delta(g),$$

provided that $|z_2| \le 1$. It remains to bound the term $k = 0$ in (4.21):

$$\frac{z_2^n}{n!}\|\varphi^{n+1} v_{\alpha'}\partial_y^{n+2} g_{\alpha-\alpha'}\|_{0,\delta} \le \Big(2\|v_{\alpha'}\|_{0,0} + \|\alpha' u_{\alpha'}\|_{0,0}\Big)\frac{z_2^n}{n!}\|\varphi^{n+2}\partial_y^{n+2} g_{\alpha-\alpha'}\|_{0,\delta}.$$

Multiplying by $e^{|\alpha| z_1}$, the corresponding sum is bounded by

$$(Gen_0(v) + \partial_{z_1} Gen_0(u))\partial_{z_2}^2 Gen_\delta(g).$$

This leads to

$$\begin{aligned}\partial_{z_2} Gen_\delta(v\partial_y g) \le{}& \partial_{z_1} Gen_0(u)\partial_{z_2} Gen_\delta(g)\\ &+ C_0(Gen_0(v) + \partial_{z_1} Gen_0(u))\partial_{z_2}^2 Gen_\delta(g).\end{aligned}$$

The bound on $\partial_{z_2} Gen_\delta(u\partial_x g)$ is similar which ends the proof of this Proposition. □

4.5 Applications of generator functions to instability

In this section, we introduce a framework to use the notion of generator functions in proving instability results. We first apply it to a toy model equation and then carry out the analysis for Euler equations. Later on, we shall use this approach to prove the instability of boundary layers.

4.5.1 A toy model equation

We will now use generator functions to prove an instability result for a toy model equation. More precisely, let us consider the following classical Hopf equation

$$\partial_t u + u\partial_z u = \alpha u \tag{4.22}$$

in the periodic setting, where $\alpha > 0$, with initial data $u(0,z) = u_1(z)$. Assume that u_1 is analytic, its generator $Gen(u_1)$ is defined on some interval $[-z_0, z_0]$. We will look for an instability solution of the form

$$u = \sum_{n\geq 1} e^{n\alpha t} u_n. \tag{4.23}$$

Putting (4.23) into (4.22), we get the recurrence relation

$$(n-1)\alpha u_n = -\sum_{1\leq k\leq n-1} u_k \partial_z u_{n-k}.$$

We will prove that the series (4.23) is convergent provided $e^{\alpha t}$ is small enough. The instability of (4.22) thus follows from the growth $e^{\alpha t}$ encoded in the series (4.23). Precisely, we have

Theorem 4.5.1. *There exists a positive T_0 such that the series (4.23) converges for every $t \leq T_0$.*

Proof. For $n \geq 2$, we compute

$$(n-1)\alpha Gen(u_n) \leq \sum_{1\leq k\leq n-1} Gen(u_k)\partial_z Gen(u_{n-k}).$$

Therefore, for $n \geq 2$, multiplying by t^{n-2}, we get

$$(n-1)\alpha Gen(u_n)t^{n-2} \leq \sum_{1\leq k\leq n-1} Gen(u_k)t^{k-1}\partial_z Gen(u_{n-k})t^{n-k-1}.$$

Summing over n, we thus obtain

$$\alpha \sum_{n\geq 2}(n-1)Gen(u_n)t^{n-2} \leq \Big(\sum_{k\geq 1} Gen(u_k)t^{k-1}\Big)\partial_z\Big(\sum_{k\geq 1} Gen(u_k)t^{k-1}\Big).$$

Let

$$G(t,z) = \sum_{k\geq 1} Gen(u_k)t^{k-1}.$$

Then

$$\alpha\partial_t G - G\partial_z G \leq 0. \tag{4.24}$$

Therefore, G satisfies an Hopf-type differential inequality. Note that G is convex, and more precisely, all its derivatives, at all orders, are non negative for $t > 0$. We would like to deduce from this inequality that G is defined on some non vanishing time interval. However two difficulties arise. First we do not know whether G converges, and second as G goes rapidly to $+\infty$, the existence time for the Hopf differential equation could be zero, since the characteristics coming from the right can reach $x = 0$ in a vanishing time.

To fix the first point, we first truncate Gen and define G_N to be

$$G_N(t,z) = \sum_{k\leq N} Gen_{N-k}(u_k)(z)t^{k-1}$$

and derive a similar Hopf inequality for G_N. For $k = 1, \cdots, n-1$, we note that $N - k \geq N - n$ and $N - n + k \geq N - n + 1$. Hence,

$$(n-1)\alpha Gen_{N-n}(u_n) \leq \sum_{1\leq k\leq n-1} Gen_{N-k}(u_k)\partial_z Gen_{N-n+k}(u_{n-k}).$$

Summing over n we get

$$\alpha\sum_{n\geq 2}(n-1)Gen_{N-n}(u_n)t^{n-2} \leq \Big(\sum_k Gen_{N-k}(u_k)t^{k-1}\Big)\partial_z\Big(\sum_k Gen_{N-k}(u_k)t^{k-1}\Big),$$

which yields the Hopf differential inequality

$$\alpha\partial_t G_N - G_N\partial_z G_N \leq 0. \tag{4.25}$$

Here, we know that $G_N(t,z)$ is defined for every t and every z. Hence (4.25) holds true for any $t \geq 0$ and any $z \geq 0$.

It remains to fix the second issue, namely to transform the inwards characteristics into outwards ones. For this we introduce

$$H_N(t,z) = G_N(t,\phi(t)z), \qquad H(t,z) = G(t,\phi(t)z).$$

A direct computation yields

$$\begin{aligned}\alpha\partial_t H_N &= \alpha\partial_t G_N + \alpha z\phi'(t)\partial_z G_N \\ &\leq \Big(\alpha z\phi'(t) + G_N\Big)\partial_z G_N \\ &\leq \Big(H_N + \alpha z\phi'(t)\Big)\phi^{-1}(t)\partial_z H_N\end{aligned} \tag{4.26}$$

We will choose ϕ such that $\phi(0) = 1$. Then

$$H_N(0, z) \leq H(0, z) = G(0, z) = Gen(u_1)(z).$$

As u_1 is holomorphic, $Gen(u_1)$ is defined near 0 on $[0, \eta_0]$ for some positive η_0. Let

$$M_0 = \sup_{0 \leq z \leq \eta_0} Gen(u_1)(z).$$

We will study (4.26) on $[0, \eta_0]$, and focus on an interval $[0, T_N]$ such that

$$\sup_{0 \leq z \leq \eta_0, 0 \leq t \leq T_N} G_N(t, z) \leq 2M_0.$$

We define $\phi(t)$ such that

$$2M_0 + \alpha\eta_0\phi'(t) = -M_0,$$

or equivalently,

$$\phi(t) = 1 - \frac{3M_0}{\alpha\eta_0} t.$$

We also take $T \leq \alpha\eta_0/6M_0$ so that $\phi(t) \geq 1/2$.

We now introduce the characteristics defined by

$$X_N(0, z) = 0,$$

$$\partial_t X_N(t, z) = -\phi^{-1}(t) H_N(t, X_N(t, z)) - \alpha z \phi^{-1}(t)\phi'(t).$$

Note that on 0 and η_0, the characteristics of (4.26) are outgoing. There is no need to prescribe a boundary condition on H_N. Moreover, the characteristics are defined for all $t \leq T$ and do not cross since both H_N and $\partial_z H_N$ are bounded for $t \leq T_N$ and $0 \leq z \leq \eta_0$ (by some constant which may depend on N). We may therefore make a change of variables and define

$$K_N(t, z) = H_N(t, X_N(t, z)).$$

Then, by definition of K_N,

$$\partial_t K_N \leq 0.$$

In particular, as K_N is positive,

$$K_N \leq \sup_{0 \leq z \leq \eta_0} H(0, z) = M_0.$$

Therefore $H_n(t, z)$ is bounded by M_0 itself for $0 \leq t \leq T$ and $z \leq \eta_0$. We can let N pass to the limit. This gives that H exists and is well defined for z and t small enough. Hence the serie (4.23) converges provided $e^{\alpha t}$ is small enough. □

4.5.2 Application to Euler equations

We now extend the previous theorem to Euler equations on $\mathbb{T}^2$. Let $U_s = (U,0)^t$ be a given shear flow. We look for solutions of Euler equations of the form $U_s + u$, which leads to

$$\partial_t \omega + \mathcal{L}\omega = Q(u,\omega) \tag{4.27}$$

where

$$\mathcal{L}\omega = U_s \cdot \nabla \omega + u \cdot \nabla \Omega_s, \qquad Q(u,\omega) = -(u\cdot\nabla)\omega.$$

Here and in what follows, velocity u is computed by vorticity ω through the Biot-Savart law $u = \nabla^\perp \Delta^{-1}\omega$. We assume that the linearized operator $\mathcal{L}$ has an unstable eigenmode, of the form $\omega_1 e^{\alpha t}$, and that

$$(H1) \qquad Gen\Big((\lambda+\mathcal{L})^{-1} f\Big) \le \frac{C}{\lambda} Gen(f) \qquad \forall \lambda, \qquad \Re\lambda > \frac{3}{2}\Re\alpha$$

for some universal constant C. We start with

$$\omega_1 = \Re(\omega_1 e^{\alpha t})$$

and iteratively construct ω_n by

$$(\alpha n + \mathcal{L})\omega_n = \sum_{1\le j\le n-1} Q(u_j, \omega_{n-j}), \tag{4.28}$$

where $u_n = \nabla^\perp\Delta^{-1}\omega_n$ is the velocity associated to vorticity ω_n.

Theorem 4.5.2. *Under Assumption (H1), there exists a positive time T_0 such that, for $t \le T_0$, the series*

$$\sum_{n\ge 0} e^{n\alpha t}\omega_n$$

converges and is a solution of (4.27).

As a consequence, U_s is nonlinearly unstable in L^∞. The proof of assumption (H1) relies on a careful of Rayleigh equation, which will be detailed later.

Proof. The proof follows a similar line as done for the toy model equation in the previous section. Indeed, by construction (4.28) and the Assumption

(H1), for $n \geq 1$, we have

$$\begin{aligned} Gen(\omega_n) &\leq \frac{C}{n\alpha} \sum_{1\leq j\leq n-1} Gen(u_j \cdot \nabla \omega_{n-j}) \\ &\leq \frac{C}{n\alpha} \sum_{1\leq j\leq n-1} \Big[Gen(\omega_j)\partial_{z_1} + Gen(\omega_j)\partial_{z_2} \Big] Gen(\omega_{n-j}), \end{aligned}$$

upon recalling the elliptic estimates

$$Gen(u_n) \leq C Gen(\omega_n).$$

For convenience, set $G^n(z_1, z_2) = Gen(\omega_n)(z_1, z_2)$. The functions $G^n(z_1, z_2)$ are well-defined for sufficiently small z_1, z_2, and in addition, there exists some universal constant C_0 so that

$$G^n \leq \frac{C_0}{n} \sum_{1\leq j\leq n-1} \Big(G^j \partial_{z_1} G^{n-j} + G^j \partial_{z_2} G^{n-j} \Big). \tag{4.29}$$

As in the previous section, for $N \geq 1$, we introduce the partial sum

$$G_N(\tau, z_1, z_2) := \sum_{n=1}^{N} G^n(z_1, z_2)\tau^{n-1},$$

for $\tau, z_1, z_2 \geq 0$. Note that G_N is a polynomial in τ, and thus well-defined for all times $\tau \geq 0$. We also note that all the coefficients $G^n(z_1, z_2)$ are positive. In particular, $G_N(\tau, z_1, z_2)$ is positive, and so are all its time derivatives (when $z_1 > 0$ and $z_2 > 0$). Moreover, $G_N(\tau, z_1, z_2)$, and all its derivatives, are increasing in N. We also observe that, at $\tau = 0$,

$$G_N(0, z_1, z_2) = G^1(z_1, z_2),$$

for all $N \geq 1$, and hence,

$$G(0, z_1, z_2) = \lim_{N\to\infty} G_N(0, z_1, z_2) = G^1(z_1, z_2).$$

Next, multiplying (11.47) by τ^{n-2} and summing up the result, we obtain the following partial differential inequality

$$\partial_\tau G_N \leq C G_N \partial_{z_1} G_N + C G_N \partial_{z_2} G_N,$$

for all $N \geq 1$. That is, the generator function satisfies an Hopf-type equation, or more precisely an Hopf differential inequality. Thus, the theorem follows from the same argument as done in the previous section. $\square$

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