Onsager type conjecture and renormalized solutions for the relativistic Vlasov–Maxwell system

Claude Bardos [*] Nicolas Besse [†] Toan T. Nguyen [‡]

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This paper is dedicated to Walter Strauss
on the occasion of his 80th birthday, as token of friendship and admiration
in particular for his contribution to the mathematical theory of Vlasov-Maxwell systems.

Abstract

In this paper we give a proof of an Onsager type conjecture on conservation of energy and entropies of weak solutions to the relativistic Vlasov–Maxwell equations. As concerns the regularity of weak solutions, say in Sobolev spaces $W^{\alpha,p}$, we determine Onsager type exponents α that guarantee the conservation of all entropies. In particular, the Onsager exponent α is smaller than $\alpha = 1/3$ established for fluid models. Entropies conservation is equivalent to the renormalization property, which have been introduced by DiPerna–Lions for studying well-posedness of passive transport equations and collisionless kinetic equations. For smooth solutions renormalization property or entropies conservation are simply the consequence of the chain rule. For weak solutions the use of the chain rule is not always justified. Then arises the question about the minimal regularity needed for weak solutions to guarantee such properties. In the DiPerna–Lions and Bouchut–Ambrosio theories, renormalization property holds under sufficient conditions in terms of the regularity of the advection field, which are roughly speaking an entire derivative in some Lebesgue spaces (DiPerna–Lions) or an entire derivative in the space of measures with finite total variation (Bouchut–Ambrosio). In return there is no smoothness requirement for the advected density, except some natural a priori bounds. Here we show that the renormalization property holds for an electromagnetic field with only a fractional space derivative in some Lebesgue spaces. To compensate this loss of derivative for the electromagnetic field, the distribution function requires an additional smoothness, typically fractional Sobolev differentiability in phase-space. As concerns the conservation of total energy, if the macroscopic kinetic energy is in L^2, then total energy is preserved.

Keywords: Relativistic Vlasov–Maxwell system, Onsager's conjecture, entropies conservation, renormalization property, energy conservation.

[*]Laboratoire J.-L. Lions, Université Pierre et Marie Curie - Paris 6 BP 187, 4 place Jussieu, 75252 Paris, Cedex 5, France (claude.bardos@gmail.com)

[†]Laboratoire J.-L. Lagrange, UMR CNRS/OCA/UCA 7293, Université Côte d'Azur, Observatoire de la Côte d'Azur, Bd de l'observatoire CS 34229, 06300 Nice, Cedex 4, France. (Nicolas.Besse@oca.eu)

[‡]Department of Mathematics, Penn State University, State College, PA, 16803, USA (nguyen@math.psu.edu)

1 Introduction

The dimensionless relativistic Vlasov–Maxwell system reads,

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_\xi f = 0, \tag{1}$$

$$\partial_t E - \nabla \times B = -j, \quad \partial_t B + \nabla \times E = 0, \tag{2}$$

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0, \tag{3}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$, and $v = \xi/\sqrt{1+|\xi|^2}$ represent time, position, momentum and velocity of particles, respectively. The distribution function of particles $f = f(t,x,\xi)$ satisfies the Vlasov equation (1) with acceleration given by the Lorentz force $F_L = E + v \times B$, while the electromagnetic field $E = E(t,x)$ and $B = B(t,x)$ satisfies Maxwell's equations (2)-(3). The coupling between the Vlasov equation and Maxwell's equations occurs through the source terms of Maxwell's equations, which are the charge density $\rho = \rho(t,x)$ and the current density $j = j(t,x)$. These densities are defined as the first ξ-moments of the phase-space density of particles f, namely,

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\, d\xi, \quad j(t,x) = \int_{\mathbb{R}^3} v f(t,x,\xi)\, d\xi. \tag{4}$$

The initial value problem associated to the system (1)-(4) requires initial conditions given by,

$$f(0,x,\xi) = f_0(x,\xi) \geq 0, \tag{5}$$

$$E(0,x) = E_0(x), \quad B(0,x) = B_0(x), \quad \nabla \cdot E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0\, d\xi, \quad \nabla \cdot B_0 = 0. \tag{6}$$

In addition for the well-posedness of Maxwell's equations (2)-(3), the densities of charge ρ and current j must satisfy a compatibility condition given by the charge conservation law,

$$\partial_t \rho + \nabla \cdot j = 0. \tag{7}$$

This continuity equation is automatically satisfied if the Vlasov equation (1) is satisfied since it can be recovered by integration in momentum variable of the Vlasov equation. Let us note that Maxwell–Gauss equations (3) are satisfied at any time if they are satisfied initially. Indeed, it is a consequence of time integration of the divergence of the Maxwell–Faraday–Ampère equations (2), in combination with the continuity equation (7) and initial conditions (6).

The Vlasov equation (1) has, at least formally, infinitely many invariants. Indeed, let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ be any smooth function. Multiplying (1) with $\mathcal{H}'(f)$ and applying the chain rule, we then obtain,

$$\partial_t \mathcal{H}(f) + v \cdot \nabla_x \mathcal{H}(f) + (E + v \times B) \cdot \nabla_\xi \mathcal{H}(f) = 0. \tag{8}$$

A solution f to (1) in the sense of distributions is said to be a renormalized solution if for any smooth nonlinear function $\mathcal{H}$, f also solves (8) in the sense of distributions. We say that the field (v, F_L) satisfies the renormalization property if any solution f to (1) in the sense of distributions is a renormalized solution. The renormalization technique appeared in the well-posedness of passive advection equations and ODEs [37], in the analysis of the Boltzmann equation [38], in the theory of weak solutions of the compressible Navier-Stokes equations [57] and in the theory of weak solutions of collisionless kinetic equations such as the Vlasov–Poisson system [34, 35]. The groundbreaking work [37] has highlighted the fundamental link between renormalized solutions to the passive transport equation,

$$\partial_t u + b \cdot \nabla u = 0, \quad u : [0,T] \times \mathbb{R}^d \to \mathbb{R}, \quad b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, \tag{9}$$

and the well-posedness theory for the associated ODE,

$$\partial_t X(t,x) = b(t, X(t,x)), \quad t \in [0,T], \quad X(0,x) = x \in \mathbb{R}^d, \tag{10}$$

where b is a non-smooth vector field. Similarly to entropy conditions for hyperbolic conservations laws, renormalization property provides additional stability under weak convergence. Indeed renormalized solutions come with a comparison principle, which allows to show uniqueness of renormalized solutions and some stability results for sequences of solutions. In return uniqueness at the PDE level (9) implies uniqueness at the ODE level (10). It was first show in [37] that the renormalization property holds provided $b \in L^1_t W^{1,p}_x$ with $p \geq 1$, plus a bounded divergence and a global space growth estimate on b (see also [55] for the case $W^{1,1}_{\text{loc}}$). Moreover, there is no additional regularity assumption for u except its boundedness or some L^p-bounds. This result was extended to $b \in L^1_t BV_x$ with $\nabla \cdot b \in L^1_{t,x}$, first in [23] for the Vlasov equation (see also [53] for a related result), and then in [9] for the general case (see also [28]). Very recently, in [11] the authors develop a local version of the DiPerna–Lions' theory under no global assumptions on the growth estimate of b. We refer the reader to [10] for a recent survey.

For the Vlasov–Poisson system when f is merely L^1, the product Ef does not belong to L^1_{loc}. Therefore higher integrability assumptions on f are needed to give a meaning to the Vlasov–Poisson equation in the sense of distributions. For example, when $d = 3$, for the term Ef to belong to L^1_{loc} one needs to have $f \in L^p$ with $p = (12+3\sqrt{5})/11$ (see for instance [34, 35]). To drop out this higher integrability hypotheses, in [34, 35] the authors considered the concept of renormalized solutions and obtained global existence provided that the total energy is finite and $f_0 \log(1 + f_0) \in L^1$. In addition, under some suitable integrability hypotheses on f, they can show that the concepts of weak and renormalized solutions are equivalent. For bounded density f, renormalization property holds because elliptic regularity of the Poisson equation leads to $E \in W^{1,p}$, with $p > 1$ (see [34, 35]). For the Vlasov–Maxwell system the only available global existence result is in [36], where the authors have constructed weak solutions for which it is not possible to show the renormalization property. Indeed, the best electromagnetic-field regularity, obtained so far for the DiPerna–Lions weak solutions, is in [22], where the authors show that the electromagnetic field (E, B) belongs to $H^s_{\text{loc}}(\mathbb{R}^+_* \times \mathbb{R}^3)$, with $s = 6/(13 + \sqrt{142})$, if the macroscopic kinetic energy is in L^2.

Regularity of rough vector field considered above, i.e Sobolev or BV vector fields, is somehow like the Lipschitz case because there is always a control (in Lebesgue spaces or in the space of measures with finite total variation) on an entire derivative of the vector field. By contrast, when b is not Lipschitz-like, the use of the chain rule is no longer justified, and many counterexamples to renormalization have been obtained in [2, 33, 29, 6, 7, 4, 5, 30, 31, 65].

Here, we show that the renormalization property holds for an electromagnetic field with only a fractional derivative in some Lebesgue spaces, i.e. $E, B \in L^\infty_t W^{\beta,q}_x$, with $0 < \beta < 1$ and $1 \leq q \leq \infty$. To compensate this loss of derivative for the electromagnetic field, the density f requires additional smoothness, typically fractional Sobolev differentiability in phase-space, i.e. $f \in L^1_t W^{\alpha,p}_{x,\xi}$, with $0 < \alpha < 1$ and $1 \leq p \leq \infty$. We determine Onsager type exponents [44] α and β, which ensure conservation of all entropies and guarantee that the renormalization property holds. As concerns the conservation of total energy, if the macroscopic kinetic energy is in L^2, we then show that total energy is preserved. A comparable work has been done in [3] for the renormalization of an active scalar transport equation.

A similar situation occurs with systems of conservation laws of continuum physics, which are endowed with natural companion laws: the so called the entropy conditions (inequality versus equality) coming from the second law of thermodynamics. In [52, 20] the authors have determined the critical regularity of weak solutions to a general system of conservation laws to satisfy an associated entropy conservation law as an equality. They obtained the famous Onsager exponent 1/3 [58]. The first result of this kind was obtained in [32] (see also [43]), where the authors have shown that weak solutions of the incompressible Euler equations conserve energy provided they possess fractional Besov differentiability of order greater than 1/3. Such result has been extended in various directions: In [41, 27, 47] the Besov criterium has been optimized; in [56, 45, 67, 39, 52] the authors have considered compressible Euler, Navier-Stokes and magnetohydrodynamic equations; works [60, 61, 18, 19, 40] include boundary effects.

2 Basic properties

In this section, we recall the basic properties of the relativistic Vlasov–Maxwell system, which are valid for any smooth solution (f, E, B), vanishing at infinity. Theses formal properties, in particular natural a priori estimates, are the key cornerstones for proving the local-in-time well-posedness of this system [48]. Consider the following set of equations,

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_\xi f = 0, \quad (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{11}$$

$$\partial_t E - \nabla \times B = -j, \quad \partial_t B + \nabla \times E = 0, \tag{12}$$

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, \xi)\, d\xi, \quad j(t, x) = \int_{\mathbb{R}^3} v f(t, x, \xi)\, d\xi, \tag{13}$$

$$\gamma = \sqrt{1 + |\xi|^2}, \quad v = \nabla_\xi \gamma = \frac{\xi}{\sqrt{1 + |\xi|^2}}. \tag{14}$$

Observe that once the current density j and initial data (E_0, B_0) are given, the Maxwell equations (12) are well defined. Indeed the Maxwell operator M defined by,

$$X \mapsto MX = \begin{pmatrix} -\nabla \times B \\ \nabla \times E \end{pmatrix}, \quad \text{with} \quad X = \begin{pmatrix} E \\ B \end{pmatrix}, \tag{15}$$

is the generator of a strongly continuous unitary group $t \mapsto S(t) := \exp(-iMt)$ in $L^2(\mathbb{R}^3)$ [66, 42, 14]. If $(E(t = 0), B(t = 0)) = (E_0, B_0) \in L^2(\mathbb{R}^3)$ and $j \in L^1(\mathbb{R}^+; L^2(\mathbb{R}^3))$, then, using the properties of the group $S(t)$ and the Duhamel formula, we can show that the the solution (E, B) to (12) belongs to $\mathscr{C}(\mathbb{R}^+; L^2(\mathbb{R}^3))$. Moreover for any $s \geq 0$, the H^s regularity is preserved, i.e. the previous statement remains valid if we replace $L^2(\mathbb{R}^3)$ by $H^s(\mathbb{R}^3)$. In the same way, once the smooth electromagnetic field (E, B) and initial data $f_0(x, \xi)$ are given, the Vlasov equation is then well defined. Indeed, introducing the characteristics curves $t \mapsto (X(t), \Xi(t))$, which are the unique and smooth solution to the ODEs,

$$\frac{dX}{dt}(t) = v(\Xi(t)), \quad \frac{d\Xi}{dt}(t) = E(t, X(t)) + v(\Xi(t)) \times B(t, X(t)), \tag{16}$$

$$X(0; 0, x, \xi) = x, \quad \Xi(0; 0, x, \xi) = \xi, \tag{17}$$

the Lagrangian solution to (11) is given by (e.g., see [24])

$$f(t,x,\xi) = f_0(X(0;t,x,\xi),\Xi(0,t,x,\xi)). \tag{18}$$

The relativistic Vlasov–Maxwell system (11)-(14) satisfies some formal conservation laws, summarized in

Proposition 1 *Let (f, E, B) be a smooth solution, vanishing at infinity, to the relativistic Vlasov–Maxwell system (11)-(14). Then the following a priori estimates hold:*

1. *(Maximum principle).* $0 \le m \le f_0 \le M < \infty$ *implies* $m \le f(t) \le M$ *for all* $t > 0$.
2. *(L^p-norm conservation). For all $t \ge 0$, and $1 \le p \le \infty$, one has,* $\|f(t)\|_{L^p(\mathbb{R}^6)} = \|f_0\|_{L^p(\mathbb{R}^6)}$.
3. *(Entropies). For any function $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+;\mathbb{R}^+)$, one has for all $t \ge 0$,*
$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{H}(f(t))\, d\xi dx = 0.$$
4. *(Energy conservation). For all $t \ge 0$ one has,*
$$\frac{d}{dt}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(\gamma(\xi)-1)f(t)\, d\xi dx + \frac{1}{2}\int_{\mathbb{R}^3}(|E(t)|^2 + |B(t)|^2)\, dx\right) = 0.$$
5. *(Momentum conservation). For all $t \ge 0$ one has,*
$$\frac{d}{dt}\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \xi f(t)\, d\xi dx + \int_{\mathbb{R}^3}(E(t)\times B(t))\, dx\right) = 0.$$

Proof. The proof is standard and can be found, for instance, in [24]. □

Remark 1 *Properties of Proposition 1 are key ingredients to obtain the global-in-time existence of weak solutions [36, 48, 59] and the local-in-time existence, uniqueness and stability of classical solutions (e.g. see [48] and references therein). Properties of Proposition 1 are also independent of other a priori invariances described below. Indeed from Maxwell–Faraday equation, $\partial_t B + \nabla\times E = 0$, we deduce that $\partial_t \nabla\cdot B = 0$, which leads to $\nabla\cdot B(t) = 0$ for all $t > 0$, if initially $\nabla\cdot B_0 = 0$. In a similar way, from Maxwell–Ampère equation, $\partial_t E - \nabla\times B = -j$, we deduce that $\partial_t(\nabla\cdot E) + \nabla\cdot j = 0$. Using the charge conservation law (7) (obtained by integration of the Vlasov equation (11) with respect to ξ) we then obtain $\partial_t(\rho - \nabla\cdot E) = 0$, which leads to $\nabla\cdot E(t) = \rho(t)$ for all $t > 0$, if initially $\nabla\cdot E_0 = \rho_0$.*

3 Renormalization property and entropies conservation

3.1 Notation

We denote by $\mathbb{R}^+$ the non-negative real numbers, by $\mathcal{D}(\mathbb{R}^d)$ the space of indefinitely differentiable with compact support, and by $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions. We also denote by $\mathcal{S}(\mathbb{R}^d)$, the space of indefinitely differentiable and rapidly decreasing functions, and $\mathcal{S}'(\mathbb{R}^d)$ the dual of $\mathcal{S}(\mathbb{R}^d)$, i.e. the space of tempered distributions. We use the notation $B^{\alpha}_{p,q}$ ($0 < \alpha < 1$, $1 \le p \le \infty$, $1 \le q \le \infty$) for Besov spaces, the definition of which, can be found e.g., in [1, 21, 63, 64]. The notation $W^{\alpha,p}$ ($0 < \alpha < 1$, $1 \le p \le \infty$) stands for the generalized Sobolev spaces of fractional order, whose precise definition can also be found e.g., in [1, 21, 63, 64]. Let us simply recall first $W^{\alpha,p}(\mathbb{R}^d) = B^{\alpha}_{p,p}(\mathbb{R}^d)$ for α positive but not an integer and $1 \le p \le \infty$, and secondly the continuous embeddings: $B^{\alpha}_{p,1}(\mathbb{R}^d) \subset W^{\alpha,p}(\mathbb{R}^d) \subset B^{\alpha}_{p,\infty}(\mathbb{R}^d)$, with $1 \le p \le \infty$. We also define the functional space L^1_γ such that,

$$L^1_\gamma = \left\{ f \ge 0 \ \text{ a.e } \ \mid \ \|f\|_{L^1_\gamma(\mathbb{R}^6)} := \int_{\mathbb{R}^6} \gamma f \, dx d\xi < +\infty \right\}. \tag{19}$$

Moreover we define the function space $\mathscr{E}$ such that,

$$\mathscr{E} = \left\{ \mathcal{H} : \mathbb{R}^+ \mapsto \mathbb{R}^+;\ \mathcal{H} \text{ is non-decreasing}, \ \ \mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+;\mathbb{R}^+), \ \ \lim_{\sigma \to +\infty} \frac{\mathcal{H}(\sigma)}{\sigma} = +\infty \right\}. \tag{20}$$

3.2 Main theorems

In this section we present our main results. For this, we need to recall the DiPerna–Lions theorem, which is the only existing result concerning the existence of global-in-time (weak) solutions to the Vlasov–Maxwell system in $\mathbb{R}^6$.

Theorem 1 *(DiPerna–Lions [36]). Let $f_0 \in L^1_\gamma \cap L^\infty(\mathbb{R}^6)$, and E_0, $B_0 \in L^2(\mathbb{R}^3)$, be initial conditions which satisfy the constraints,*

$$\nabla \cdot B_0 = 0, \quad \nabla \cdot E_0 = \rho_0 = \int_{\mathbb{R}^3} f_0 \, d\xi, \quad \textit{in } \ \mathcal{D}'(\mathbb{R}^3).$$

Then, there exists a global-in-time weak solution of the relativistic Vlasov–Maxwell system, i.e. there exists functions,

$$f \in L^\infty\big(\mathbb{R}^+; L^1_\gamma \cap L^\infty(\mathbb{R}^6)\big), \quad E,\, B \in L^\infty\big(\mathbb{R}^+; L^2(\mathbb{R}^3)\big), \quad \rho,\, j \in L^\infty\big(\mathbb{R}^+; L^{4/3}(\mathbb{R}^3)\big), \tag{21}$$

such that (f, E, B) satisfy (11)-(12) *in the sense of distributions, with ρ, j defined in terms of* (13). *Constraints equations* (3) *and the charge conservation law* (7) *are statisfied in the sense of distributions.*

In addition, the mapping $t \mapsto f(t)$ (resp. $t \mapsto (E(t), B(t))$) is continuous with respect to the following topologies: the standard topology in the space of distributions $\mathcal{D}'(\mathbb{R}^6)$ (resp. $\mathcal{D}'(\mathbb{R}^3)$), the weak topology of $L^2(\mathbb{R}^6)$ (resp. $L^2(\mathbb{R}^3)$), and the strong topology of $H^{-s}(\Omega)$ for any $s > 0$ and any bounded subset Ω of $\mathbb{R}^6$ (resp. $\mathbb{R}^3$).

Futhermore, the total mass,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t) dx d\xi,$$

is independent of time, and one has,

$$\|f(t)\|_{L^p(\mathbb{R}^6)} = \|f_0\|_{L^p(\mathbb{R}^6)} \ \ \textit{a.e.} \ \ t \ge 0, \ \ \textit{for} \ \ 1 \le p \le +\infty, \qquad \textit{and} \qquad \mathcal{E}(t) = \mathcal{E}_0 < \infty \ \ \textit{a.e.} \ \ t \ge 0,$$

with the definition,

$$\mathcal{E}(t) := \int_{\mathbb{R}^6} \gamma f(t) \, d\xi dx + \frac{1}{2} \int_{\mathbb{R}^3} (|E(t)|^2 + |B(t)|^2) \, dx. \tag{22}$$

Remark 2 *1. The conservation of mass and all the L^p norms was in fact proved in [59].*

2. Using lower semi-continuity, weak solutions of Theorem 1 satisfy, for all $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+;\mathbb{R}^+)$,

$$\int_{\mathbb{R}^6} \mathcal{H}(f(t))\, d\xi dx \leq \int_{\mathbb{R}^6} \mathcal{H}(f_0)\, d\xi dx, \qquad \textit{for} \ \ t \geq 0.$$

Now we intend to produce supplementary sufficient regularity conditions, which will imply the validity of supplementary conservation laws. As the first step this is the aim of Theorem 2 below: indeed we first give sufficient regularity hypotheses which couple the regularity of the distribution function f with the regularity of the electromagnetic field (E, B). In the second step, we use Theorem 2 and the results of [22] on the regularity of DiPerna–Lions weak solutions, to obtain Corollary 1 below, which involves only sufficient regularity condition on the distribution function f. As concerns the renormalization property and entropies conservation, we have,

Theorem 2 *Let (f, E, B) be a weak solution of the relativistic Vlasov–Maxwell system* (11)-(14)*, given by Theorem 1. Assume that with,*

$$\alpha, \beta \in \mathbb{R}, \qquad 0 < \alpha,\, \beta < 1, \qquad \textit{and} \qquad \alpha\beta + \beta + 3\alpha - 1 > 0, \tag{23}$$

this weak solution satisfies for some $(p, q) \in \mathbb{N}_*^2$ with,

$$\begin{aligned} &\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1 \quad \textit{if} \quad 1 \leq p, q < \infty, \\ &1 \leq r < \infty \ \textit{is arbitrary if}\ p = q = \infty, \end{aligned} \tag{24}$$

the supplementary regularity hypotheses,

$$f \in L^\infty\big(0, T; W^{\alpha,p}(\mathbb{R}^6)\big), \qquad \textit{and} \qquad E, B \in L^\infty\big(0, T; W^{\beta,q}(\mathbb{R}^3)\big). \tag{25}$$

Then for any entropy function $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+;\mathbb{R}^+)$, we have the renormalization property,

$$\partial_t(\mathcal{H}(f)) + \nabla_x \cdot (v\mathcal{H}(f)) + \nabla_\xi \cdot (F_L \mathcal{H}(f)) = 0, \quad \textit{in} \ \ \mathcal{D}'((0,T) \times \mathbb{R}^6). \tag{26}$$

Moreover, if $\mathcal{H} \in \mathscr{E}$ and the map,

$$t \mapsto f(t, \cdot, \cdot) \ \textit{is uniformly integrable in}\ \mathbb{R}^6, \ \textit{for a.e.}\ t \in [0; T], \tag{27}$$

then we have the local entropy conservation laws,

$$\partial_t \left(\int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f) \right) + \nabla_x \cdot \left(\int_{\mathbb{R}^3} d\xi\, v\mathcal{H}(f) \right) = 0, \quad \textit{in} \ \ \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{28}$$

$$\partial_t \left(\int_{\mathbb{R}^3} dx\, \mathcal{H}(f) \right) + \nabla_\xi \cdot \left(\int_{\mathbb{R}^3} dx\, F_L\mathcal{H}(f) \right) = 0, \quad \textit{in} \ \ \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{29}$$

and the global entropy conservation law,

$$\int_{\mathbb{R}^6} \mathcal{H}(f(t, x, \xi))\, d\xi dx = \int_{\mathbb{R}^6} \mathcal{H}(f(s, x, \xi))\, d\xi dx, \qquad \textit{for} \ \ 0 < s \leq t < T. \tag{30}$$

The proof of Theorem 2 is postponed to Section 3.3. A few remarks are now in order.

Remark 3 *In fact, Theorem 2 is also true for the Vlasov–Poisson and the non-relativistic Vlasov–Maxwell systems, under the same regularity assumptions.*

Remark 4 *1. In fact, Theorem 2 still holds when we replace Sobolev spaces $W^{\alpha,p}$ (resp. $W^{\beta,q}$) by Besov spaces $B^{\alpha+\epsilon}_{p,\infty}$ (resp. $B^{\beta}_{q,\infty}$), with $\epsilon > 0$. Indeed, even if Besov spaces $B^{\alpha}_{p,\infty}$ do not share the restriction property (needed for proving commutator estimates of Lemma 2), we still have the following result (see [54, 15, 26]): let $N \geq 2$, $1 \leq d < N$, $0 < p < q \leq \infty$, $\alpha' > N(1/p - 1)_+$, and $f \in B^{\alpha'}_{p,\infty}(\mathbb{R}^N)$. Then,*

$$f(\cdot, y) \in \bigcap_{\alpha<\alpha'} B^{\alpha}_{p,\infty}(\mathbb{R}^d), \quad \text{for a.e } \; y \in \mathbb{R}^{N-d}.$$

Therefore, in the Besov-spaces framework, replacing α by $\alpha+\epsilon$ with $\epsilon > 0$ in (23), we observe that the condition $\alpha\beta + \beta + 3\alpha - 1 > 0$ keeps the same, whereas the phase-space regularity of f is slightly better than $B^{\alpha}_{p,\infty}$. Since the interpolation between $B^{\alpha+\epsilon}_{p,\infty}$ and $B^{\alpha}_{p,p}$ is $B^{\alpha'}_{p,r}$, with $\alpha < \alpha' < \alpha + \epsilon$, and $1 \leq r \leq \infty$, (e.g., Theorem 6.4.5 in [21]), we then have $B^{\alpha+\epsilon}_{p,\infty} \subset W^{\alpha,p}$.

2. Theorem 2 also includes the Hölder spaces where,

$$f \in L^\infty\big(0, T; \mathscr{C}^{0,\alpha}(\mathbb{R}^6)\big), \qquad \text{and} \qquad E, B \in L^\infty\big(0, T; \mathscr{C}^{0,\beta}(\mathbb{R}^3)\big).$$

It corresponds to case where $p = q = \infty$ in (4), since $\mathscr{C}^{0,\alpha} = W^{\alpha,\infty}$.

Remark 5 *Our result is almost in agreement with the structure-function scaling exponents derived in the study of dissipative anomalies in nearly collisionless plasma turbulence [44].*

1. *Here, the rigorous analysis is purely deterministic and regularity conditions (23)-(25) give a sufficient condition for the conservation of entropies for any individual solution as in [44]. In other words, by contraposition, a necessary condition for anomalous dissipation/non-conservation of entropies is, $\alpha\beta + \beta + 3\alpha - 1 < 0$ with $0 < \alpha, \beta < 1$. Nevertheless, this condition is not sufficient. Indeed, as in fluid mechanics with the Onsager critical regularity exponent $1/3$ [16, 62, 17], this necessary condition does not rule out the existence of some solutions that are less regular than the critical regularity (exponent) and that also satisfy the absence of anomalous entropy dissipation.*

2. *In [44] the author obtains, in a particular case, the critical exponent value $\alpha = \sqrt{5} - 2$, assuming that $f \in L^\infty(0, T; B^{\alpha}_{p,\infty}(\mathbb{R}^3_x; B^{\alpha}_{p,\infty}(\mathbb{R}^3_\xi)))$ and $E, B \in L^\infty(0, T; B^{\alpha}_{p,\infty}(\mathbb{R}^3))$, with $p \geq 3$. From Remark 4 on the restriction property of Besov spaces, in order to obtain $f \in L^\infty(0, T; B^{\alpha}_{p,\infty}(\mathbb{R}^3_x; B^{\alpha}_{p,\infty}(\mathbb{R}^3_\xi)))$, we must require the distribution function f to belong to the functional space $L^\infty(0, T; B^{\alpha+\epsilon}_{p,\infty}(\mathbb{R}^6))$, with $\epsilon > 0$. Now, taking $\alpha = \beta$, the condition $\alpha\beta + \beta + 3\alpha - 1 > 0$ in (23) becomes $\alpha^2 + 4\alpha - 1 > 0$, which is satisfied for $\alpha > \sqrt{5} - 2$. We then recover the same critical exponent value $\alpha = \sqrt{5} - 2$, but for $f \in L(0, T; W^{p,\alpha}(\mathbb{R}^6))$ and $E, B \in L^\infty(0, T; W^{q,\alpha}(\mathbb{R}^3))$, with $1/p + 1/q \leq 1$. Therefore our regularity conditions (23)-(25) are weaker, but less restrictive that those of [44]. Indeed we have $B^{\alpha+\epsilon}_{p,\infty} \subset W^{\alpha,p}$, $\forall \epsilon > 0$, and the condition $1/p + 1/q \leq 1$ is less restrictive than the condition $p = q \geq 3$.*

3. *In [44] the author obtains a refined version of the condition (23), by considering anisotropic regularity for the distribution function f between the space of velocities and the physical*

space, namely $f \in L^\infty(0,T; B^\kappa_{p,\infty}(\mathbb{R}^3_x; B^\sigma_{p,\infty}(\mathbb{R}^3_\xi)))$. From Remark 4 on the restriction property of Besov spaces, this anisotropic regularity implies that $f \in L^\infty(0,T; B^{\alpha+\epsilon}_{p,\infty}(\mathbb{R}^6))$, with $\alpha := \max\{\kappa, \sigma\}$ and $\epsilon > 0$. This regularity condition is still more restrictive than our regularity condition, namely $f \in L^\infty(0,T; W^{\alpha,p}(\mathbb{R}^6))$ with the same index α. In addition anisotropic regularity in phase space is questionable because of the following physical argument. Phase-space turbulence involves typical structures known as vortices that are the result of the filamentation and the trapping (or wave-particle synchronization) phenomena. The fact that characteristic curves roll up in phase space seems to contradict that phase-space regularity is anisotropic between the space of velocities and the physical space. On the contrary, this mixing motion must propagate regularity versus singularities from one direction to another. By constrast, anisotropic regularity between the electromagnetic field (E, B) and the distribution f is justified and crucial, because the velocity integration of f can lead to additional regularity in the physical space for the moments such as charge and current densities, and hence for the electromagnetic field (through Maxwell's equations). This is the essence of averaging lemma [36] and the spirit of regularity results obtained for the Diperna–Lions weak solutions [25, 22]. This anisotropy of regularity is handled both here and in [44].

Remark 6 *In the non-self-consistent case, i.e. when the Lorentz force F_L is a given external force, renormalization property* (26) *implies straightforwardly the uniqueness of weak solutions of Theorem 2, if such solutions exist. Indeed, let f^i, $i = 1, 2$, be two solutions of the Vlasov equation* (11)*, with initial conditions f^i_0, $i = 1, 2$, and where the electromagnetic field (E, B) is prescribed. Such solutions satisfy the regularity properties of Theorem 2, in particular* (25)*. Setting $f = f^1 - f^2$, and taking $\mathcal{H}(\cdot) = (\cdot)^2$ ($\mathcal{H} \in \mathscr{E}$), we obtain from Theorem 2,*

$$\partial_t(\mathcal{H}(f)) + \nabla_x \cdot (v\mathcal{H}(f)) + \nabla_\xi \cdot (F_L \mathcal{H}(f)) = 0, \quad in \;\; \mathcal{D}'((0,T) \times \mathbb{R}^6),$$

and

$$\int_{\mathbb{R}^6} \mathcal{H}(f(t))\, d\xi dx = \int_{\mathbb{R}^6} \mathcal{H}(f_0)\, d\xi dx.$$

Therefore, taking $f^1_0 = f^2_0$, i.e $f_0 = f^1 - f^2 = 0$, we obtain $f = 0$ a.e., i.e. $f_1 = f_2$ a.e.. In a similar way we can show the following comparison principle: $f^1_0 \le f^2_0$ a.e. implies $f^1 \le f^2$ a.e.. Two open issues remain. The first one is the uniqueness of solutions of Theorem 2, which corresponds to the self-consistent case. Of course the existence of solutions of Theorem 2 is also an open big problem. Following the program of [36], the second one is the existence and uniqueness of corresponding Lagrangian solutions, i.e. solutions constructed from almost-everywhere-well-defined caracteristics curves, as in the smooth framework (16)-(18).

Remark 7 *Another open issue is the case of bounded domains in space, with specular reflection and/or absorbing conditions [49]. This is not an easy task since, for such natural boundary conditions, some singularities could occur at the boundary and propagate inside the domain [50, 51].*

From Theorem 2 and the result of [22] on the regularity of the DiPerna–Lions weak solutions, we deduce the following corollary, which involves hypotheses concerning only the distribution function f.

Corollary 1 *Let $\beta = 6/(13 + \sqrt{142})$, and $\alpha \in \mathbb{R}$ solution to,*

$$\alpha\beta + \beta + 3\alpha - 1 > 0, \qquad and \qquad 0 < \alpha < 1. \tag{31}$$

Let (f, E, B) be a weak solution to the relativistic Vlasov–Maxwell system (11)-(14)*, given by Theorem 1. Assume the additional hypotheses: initial conditions (E_0, B_0) belong to $H^1(\mathbb{R}^3)$, the distribution function f satisfies the supplementary integrability condition,*

$$\int_{\mathbb{R}^3} \gamma f \, d\xi \in L^\infty\big(0, T; L^2(\mathbb{R}^3)\big), \tag{32}$$

and the regularity assumption,

$$f \in L^\infty\big(0, T; H^\alpha(\mathbb{R}^6)\big). \tag{33}$$

Then for any entropy function $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^+)$, renormalization property (26) *holds. Moreover, if $\mathcal{H} \in \mathscr{E}$ and the map $t \mapsto f(t, \cdot, \cdot)$ is uniformly integrable in $\mathbb{R}^6$ for a.e. $t \in \mathbb{R}^+$, then local entropy conservation laws* (28)-(29)*, as well as, global entropy conservation law* (30) *hold.*

Proof of Corollary 1. Using assumptions $E_0, B_0 \in H^1(\mathbb{R}^3)$ and (32), from Theorem 1.1 of [22], we obtain that the electromagnetic field (E, B) belongs to $H^\beta_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^3)$, with $\beta = 6/(13 + \sqrt{142})$. Setting $p = q = 2$ and $\beta = 6/(13 + \sqrt{142})$ in the hypotheses of Theorem 2, and using assumption (33) under constraints (31), we obtain from Theorem 2 the desired result. □

Remark 8 *From Corollary 1, we deduce that for α such that,*

$$1 > \alpha > \frac{1 - \beta}{3 + \beta} = \frac{7 + \sqrt{142}}{45 + 3\sqrt{142}} \simeq 0.234, \tag{34}$$

the Vlasov equation (11)*, which is a first-order conservation law in the phase-space $\mathbb{R}^6$, has an infinite number of conserved entropies. A similar situation occurs with general systems of conservation laws, which are studied in [20] within the regularity framework of Hölder spaces $\mathscr{C}^{0,\alpha}$. Nevertheless in [20], the authors show conservation of entropies under the sufficient condition $\alpha > 1/3$ (the famous Onsager exponent [58]), which is more restrictive than the present result, from two points of view. First, Sobolev spaces H^α are less regular than Hölder spaces $\mathscr{C}^{0,\alpha}$, for the same α. Secondly our index α is smaller than $1/3$. An explanation to such discrepancy, comes from our commutator estimates which exploit the anisotropy between the velocity and physical spaces, whereas commutator estimates in [52, 20] use some Taylor expansions, which does not advantage a particular direction of space. Finally, we observe that the critical exponent $\alpha = (7 + \sqrt{142})/(45 + 3\sqrt{142})$, which is smaller that $\sqrt{5} - 2$, can not be retrieved with the method of [44], since the latter is obtained under the condition $p = q \geq 3$ and hence can not deal with the case $p = q = 2$.*

3.3 Proof of Theorem 2

Before giving the proof of Theorem 2, we first introduce some standard regularization operators and we recall their main properties. Using a smooth non-negative function ϱ such that,

$$\tau \mapsto \varrho(\tau) \geq 0, \quad \varrho \in \mathcal{D}(\mathbb{R}), \quad \text{supp}(\varrho) \subset]-1, 1[, \quad \int_{\mathbb{R}} \varrho(\tau) d\tau = 1, \tag{35}$$

one define the radially-symmetric compactly-supported Friedrichs mollifier $z \mapsto \varrho_\epsilon(z)$, given by

$$\mathbb{R}^d \longrightarrow \mathbb{R}^+ \tag{36}$$

$$z \longmapsto \varrho_\epsilon(z) = \frac{1}{\epsilon^d} \varrho\left(\frac{|z|}{\epsilon}\right), \quad \epsilon > 0. \tag{37}$$

For any distribution $f \in \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^6)$, we define its $\mathscr{C}^\infty$-regularization by

$$f^{\eta,\varepsilon,\delta}(t,x,\xi) = \varrho_\eta(t) \underset{t}{*} \varrho_\varepsilon(x) \underset{x}{*} \varrho_\delta(\xi) \underset{\xi}{*} f(t,x,\xi), \tag{38}$$

where the operator $*$ denotes the standard convolution product. We denote by $\langle \cdot, \cdot \rangle$ the dual bracket between spaces $\mathcal{D}'$ and $\mathcal{D}$. Using previous definitions, we have for the regularization operator $(\cdot)^\epsilon$ the following standard properties (see, e.g., [8]), which are summarized in

Lemma 1 *1. For any distribution $f \in \mathcal{D}'(\mathbb{R}^d)$, we have,*

$$\langle f^\epsilon, g \rangle = \langle f, g^\epsilon \rangle, \quad g \in \mathcal{D}(\mathbb{R}^d).$$

2. For any function $f \in L^1 \cap L^\infty \cap W^{\alpha,p}(\mathbb{R}^d)$, with $0 < \alpha < 1$ and $1 \le p \le \infty$, there exists a constant C such that,

$$\begin{aligned}
&\|f^\epsilon\|_{L^q(\mathbb{R}^d)} \le \|f\|_{L^q(\mathbb{R}^d)}, \quad 1 \le q \le \infty,\\
&\|f^\epsilon\|_{W^{\alpha,p}(\mathbb{R}^d)} \le \|f\|_{W^{\alpha,p}(\mathbb{R}^d)},\\
&\|f^\epsilon - f\|_{L^p(\mathbb{R}^d)} \le C\epsilon^\alpha \|f\|_{W^{\alpha,p}(\mathbb{R}^d)},\\
&\|\nabla f^\epsilon\|_{L^p(\mathbb{R}^d)} \le C\epsilon^{\alpha-1} \|f\|_{W^{\alpha,p}(\mathbb{R}^d)}.
\end{aligned}$$

3. For any function $f \in B^\alpha_{p,\infty}(\mathbb{R}^d)$, with $0 < \alpha < 1$ and $1 \le p \le \infty$, there exists a constant C such that,

$$\|f(\cdot - z) - f(\cdot)\|_{L^p(\mathbb{R}^d)} \le C|z|^\alpha \|f\|_{B^\alpha_{p,\infty}(\mathbb{R}^d)}.$$

Proof. Since the proof is elementary, it is left to the reader. □

In order to prove Theorem 2, we use some commutator estimates which are given by

Lemma 2 *Let (f, E, B) be a weak solution of the relativistic Vlasov–Maxwell system (11)-(14), given by Theorem 1, satisfying the regularity assumptions (23)-(25) of Theorem 2. Let us recall that $F_L := E + v \times B$ is the Lorentz force field. Then there exist a constant C_{fs} depending on $\|f\|_{L^\infty(0,T;W^{\alpha,p}(\mathbb{R}^6))}$ and a constant C_{fl} depending on $\|f\|_{L^\infty(0,T;W^{\alpha,p}(\mathbb{R}^6))}$, $\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}$ and $\|B\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}$ such that,*

$$\big\|\nabla_x \cdot \big((vf)^{\eta,\varepsilon,\delta} - v^\delta f^{\eta,\varepsilon,\delta}\big)\big\|_{L^1(0,T;L^p(\mathbb{R}^6))} \le C_{fs}\delta^{\alpha+1}\varepsilon^{\alpha-1}, \tag{39}$$

and

$$\big\|\nabla_\xi \cdot \big((F_L f)^{\eta,\varepsilon,\delta} - F_L^{\eta,\varepsilon,\delta} f^{\eta,\varepsilon,\delta}\big)\big\|_{L^1\left(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x))\right)} \le C_{fl}(\varepsilon^{\beta+\alpha}\delta^{\alpha-1} + \delta^\alpha), \tag{40}$$

where α, β, p, q and r satisfy relations (23)-(24).

Remark 9 *The precise estimates obtained in Lemma 2 seem to be compulsory to obtain the precise Onsager exponents α, β in the main theorem, Theorem 2, instead of the general exponent $1/3$ established for fluid models.*

Proof. We start with two basic estimates, which will be often used along the proof. Using the fundamental theorem of calculus and,

$$|\nabla_\xi v| = \Big|\frac{I_3}{(1+|\xi|^2)^{1/2}} - \frac{\xi\otimes\xi}{(1+|\xi|^2)^{3/2}}\Big| \le \frac{2}{\sqrt{1+|\xi|^2}} \le 2, \tag{41}$$

we obtain the first basic estimate,

$$|v(\xi-w) - v(\xi)| \le |w| \int_0^1 |\nabla v(\xi - \tau w)|\, d\tau \le 2|w|. \tag{42}$$

Using the fundamental theorem of calculus twice, we obtain componentwise,

$$\begin{aligned} v_i - v_i^\delta &= \int_{\mathbb{R}^3} dw\, \varrho_\delta(w)(v_i(\xi) - v_i(\xi - w)) \\ &= \sum_j \int_{\mathbb{R}^3} dw\, \varrho_\delta(w) w_j \int_0^1 d\tau\, \partial_j v_i(\xi - \tau w) \\ &= \sum_j \partial_j v_i(\xi) \int_{\mathbb{R}^3} dw\, \varrho_\delta(w) w_j \\ &\quad + \sum_{j,k} \int_{\mathbb{R}^3} dw\, \varrho_\delta(w) w_j w_k \int_0^1 d\tau \int_0^1 ds\, \partial^2_{jk} v_i(\xi - s\tau w). \end{aligned} \tag{43}$$

Since the smooth function ϱ is radially symmetric and compactly supported, we have,

$$\int_{\mathbb{R}^3} dw\, \varrho(w) w_i = 0, \quad \text{and} \quad \int_{\mathbb{R}^3} dw\, \varrho(w)|w_i||w_j| \le C_\varrho < +\infty, \quad \forall i, j, \in \{1,2,3\}, \tag{44}$$

where C_ϱ is a numerical constant depending only on the function ϱ. Using the first equality in (44), the first term of the right-hand side of (43) vanishes. Using the second inequality of (44), and

$$|\nabla^2_{jk} v_i(\xi)| = \Big|\frac{\delta_{ij}\xi_k}{(1+|\xi|^2)^{3/2}} + \frac{\delta_{jk}\xi_i}{(1+|\xi|^2)^{3/2}} + \frac{\delta_{ik}\xi_j}{(1+|\xi|^2)^{3/2}} - \frac{3\xi_i\xi_j\xi_k}{(1+|\xi|^2)^{3/2}}\Big| \le \frac{6}{1+|\xi|^2} \le 6,$$

we obtain from (43) the second basic estimate,

$$|v - v^\delta| \le 6 C_\varrho \delta^2. \tag{45}$$

We now deal with commutator estimate (39) for the free-streaming term. We define,

$$\begin{aligned} r_{\eta,\varepsilon,\delta}(f,g)(t,x,\xi) = \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)\varrho_\delta(w) \\ (f(t-\tau, x-y, \xi-w) - f(t,x,\xi))(g(t-\tau, x-y, \xi-w) - g(t,x,\xi)). \end{aligned} \tag{46}$$

Using (46), it is easy to check that,

$$(vf)^{\eta,\varepsilon,\delta} = v^\delta f^{\eta,\varepsilon,\delta} + r_{\eta,\varepsilon,\delta}(v,f) - (f - f^{\eta,\varepsilon,\delta})(v - v^\delta). \tag{47}$$

Observing that,

$$r_{\eta,\varepsilon,\delta}(v,f) = r_\delta(v, f^{\eta,\varepsilon}) + (f - f^{\eta,\varepsilon})(v - v^\delta),$$

Eq. (47) becomes,

$$(vf)^{\eta,\varepsilon,\delta} - v^\delta f^{\eta,\varepsilon,\delta} = r_\delta(v, f^{\eta,\varepsilon}) - ((f^{\eta,\varepsilon})^\delta - f^{\eta,\varepsilon})(v - v^\delta). \tag{48}$$

Using estimate (42), Lemma 1, continuous embedding $W^{\alpha,p}(\mathbb{R}^d) \subset B^\alpha_{p,\infty}(\mathbb{R}^d)$ with $1 \leq p \leq \infty$, the restriction property for Sobolev spaces $W^{\alpha,p}(\mathbb{R}^d)$ (see Remark 4), and regularity assumptions (23)-(25), we obtain,

$$\begin{aligned}
\left\|\nabla_x \cdot r_\delta(v, f^{\eta,\varepsilon})\right\|_{L^1(0,T;L^p(\mathbb{R}^6))} &\leq \int_0^T dt \int_{\mathbb{R}^3} dw\, \varrho_\delta(w) \left\|(v(\xi - w) - v(\xi))\right. \\
&\quad \left.\cdot (\nabla_x f^{\eta,\varepsilon}(t, x, \xi - w) - \nabla_x f^{\eta,\varepsilon}(t, x, \xi))\right\|_{L^p(\mathbb{R}^6_{x\xi})} \\
&\leq C \int_0^T dt \int_{\mathbb{R}^3} dw\, \varrho_\delta(w)|w|^{\alpha+1} \|\nabla_x f^{\eta,\varepsilon}(t)\|_{L^p(\mathbb{R}^3_x;B^\alpha_{p,\infty}(\mathbb{R}^3_\xi))} \\
&\leq C\delta^{\alpha+1} \int_0^T dt\, \|\nabla_x f^{\eta,\varepsilon}(t)\|_{L^p(\mathbb{R}^3_x;W^{\alpha,p}(\mathbb{R}^3_\xi))} \\
&\leq C\varepsilon^{\alpha-1}\delta^{\alpha+1} \int_0^T dt\, \|f^\eta(t)\|_{W^{\alpha,p}(\mathbb{R}^3_x;W^{\alpha,p}(\mathbb{R}^3_\xi))} \\
&\leq C\varepsilon^{\alpha-1}\delta^{\alpha+1} \int_0^T dt\, \varrho_\eta(t) * \|f(t)\|_{W^{\alpha,p}(\mathbb{R}^6)} \\
&\leq C\varepsilon^{\alpha-1}\delta^{\alpha+1} \|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}.
\end{aligned} \tag{49}$$

Using estimate (45), Lemma 1, the restriction property for Sobolev spaces $W^{\alpha,p}(\mathbb{R}^d)$, and regularity assumptions (23)-(25), we obtain,

$$\begin{aligned}
\left\|\nabla_x \cdot ((f^{\eta,\varepsilon})^\delta - f^{\eta,\varepsilon})(v - v^\delta))\right\|_{L^1(0,T;L^p(\mathbb{R}^6))} &\leq \|v - v^\delta\|_{L^\infty(\mathbb{R}^3)} \left\|(\nabla_x f^{\eta,\varepsilon})^\delta - \nabla_x f^{\eta,\varepsilon}\right\|_{L^1(0,T;L^p(\mathbb{R}^6))} \\
&\leq C\delta^{\alpha+1} \|\nabla_x f^{\eta,\varepsilon}\|_{L^1(0,T;L^p(\mathbb{R}^3_x;W^{\alpha,p}(\mathbb{R}^3_\xi)))} \\
&\leq C\varepsilon^{\alpha-1}\delta^{\alpha+1} \|f^\eta\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^3_x;W^{\alpha,p}(\mathbb{R}^3_\xi)))} \\
&\leq C\varepsilon^{\alpha-1}\delta^{\alpha+1} \|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}.
\end{aligned} \tag{50}$$

Using (49)-(50), we obtain from (48), commutator estimate (39). We continue with commutator estimate (40) for the Lorentz force term. Using definition (46), we first make the following decomposition,

$$(F_L f)^{\eta,\varepsilon,\delta} - F_L^{\eta,\varepsilon,\delta} f^{\eta,\varepsilon,\delta} = T_E + T_B, \tag{51}$$

where

$$T_E = (Ef^\delta)^{\eta,\varepsilon} - E^{\eta,\varepsilon}(f^\delta)^{\eta,\varepsilon} = r_{\eta,\varepsilon}(E, f^\delta) - (E - E^{\eta,\varepsilon})(f^\delta - (f^\delta)^{\eta,\varepsilon}), \tag{52}$$

and

$$T_B = (v \times Bf)^{\varepsilon,\delta} - v^\delta \times B^\varepsilon f^{\varepsilon,\delta}. \tag{53}$$

Let us first deal with the term T_E. Passing to the limit $\eta \to 0$ in $r_{\eta,\varepsilon}(E, f^\delta)$, which can be justified by the Lebesgue dominated convergence theorem and regularity assumptions (23)-(25), we obtain,

$$\begin{aligned}
\left\|\nabla_\xi \cdot r_{\eta,\varepsilon}(E, f^\delta)\right\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} &\leq \left\|\nabla_\xi \cdot r_\varepsilon(E, f^\delta)\right\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} \\
&\leq \int_{\mathbb{R}^3} dy\, \varrho_\varepsilon(y) \|(E(t, x - y) - E(t, x)) \\
&\quad \cdot (\nabla_\xi f^\delta(t, x - y, \xi) - \nabla_\xi f^\delta(t, x, \xi))\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))}.
\end{aligned} \tag{54}$$

Using Hölder inequality, Lemma 1, continuous embedding $W^{\alpha,p}(\mathbb{R}^d) \subset B^\alpha_{p,\infty}(\mathbb{R}^d)$ with $1 \le p \le \infty$, the restriction property for Sobolev spaces $W^{\alpha,p}(\mathbb{R}^d)$, and regularity assumptions (23)-(25), we obtain from (54),

$$\begin{aligned}
\left\|\nabla_\xi \cdot r_{\eta,\varepsilon}(E,f^\delta)\right\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} &\le \int_{\mathbb{R}^3} dy\, \varrho_\varepsilon(y) \|E(t,x-y)-E(t,x)\|_{L^\infty(0,T;L^q(\mathbb{R}^3_x))} \\
&\quad \|\nabla_\xi f^\delta(t,x-y,\xi) - \nabla_\xi f^\delta(t,x,\xi)\|_{L^1(0,T;L^p(\mathbb{R}^6_{x\xi}))} \\
&\le C\int_{\mathbb{R}^3} dy\, \varrho_\varepsilon(y)|y|^{\alpha+\beta} \\
&\quad \|E\|_{L^\infty(0,T;B^\beta_{q,\infty}(\mathbb{R}^3))}\|\nabla_\xi f^\delta\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;B^\alpha_{p,\infty}(\mathbb{R}^3_x)))} \\
&\le C\varepsilon^{\alpha+\beta}\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|\nabla_\xi f^\delta\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;W^{\alpha,p}(\mathbb{R}^3_x)))} \\
&\le C\varepsilon^{\alpha+\beta}\delta^{\alpha-1}\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^3_\xi;W^{\alpha,p}(\mathbb{R}^3_x)))} \\
&\le C\varepsilon^{\alpha+\beta}\delta^{\alpha-1}\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \qquad (55)
\end{aligned}$$

Using the Lebesgue dominated convergence theorem and regularity assumptions (23)-(25), we can pass to the limit $\eta \to 0$ in the term, $(E - E^{\eta,\varepsilon})(f^\delta - (f^\delta)^{\eta,\varepsilon})$, to obtain,

$$\begin{aligned}
&\left\|\nabla_\xi \cdot \left((E-E^{\eta,\varepsilon})(f^\delta - (f^\delta)^{\eta,\varepsilon})\right)\right\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} \\
&\qquad \le \left\|\nabla_\xi \cdot \left((E-E^{\varepsilon})(f^\delta - (f^\delta)^{\varepsilon})\right)\right\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))}. \qquad (56)
\end{aligned}$$

Using Hölder inequality, Lemma 1, continuous embedding $W^{\alpha,p}(\mathbb{R}^d) \subset B^\alpha_{p,\infty}(\mathbb{R}^d)$ with $1 \le p \le \infty$, the restriction property for Sobolev spaces $W^{\alpha,p}(\mathbb{R}^d)$, and regularity assumptions (23)-(25), we obtain from (56),

$$\begin{aligned}
&\left\|\nabla_\xi \cdot \left((E-E^{\eta,\varepsilon})(f^\delta - (f^\delta)^{\eta,\varepsilon})\right)\right\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} \\
&\qquad \le \|E-E^\varepsilon\|_{L^\infty(0,T;L^q(\mathbb{R}^3))}\|\nabla_\xi f^\delta - (\nabla_\xi f^\delta)^\varepsilon\|_{L^1(0,T;L^p(\mathbb{R}^6))} \\
&\qquad \le C\varepsilon^{\alpha+\beta}\|E\|_{L^\infty(0,T;B^\beta_{q,\infty}(\mathbb{R}^3))}\|\nabla_\xi f^\delta\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;B^\alpha_{p,\infty}(\mathbb{R}^3_x)))} \\
&\qquad \le C\varepsilon^{\alpha+\beta}\delta^{\alpha-1}\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^3_\xi;W^{\alpha,p}(\mathbb{R}^3_x)))} \\
&\qquad \le C\varepsilon^{\alpha+\beta}\delta^{\alpha-1}\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \qquad (57)
\end{aligned}$$

From (54) and (57), we obtain,

$$\left\|\nabla_\xi \cdot T_E\right\|_{L^1(0,T;L^p(\mathbb{R}^3_x;L^r(\mathbb{R}^3_x)))} \le C\varepsilon^{\alpha+\beta}\delta^{\alpha-1}\|E\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \qquad (58)$$

We now deal with the Term T_B, given by (53), and which can be recast as,

$$\begin{aligned}
T_B &= \int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)\varrho_\delta(w)[v(\xi-w)-v(\xi)] \times B(t-\tau,x-y)f(t-\tau,x-y,\xi-w) \\
&\quad + \ v \times [(Bf^\delta)^{\eta,\varepsilon} - B^{\eta,\varepsilon}(f^\delta)^{\eta,\varepsilon}] \ + \ (v-v^\delta)\times B^{\eta,\varepsilon}(f^\delta)^{\eta,\varepsilon} = T_{B1}+T_{B2}+T_{B3}. \qquad (59)
\end{aligned}$$

The term $\nabla_\xi \cdot T_{B1}$ can be decomposed as,

$$\nabla_\xi \cdot T_{B1} = \int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)\nabla_w\varrho_\delta(w)\cdot([v(\xi-w)-v(\xi)]\times B(t-\tau,x-y))f(t-\tau,x-y,\xi)+ \int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)\nabla_w\varrho_\delta(w)\cdot([v(\xi-w)-v(\xi)]\times B(t-\tau,x-y)) (f(t-\tau,x-y,\xi-w)-f(t-\tau,x-y,\xi)) = T_{B11}+T_{B12}. \quad (60)$$

Using integration by parts, we observe that,

$$T_{B11} = \int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)\varrho_\delta(w) \nabla_w\cdot([v(\xi-w)-v(\xi)]\times B(t-\tau,x-y))f(t-\tau,x-y,\xi)=0, \quad (61)$$

because $\nabla_w\cdot([v(\xi-w)-v(\xi)]\times B(t,x-y))=0$. Using Hölder inequality, estimate (42), Lemma 1, continuous embedding $W^{\alpha,p}(\mathbb{R}^d)\subset B^\alpha_{p,\infty}(\mathbb{R}^d)$ with $1\le p\le\infty$, the restriction property for Sobolev spaces $W^{\alpha,p}(\mathbb{R}^d)$, and regularity assumptions (23)-(25), we obtain,

$$\begin{aligned}\big\|T_{B12}\big\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} &\le 2\int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)|\nabla_w\varrho_\delta(w)||w| \\ &\quad \|B(t-\tau,x-y)(f(t-\tau,x-y,\xi-w)-f(t-\tau,x-y,\xi))\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} \\ &\le 2\int_0^T d\tau \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} dw\, \varrho_\eta(\tau)\varrho_\varepsilon(y)|\nabla_w\varrho_\delta(w)||w| \\ &\quad \|B(t-\tau,x-y)\|_{L^\infty(0,T;L^q(\mathbb{R}^3_x))}\|f(t-\tau,x-y,\xi-w)-f(t-\tau,x-y,\xi)\|_{L^1(0,T;L^p(\mathbb{R}^6_{x\xi}))} \\ &\le C\int_{\mathbb{R}^3} dw\, |\nabla_w\varrho_\delta(w)||w|^{\alpha+1}\|B\|_{L^\infty(0,T;L^q(\mathbb{R}^3_x))}\|f\|_{L^1(0,T;L^p(\mathbb{R}^3_x;B^\alpha_{p,\infty}(\mathbb{R}^3_\xi)))} \\ &\le C\delta^\alpha\|B\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \end{aligned} \quad (62)$$

In the similar way we have obtained estimate (58) for $\nabla_\xi\cdot T_E$, we also obtain for $\nabla_\xi\cdot T_{B2}$,

$$\big\|\nabla_\xi\cdot T_{B2}\big\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} \le C\varepsilon^{\alpha+\beta}\delta^{\alpha-1}\|B\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \quad (63)$$

Using estimate (45), Hölder inequality and Lemma 1, we obtain for $\nabla_\xi\cdot T_{B3}$,

$$\begin{aligned}\big\|\nabla_\xi\cdot T_{B3}\big\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} &\le \big\|(v-v^\delta)\times B^{\eta,\varepsilon}(\nabla_\xi f^\delta)^{\eta,\varepsilon}\big\|_{L^1(0,T;L^p(\mathbb{R}^3_\xi;L^r(\mathbb{R}^3_x)))} \\ &\le C\delta^2\|B^{\eta,\varepsilon}\|_{L^\infty(0,T;L^q(\mathbb{R}^3_x))}\|\nabla_\xi f^{\eta,\delta,\varepsilon}\|_{L^1(0,T;L^p(\mathbb{R}^6))} \\ &\le C\delta^{\alpha+1}\|B\|_{L^\infty(0,T;L^q(\mathbb{R}^3_x))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \end{aligned} \quad (64)$$

Gathering estimates (61)-(64), we obtain from decompositions (59)-(60),

$$\big\|\nabla_\xi\cdot T_B\big\|_{L^1(0,T;L^p(\mathbb{R}^3_x;L^r(\mathbb{R}^3_x)))} \le C(\varepsilon^{\alpha+\beta}\delta^{\alpha-1}+\delta^\alpha)\|B\|_{L^\infty(0,T;W^{\beta,q}(\mathbb{R}^3))}\|f\|_{L^1(0,T;W^{\alpha,p}(\mathbb{R}^6))}. \quad (65)$$

Eventually, from (58) and (65), we obtain commutator estimate (40), which ends the proof of Lemma 2 $\square$

Proof of Theorem 2. Let us now give the proof of the main theorem. The weak formulation for the Vlasov equation reads,

$$\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, f(\partial_t \Psi + v \cdot \nabla_x \Psi + F_L \cdot \nabla_\xi \Psi) = 0, \quad \forall \Psi \in \mathcal{D}((0,T) \times \mathbb{R}^6), \tag{66}$$

with $F_L := E + v \times B$. Let us note that all integrals in (66) have a sense since for DiPerna–Lions weak solutions [36] we have $f \in L^\infty(0,T; L^2(\mathbb{R}^6))$, and E, $B \in L^\infty(0,T; L^2(\mathbb{R}^3))$. We choose in (66) the test function,

$$\Psi = \Psi_{\varepsilon,\delta} = (\mathcal{H}'(f^{\eta,\varepsilon,\delta})\Phi)^{\eta,\varepsilon,\delta} \in \mathcal{D}((0,T) \times \mathbb{R}^6), \tag{67}$$

with $\Phi \in \mathcal{D}((0,T) \times \mathbb{R}^6)$ and $\mathcal{H} \in \mathscr{C}^1(\mathbb{R}^+; \mathbb{R}^+)$. Using the first property of Lemma 1 and successive integrations by parts, we obtain from (66)-(67),

$$\begin{aligned} \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \Big\{ & \mathcal{H}(f^{\eta,\varepsilon,\delta})(\partial_t \Phi + v^\delta \cdot \nabla_x \Phi + F_L^{\eta,\varepsilon,\delta} \cdot \nabla_\xi \Phi) \\ & + \Phi \mathcal{H}'(f^{\eta,\varepsilon,\delta}) \Big[\nabla_x \cdot \big((vf)^{\eta,\varepsilon,\delta} - v^\delta f^{\eta,\varepsilon,\delta} \big) + \nabla_\xi \cdot \big((F_L f)^{\eta,\varepsilon,\delta} - F_L^{\eta,\varepsilon,\delta} f^{\eta,\varepsilon,\delta} \big) \Big] \Big\} = 0, \end{aligned} \tag{68}$$

for all $\Phi \in \mathcal{D}((0,T) \times \mathbb{R}^6)$. We now establish the renormalized Vlasov equation (26). Using regularity assumptions (23)-(25), (67), Lemma 1 and 2, we obtain from (68),

$$\begin{aligned} \left| \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta})(\partial_t \Phi + v^\delta \cdot \nabla_x \Phi + F_L^{\eta,\varepsilon,\delta} \cdot \nabla_\xi \Phi) \right| & \\ \leq C_* \left(\delta^{\alpha+1} \varepsilon^{\alpha-1} + \varepsilon^{\alpha+\beta} \delta^{\alpha-1} + \delta^\alpha \right), & \end{aligned} \tag{69}$$

where C_* depends on $\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^6))}$, C_{fs}, C_{fl}, $\mathcal{H}$, and Φ. Balancing contributions coming from the free-streaming and Lorentz force terms in the right-hand side of (69), we obtain,

$$\varepsilon^{\alpha-1}\delta^2 - \delta - \varepsilon^{\alpha+\beta} = 0, \tag{70}$$

and estimate (69) becomes,

$$\left| \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta})(\partial_t \Phi + v^\delta \cdot \nabla_x \Phi + F_L^{\eta,\varepsilon,\delta} \cdot \nabla_\xi \Phi) \right| \leq C_* \eta, \tag{71}$$

with the definition,

$$\eta := \varepsilon^{\alpha-1}\delta^{\alpha+1}.$$

Solving quadratic equation (70) in δ, the only positive solution is given by,

$$\delta = \frac{1 + \sqrt{1 + 4\varepsilon^{2\alpha+\beta-1}}}{2\varepsilon^{\alpha-1}}.$$

Two cases are to be considered according to the value of α and β:

i) $2\alpha+\beta-1<0$. We then have $\delta \simeq \varepsilon^{(\beta+1)/2}$, and $\eta \simeq \varepsilon^{(\alpha\beta+\beta+3\alpha-1)/2} \to 0$ as $\varepsilon \to 0$ if $\alpha\beta+\beta+3\alpha-1>0$.

ii) $2\alpha+\beta-1 \geq 0$. We then have $\delta \simeq \varepsilon^{1-\alpha}$, and $\eta \simeq \varepsilon^{\alpha(1-\alpha)} \to 0$ as $\varepsilon \to 0$ if $0<\alpha<1$.

Assuming that the free-streaming contribution dominates the Lorentz-force contribution, this implies $\varepsilon^{\alpha-1}\delta^2-\delta-\varepsilon^{\alpha+\beta} \gg 0$, which leads to a contradiction as $\delta \to 0$. On the contrary, assuming that the Lorentz-force contribution dominates the free-streaming contribution, this implies $\varepsilon^{\alpha-1}\delta^2-\delta-\varepsilon^{\alpha+\beta} \ll 0$, which leads also to a contradiction as first $\delta \to 0$ and next $\varepsilon \to 0$. In conclusion, if $\alpha\beta+\beta+3\alpha-1>0$, then the right-hand side of (71) vanishes as $(\varepsilon,\delta) \to 0$, and we obtain the renormalized Vlasov equation (26).

We continue with the local-in-space entropy conservation law (28). For this purpose, we first restrict entropy functions $\mathcal{H}$ to the set $\mathscr{E}$, defined by (20), and secondly we take in (71) a test function Φ such that,

$$\Phi(t,x,\xi)=\Lambda(t,x)\Theta(\xi), \quad \text{with } \Lambda \in \mathcal{D}((0,T)\times\mathbb{R}^3), \text{ and } \Theta \in \mathcal{D}(\mathbb{R}^3).$$

We then choose the test function Θ such that,

$$\Theta(\xi)=\Theta_R(\xi):=\theta(\xi/R), \quad \text{with } R>0.$$

Here the function $\theta \in \mathcal{D}(\mathbb{R}^3)$ is such that $\operatorname{supp}(\theta) \subset B_{\mathbb{R}^3}(0,2)$, $\theta \equiv 1$ on $B_{\mathbb{R}^3}(0,1)$ and $0 \leq \theta \leq 1$ on $B_{\mathbb{R}^3}(0,2) \setminus B_{\mathbb{R}^3}(0,1)$. We then have,

$$\Theta_R \longrightarrow 1, \text{ a.e as } R \to +\infty, \text{ and } \nabla_\xi \Theta_R \longrightarrow 0, \text{ a.e as } R \to +\infty. \tag{72}$$

From the uniform integrability assumption (27), and the de La Vallée Poussin theorem, there exists a constant $C_{\mathcal{H}}>0$, independent of (ε,δ), but depending on $\mathcal{H}$ such that,

$$\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f^{\eta,\varepsilon,\delta}) \leq C_{\mathcal{H}} < +\infty, \quad \forall \mathcal{H} \in \mathscr{E}. \tag{73}$$

Using estimate (73), regularity assumptions (23)-(25) and property (72), we obtain from the Lebesgue dominated convergence theorem that,

$$\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f^{\eta,\varepsilon,\delta})\partial_t\Lambda\Theta_R \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f^{\eta,\varepsilon,\delta})\partial_t\Lambda, \text{ as } R \to +\infty, \tag{74}$$

$$\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f^{\eta,\varepsilon,\delta})v^\delta \cdot \nabla_x\Lambda\Theta_R \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f^{\eta,\varepsilon,\delta})v^\delta \cdot \nabla_x\Lambda, \text{ as } R \to +\infty, \tag{75}$$

and

$$\mathcal{R}_1 := \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi\, \mathcal{H}(f^{\eta,\varepsilon,\delta})F_L^{\eta,\varepsilon,\delta} f^{\eta,\varepsilon,\delta} \cdot \nabla_\xi\Theta_R\Lambda \longrightarrow 0 \text{ as } R \to +\infty. \tag{76}$$

Limits (74)-(76) are uniform in $(\eta,\varepsilon,\delta)$, and in addition there exists a constant $\kappa_1>0$, independent of $(\eta,\varepsilon,\delta)$, but depending on $\|f\|_{L^\infty(0,T;L^2\cap L^\infty(\mathbb{R}^6))}$, $\|B\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}$, $\|E\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}$, Λ and θ such that,

$$|\mathcal{R}_1| \leq \kappa_1 R^{-1}. \tag{77}$$

Using (74)-(77), we obtain from (71),

$$\left| \int_0^T dt \int_{\mathbb{R}^3} dx \ (\partial_t \Lambda + v^\delta \cdot \nabla_x \Lambda) \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta}) \right| \leq C_* \eta + \kappa_1 R^{-1}. \tag{78}$$

Under the condition, $\alpha\beta + \beta + 3\alpha - 1 > 0$, the right-hand side of (78) vanishes as $(\eta, \varepsilon, \delta) \to 0$ and $R \to +\infty$, and we obtain from (78) the local-in-space conservation law (28). In a similar way, by interchanging the role of the test functions Λ and Θ, we obtain local-in-momentun conservation law (29).

We pursue with global entropy conservation law (30). For this aim, we first take in (78) a test function Λ such that,

$$\Lambda(t, x) = \varphi(t)\Lambda(x), \quad \text{with} \ \ \varphi \in \mathcal{D}((0, T)), \ \ \text{and} \ \ \Lambda \in \mathcal{D}(\mathbb{R}^3).$$

We then choose the test function Λ such that,

$$\Lambda(x) = \Lambda_R(x) := \lambda(x/R), \ \ \text{with} \ \ R > 0.$$

Here the function $\lambda \in \mathcal{D}(\mathbb{R}^3)$ is such that $\text{supp}(\lambda) \subset B_{\mathbb{R}^3}(0, 2)$, $\lambda \equiv 1$ on $B_{\mathbb{R}^3}(0, 1)$ and $0 \leq \lambda \leq 1$ on $(B_{\mathbb{R}^3}(0, 2) \setminus B_{\mathbb{R}^3}(0, 1))$. We then have,

$$\Lambda_R \longrightarrow 1, \ \ \text{a.e as} \ \ R \to +\infty, \ \ \text{and} \ \ \nabla_x \Lambda_R \longrightarrow 0, \ \ \text{a.e as} \ \ R \to +\infty. \tag{79}$$

Using estimate (73), regularity assumptions (23)-(25) and property (79), we obtain from the Lebesgue dominated convergence theorem that,

$$\int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta}) \partial_t \varphi \Lambda_R \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta}) \partial_t \varphi, \ \ \text{as} \ \ R \to +\infty, \tag{80}$$

$$\mathcal{R}_2 := \int_0^T dt \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta}) v^\delta \cdot \nabla_x \Lambda_R \varphi \longrightarrow 0, \ \ \text{as} \ \ R \to +\infty. \tag{81}$$

Limits (80)-(81) are uniform in $(\eta, \varepsilon, \delta)$, and in addition there exists a constant $\kappa_2 > 0$, independent of $(\eta, \varepsilon, \delta)$, but depending on $C_{\mathcal{H}}$, φ, and λ such that,

$$|\mathcal{R}_2| \leq \kappa_2 R^{-1}. \tag{82}$$

Using (80)-(82), we obtain from (78),

$$\left| \int_0^T dt \, \partial_t \varphi \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} d\xi \, \mathcal{H}(f^{\eta,\varepsilon,\delta}) \right| \leq C_* \eta + \kappa_1 R^{-1} + \kappa_2 R^{-1}. \tag{83}$$

Under the condition, $\alpha\beta + \beta + 3\alpha - 1 > 0$, the right-hand side of (83) vanishes as $(\eta, \varepsilon, \delta) \to 0$ and $R \to +\infty$, and we obtain from (83) the global entropy conservation law (30). This ends the proof of Theorem 2 $\square$

4 Energy conservation

As concerns conservation of total energy we have,

Theorem 3 *Let* (f, E, B) *be a weak solution to the relativistic Vlasov–Maxwell system* (11)-(14), *given by Theorem 1. If the macroscopic kinetic energy density satisfies the supplementary integrability condition,*

$$\int_{\mathbb{R}^3} \gamma f \, d\xi \in L^\infty\big(0,T; L^2(\mathbb{R}^3)\big), \tag{84}$$

then, using definition (22), *we have the local conservation law of total energy,*

$$\partial_t \mathcal{E} + \nabla \cdot \left(\int_{\mathbb{R}^3} \gamma f v \, d\xi + E \times B \right) = 0, \qquad in \ \ \mathcal{D}'((0,T) \times \mathbb{R}^3), \tag{85}$$

and the global conservation law of total energy,

$$\mathcal{E}(t) = \mathcal{E}(s), \quad for \ \ 0 < s \le t < T. \tag{86}$$

Remark 10 *Under assumption* (84)*, it has been proved in [22] that the electromagnetic field* (E, B) *belongs to* $H^s_{\text{loc}}(\mathbb{R}^+_* \times \mathbb{R}^3)$*, with* $s = 6/(13 + \sqrt{142})$*. Then, such solutions satisfy the conservation laws* (85)-(86).

Proof. Choosing in the weak formulation (66) the test function,

$$\Psi(t,x,\xi) = \Lambda(t,x)\Theta(\xi)\gamma(\xi) \in \mathcal{D}((0,T) \times \mathbb{R}^6), \quad \text{with} \ \ \Lambda \in \mathcal{D}((0,T) \times \mathbb{R}^3), \ \ \text{and} \ \Theta \in \mathcal{D}(\mathbb{R}^3), \tag{87}$$

and using $\nabla_\xi \gamma = v$, we obtain,

$$\begin{aligned} &\int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi \, \gamma f \Theta \right) \partial_t \Lambda + \int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi \, \gamma f v \Theta \right) \cdot \nabla_x \Lambda \\ &\qquad + \int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi \, f v \cdot E \Theta \right) \Lambda + \int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi \, \gamma f F_L \cdot \nabla_\xi \Theta \right) \Lambda = 0. \end{aligned} \tag{88}$$

We now establish the local conservation law of total energy. For this we take in (88) a test function Θ such that,

$$\Theta(\xi) = \Theta_R(\xi) := \theta(\xi/R), \quad \text{with} \ \ R > 0.$$

Here the function $\theta \in \mathcal{D}(\mathbb{R}^3)$ is such that $\text{supp}(\theta) \subset B_{\mathbb{R}^3}(0,2)$, $\theta \equiv 1$ on $B_{\mathbb{R}^3}(0,1)$ and $0 \le \theta \le 1$ on $B_{\mathbb{R}^3}(0,2) \setminus B_{\mathbb{R}^3}(0,1)$. We then have,

$$\Theta_R \longrightarrow 1, \ \ \text{a.e as} \ \ R \to +\infty, \ \ \text{and} \ \ \nabla_\xi \Theta_R \longrightarrow 0, \ \ \text{a.e as} \ \ R \to +\infty. \tag{89}$$

Using (89) and regularity properties (21), we obtain from the Lebesgue dominated convergence theorem,

$$\int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi \, \gamma f \Theta_R \right) \partial_t \Lambda \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi \, \gamma f \right) \partial_t \Lambda, \ \ \text{as} \ R \to \infty, \tag{90}$$

and

$$\int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi\, \gamma f v \Theta_R \right) \cdot \nabla_x \Lambda \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi\, \gamma f v \right) \cdot \nabla_x \Lambda, \quad \text{as } R \to \infty. \tag{91}$$

Using assumption (84), regularity properties (21) and Hölder inequality, we obtain,

$$\begin{aligned} \left| \int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi\, \gamma f F_L \cdot \nabla_\xi \Theta_R \right) \Lambda \right| &\leq C R^{-1} \|\nabla \theta\|_{L^\infty} \|\Lambda\|_{L^\infty} \\ \Big\| \int_{\mathbb{R}^3} d\xi\, \gamma f \Big\|_{L^\infty(0,T;L^{r'}(\mathbb{R}^3))} &(\|E\|_{L^\infty(0,T;L^r(\mathbb{R}^3))} + \|B\|_{L^\infty(0,T;L^r(\mathbb{R}^3))}) \longrightarrow 0, \quad \text{as } R \to \infty, \end{aligned} \tag{92}$$

with $1/r + 1/r' = 1$ and setting $r = 2$. We now claim that,

$$|f v \cdot E \Lambda| \leq |\Lambda| |E| f \ \in L^\infty(0,T; L^1(\mathbb{R}^6)) \quad \text{if} \quad \int_{\mathbb{R}^3} d\xi\, \gamma f \in L^\infty(0,T; L^{3/2}(\mathbb{R}^3)). \tag{93}$$

Indeed using interpolation Lemma 2.3 in [22], we obtain,

$$\Big\| \int_{\mathbb{R}^3} d\xi\, f \Big\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq 9 \|f\|_{L^\infty}^{1/4} \Big\| \int_{\mathbb{R}^3} d\xi\, \gamma f \Big\|_{L^\infty(0,T;L^{3/2}(\mathbb{R}^3))}^{3/4}. \tag{94}$$

Therefore (93) results from (94) and Cauchy-Schwarz inequality. We notice that the $L^{3/2}$-integrability condition in (93) results from regularity properties (21), assumption (84) and standard interpolation results between Lebesgue spaces. Using (89) we have $f v \cdot E \Theta_R \Lambda \to f v \cdot E \Lambda$ a.e. as $R \to +\infty$. Moreover using (93), we obtain, from the Lebesgue dominated convergence theorem,

$$\int_0^T dt \int_{\mathbb{R}^3} dx \left(\int_{\mathbb{R}^3} d\xi\, f v \cdot E \Theta_R \right) \Lambda \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx\, j \cdot E \Lambda, \quad \text{as } R \to \infty. \tag{95}$$

Using the weak formulation of the Maxwell equation, we obtain,

$$\int_0^T dt \int_{\mathbb{R}^3} dx\, j \cdot E \Lambda = \int_0^T dt \int_{\mathbb{R}^3} dx\, \frac{|E|^2 + |B|^2}{2} \partial_t \Lambda + \int_0^T dt \int_{\mathbb{R}^3} dx\, E \times B \cdot \nabla_x \Lambda. \tag{96}$$

Using (90)-(92) and (95)-(96), we obtain from (88),

$$\begin{aligned} \int_0^T dt \int_{\mathbb{R}^3} dx \left\{ \left(\int_{\mathbb{R}^3} d\xi\, \gamma f \right) + \frac{|E|^2 + |B|^2}{2} \right\} \partial_t \Lambda \\ + \int_0^T dt \int_{\mathbb{R}^3} dx \left\{ \left(\int_{\mathbb{R}^3} d\xi\, \gamma f v \right) + E \times B \right\} \cdot \nabla_x \Lambda = 0, \end{aligned} \tag{97}$$

which gives the local conservation law of total energy (85). We continue by deriving the global conservation law of total energy. For this we take in (97) a test function Λ such that,

$$\Lambda(t,x) = \varphi(t) \Lambda(x), \quad \text{with } \ \varphi \in \mathcal{D}((0,T)), \ \text{ and } \ \Lambda \in \mathcal{D}(\mathbb{R}^3).$$

We then choose the test function Λ such that,

$$\Lambda(x) = \Lambda_R(x) := \lambda(x/R), \quad \text{with } \ R > 0.$$

Here the function $\lambda \in \mathcal{D}(\mathbb{R}^3)$ is such that $\text{supp}(\lambda) \subset B_{\mathbb{R}^3}(0,2)$, $\lambda \equiv 1$ on $B_{\mathbb{R}^3}(0,1)$ and $0 \le \lambda \le 1$ on $(B_{\mathbb{R}^3}(0,2) \setminus B_{\mathbb{R}^3}(0,1))$. We then have,

$$\Lambda_R \longrightarrow 1, \quad \text{a.e as} \quad R \to +\infty, \quad \text{and} \quad \nabla_x \Lambda_R \longrightarrow 0, \quad \text{a.e as} \quad R \to +\infty. \tag{98}$$

Using (98) and regularity properties (21), especially $f \in L^\infty(0,T; L^1_\gamma(\mathbb{R}^6))$ and $E,\, B \in L^\infty(0,T; L^2(\mathbb{R}^3))$, we obtain from the Lebesgue dominated convergence theorem,

$$\int_0^T dt \int_{\mathbb{R}^3} dx \left\{ \left(\int_{\mathbb{R}^3} d\xi\, \gamma f \right) + \frac{|E|^2 + |B|^2}{2} \right\} \Lambda_R \partial_t \varphi \longrightarrow \int_0^T dt \int_{\mathbb{R}^3} dx \left\{ \left(\int_{\mathbb{R}^3} d\xi\, \gamma f \right) + \frac{|E|^2 + |B|^2}{2} \right\} \partial_t \varphi, \quad \text{as} \quad R \to +\infty, \tag{99}$$

and

$$\int_0^T dt \int_{\mathbb{R}^3} dx \left\{ \left(\int_{\mathbb{R}^3} d\xi\, \gamma f v \right) + E \times B \right\} \cdot \nabla_x \Lambda_R \varphi \longrightarrow 0, \quad \text{as} \quad R \to +\infty. \tag{100}$$

Using (99)-(100), and passing to the limit $R \to +\infty$ in (97), with $\Lambda(t,x) = \varphi(t)\lambda(x/R)$, we obtain,

$$\int_0^T dt\, \partial_t \varphi \int_{\mathbb{R}^3} dx \left\{ \left(\int_{\mathbb{R}^3} d\xi\, \gamma f \right) + \frac{|E|^2 + |B|^2}{2} \right\} = 0, \tag{101}$$

which gives the global conservation law of total energy (86). □

Remark 11 *1. If $E,\, B \in L^\infty(0,T; L^\infty(\mathbb{R}^6))$, we observe that the proof of Theorem 3 remains valid without condition (84), and then local and global conservation of total energy (85)-(86) are satisfied.*

2. Using the continuous embedding $W^{\beta,q}(\mathbb{R}^3) \subset L^{3q/(3-\beta q)}(\mathbb{R}^3)$, with $\beta q < 3$, we observe that if $E,\, B \in L^\infty(0,T; L^2 \cap W^{\beta,q}(\mathbb{R}^3))$, and

$$\int_{\mathbb{R}^3} d\xi\, \gamma f \in L^\infty\big(0,T; L^{3q/((3+\beta)q-3)}(\mathbb{R}^3)\big),$$

then estimates (92) and (95) still hold. Therefore local and global conservation of total energy (85)-(86) are satisfied.

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