On global stability of optimal rearrangement maps

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Abstract

We study the nonlocal vectorial transport equation $\partial_t y + (\mathbb{P}y \cdot \nabla)y = 0$ on bounded domains of $\mathbb{R}^d$ where $\mathbb{P}$ denotes the Leray projector. This equation was introduced to obtain the unique optimal rearrangement of the initial map y_0 as its steady states ([1, 10, 4]). We rigorously justify this expectation by proving that for initial maps y_0 sufficiently close to maps with strictly convex potential, the solutions y are global in time and converge exponentially fast to the optimal rearrangement of y_0 as time tends to infinity.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ equiped with the Lebesgue measure. Two L^2 maps $y_1, y_2 : \Omega \to \mathbb{R}^d$ are rearrangements of each other if they define the same image measure of the Lebesgue measure, i.e.

$$\int_\Omega f(y_1(x))dx = \int_\Omega f(y_2(x))dx$$

for all compactly supported continuous function $f : \mathbb{R}^d \to \mathbb{R}$. A celebrated theorem due to Brenier [3] asserts that for each L^2 map $y_0 : \Omega \to \mathbb{R}^d$ there exists a unique rearrangement y^* with *convex potential*, i.e. $y^* = \nabla p^*$ for some convex function p^*. Moreover, among all possible rearrangements of y_0, y^* minimizes the quadratic cost function

$$\int_\Omega |y(x) - x|^2 dx.$$

We shall refer to y^* as the *optimal rearrangement of* y_0. Finding the unique optimal rearrangement y^* for a given map y_0 is thus among the main concerns in optimal transport theory. As an attempt to get the optimal rearrangement y^* of y_0 as an *equilibrium state*

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in the infinite time of a dynamical system that could be efficiently solved by computer, Angenent, Haker, and Tannenbaum [1] (see also McCann [10, 11] and Brenier [4]) proposed the following nonlocal vectorial transport model (AHT)

$$\begin{aligned} \partial_t y + u \cdot \nabla y = 0, \\ u = \mathbb{P} y, \end{aligned} \tag{1.1}$$

where $y = y(x,t) \in \mathbb{R}^d$, $x \in \Omega \subset \mathbb{R}^d$, $t \geq 0$, and $\mathbb{P}$ denotes the classical Leray projector onto the space of divergence-free vector fields. Throughout the paper, we take Ω a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with smooth boundary. The Leray projector $u = \mathbb{P}y$ is defined as follows. For a given map $y : \Omega \to \mathbb{R}^d$, we construct the potential p that solves

$$\begin{cases} \Delta p = \nabla \cdot y & \text{in} \quad \Omega \\ \dfrac{\partial p}{\partial n} = y \cdot n & \text{on} \quad \partial\Omega \end{cases} \tag{1.2}$$

where n is the unit outward normal to $\partial\Omega$. Then we define

$$\mathbb{P}y = y - \nabla p.$$

As a consequence of the definition, the velocity $u = \mathbb{P}y$ is tangent to the boundary,

$$u \cdot n = 0 \qquad \text{on} \quad \partial\Omega. \tag{1.3}$$

Interestingly, the AHT model (1.1) can also be obtained as the zero inertial limit of generalized (damped) Euler-Boussinesq equations in convection theory [4, 6]. In addition, by specifying $y(x) = (0, \rho(x))$, (1.1) reduces to the incompressible porous media (IPM) equations

$$\begin{cases} \partial_t \rho + (u \cdot \nabla)\rho = 0, & x \in \Omega \subset \mathbb{R}^2, \\ u + \nabla p = (0, \rho)^T. \end{cases} \tag{1.4}$$

Here ρ plays the role of fluid density. Stability of the special solution $\rho_*(x_1, x_2) = x_2$ of (1.4) has been proved in [8] for $\Omega = \mathbb{T}^2$ or $\mathbb{R}^2$, and in [7] for $\Omega = \mathbb{T} \times (-L, L)$ which posses two horizontal boundaries. The presence of boundaries, though only flat boundaries, makes the proof in [7] more involved.

Following [4] let us explain why (1.1) is expected to capture the optimal rearrangement of initial maps as steady states in infinite time. First, since the velocity u is divergence-free and tangent to the boundary, we have

$$\frac{d}{dt} \int_\Omega f(y(x,t))dx = 0 \quad \forall t > 0$$

for any compactly supported continuous function $f : \mathbb{R}^d \to \mathbb{R}$. Integrating this in time we obtain that each $y(t)$, $t > 0$ is a rearrangement of $y(0)$. Second, it is readily checked that the balance law

$$\frac{d}{dt} \int_\Omega \frac{1}{2} |y - x|^2 \, dx = - \int_\Omega |u|^2 \, dx$$

holds. In particular, steady states must be gradients since their Leray projections vanish. Conversely, all gradients are clearly steady states of (1.1). Now if y is global and the infinite-time limit y_∞ of y exists (in a sufficiently strong topology) then the integral $\int_0^\infty \int_\Omega |u|^2 dxdt$ is finite. Consequently, u vanishes as $t \to \infty$ and thus y_∞ must be a gradient, $y_\infty = \nabla p_\infty$. If we have in addition that p_∞ is a convex function, then coupling with the fact that y_∞ is a rearrangement of $y(0)$ we conclude by virtue of the aforementioned theorem of Bernier that y_∞ is the unique optimal rearrangement of $y(0)$. The remaining issues in the above argument are global existence and long time behavior for (1.1). On the other hand, the objects that we expect (1.1) to capture in infinite time are maps with convex potential. A natural problem then is:

Are maps with convex potential globally stable?

Our goal in the present paper is to prove that maps with *strictly convex* potential are globally stable. Precisely, our main theorem reads as follows.

Theorem 1.1. *Let $s > 1 + \frac{d}{2}$ be an integer with $d \geq 2$. Let Ω be a C^∞ bounded domain in $\mathbb{R}^d$. Consider $y_* = \nabla p_*$ for some strictly convex function $p_* : \Omega \to \mathbb{R}$ whose Hessian satisfies*

$$\nabla^2 p_*(x) \geq \theta_0 Id \quad \forall x \in \Omega, \quad \theta_0 > 0. \tag{1.5}$$

Then, there exists a small positive number ε depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ such that for all $y_0 \in H^s(\Omega)$ with $\|y_0 - y_*\|_{H^s(\Omega)} \leq \varepsilon$, problem (1.1) has a unique global solution y. In addition, there is a positive constant C depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ so that

$$\|y(t) - y_*\|_{H^s(\Omega)} \leq C\|y_0 - y_*\|_{H^s(\Omega)} \tag{1.6}$$

and

$$\|\mathbb{P}y(t)\|_{H^s(\Omega)} \leq C\|y_0 - y_*\|_{H^s(\Omega)} e^{-\frac{\theta_0}{C}t} \tag{1.7}$$

for all $t \geq 0$. Moreover, there exists a strictly convex function p_∞ such that

$$\|y(t) - \nabla p_\infty\|_{H^{s-1}(\Omega)} \leq Ce^{-\frac{\theta_0 t}{C}} \quad \forall t \geq 0. \tag{1.8}$$

In particular, ∇p_∞ is the optimal rearrangement of y_0.

Remark 1.2. The domain Ω need not be C^∞ but only $C^{[s]+n_0}$ for a sufficiently large integer n_0. Our results in this paper also apply to the case when $\Omega = \mathbb{T}^d$.

The estimate (1.8) exhibits the exponential convergence towards the optimal rearrangement of y_0 provided that y_0 is sufficiently close to a map with strictly convex potential. This justifies the efficiency of the AHT model (1.1). Theorem 1.1 also provides the first class of time-dependent global solutions to this nonlocal vectorial transport equation for which the issues of global regularity and finite-time blowup remain open.

Let $y_* = \nabla p_*$ be a steady state of (1.1) where p_* satisfies the strict convexity condition (1.5). Introduce the perturbation $z = y - y_*$. Noting that $\mathbb{P}y_* = 0$, equation (1.1) yields

$$\begin{aligned} \partial_t z + u\cdot\nabla y_* + u\cdot\nabla z = 0,\\ u = \mathbb{P}z, \end{aligned} \tag{1.9}$$

where $u \cdot n = 0$ on $\partial\Omega$. In order to obtain the global stability, some form of decay is needed. Since z is transported, it is not expected to decay. Our idea is to obtain decay for the divergence-free part u of z. Indeed, taking Leray's projection of (1.9) one finds that u obeys

$$\partial_t u + \mathbb{P}(u\cdot\nabla y_*) + \mathbb{P}(u\cdot\nabla z) = 0. \tag{1.10}$$

An L^2 energy estimate combined with the strict convexity of p_* and the fact that $\mathbb{P}$ is self-adjoint in L^2 shows that u decays exponentially when measured in L^2. We need however decay of high Sobolev norms of u in order to close the nonlinear iteration. In performing a direct H^s energy estimate for u at the level of (1.10), there are at least two difficulties:

(i) the term $u\cdot\nabla z$ would induce a loss of derivatives due to the presence of ∇z;

(ii) to reveal the damping mechanism due to $\nabla y_* = \nabla^2 p_* \geq \theta_0\mathrm{Id}$, one needs to make appear the term $D^s u\cdot\nabla y_*$ where D^s denotes any partial derivatives of order s. However, in the presence of boundaries, D^s do not commute with $\mathbb{P}$. Moreover, in general the commutator $[D^s, \mathbb{P}]$ does not exhibit a gain of derivative, and hence is of the same order as the damping term.

To handle (i) we commute $\mathbb{P}$ with $u\cdot\nabla$ as follows

$$\partial_t u + \mathbb{P}(u\cdot\nabla y_*) + u\cdot\nabla u + [\mathbb{P}, u\cdot\nabla]z = 0. \tag{1.11}$$

The new nonlinear term $u\cdot\nabla u$ is now an advection term, and thus does not induce any loss of derivatives. However, a gain of one derivative in $[\mathbb{P}, u\cdot\nabla]z$ is then needed. As mentioned in (ii), such a gain is not true in general for $[\mathbb{P}, \partial_j]$. Interestingly, if one replaces partial derivatives ∂_j with $u\cdot\nabla$, this holds even in domains with boundary, provided only that u is tangent to the boundary. This is the content of the next theorem, which is of independent interest.

Theorem 1.3. *Let $s > 1 + \frac{d}{2}$ be an integer with $d \geq 2$, and Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary. Let $\mathbb{P}$ denote the Leray projector associated to Ω. Then, for any vector fields u, $z \in H^s(\Omega;\mathbb{R}^d)$ with $u\cdot n|_{\partial\Omega} = 0$, the commutator estimate*

$$\|[\mathbb{P}, u\cdot\nabla]z\|_{H^s(\Omega)} \leq C\|u\|_{H^s(\Omega)}\|z\|_{H^s(\Omega)} \tag{1.12}$$

holds for some universal constant C.

Regarding the difficulty (ii), we observe that "tangential derivatives" commute nicely with the Leray projector while "normal derivatives" do not. We then introduce a boundary adapted system of derivatives $\mathcal{D}^s$ (see Section 2.1) which are defined everywhere and become the usual tangential and normal derivatives when restricted to the boundary. Next, to avoid the commutator $[\mathcal{D}^s, \mathbb{P}]$ when dealing with the nonlocal term $\mathbb{P}(u \cdot \nabla y_*)$ we write

$$\int_\Omega \mathcal{D}^s u \cdot \mathcal{D}^s \mathbb{P}(u \cdot \nabla y_*)dx = \int_\Omega \mathcal{D}^s u \cdot \mathcal{D}^s (u \cdot \nabla y_*)dx + \int_\Omega \mathcal{D}^s u \cdot \mathcal{D}^s (\mathbb{P} - \mathrm{Id})(u \cdot \nabla y_*)dx$$

and notice a special structure in the second integral. This allows us to prove a *hierarchy of estimates* for the velocity u, ordered by *the number of normal derivatives* in $\mathcal{D}^s$, and hence to close our nonlinear iteration.

For the proof of Theorem 1.1 we will need the local well-posedness of the AHT model (1.1) in Sobolev spaces:

Theorem 1.4. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$ with smooth boundary. Let $s > 1 + \frac{d}{2}$ be an integer. Then for any initial data $z_0 \in H^s(\Omega)$, there exist a positive time T depending only on $\|z_0\|_{H^s(\Omega)}$ and a unique solution $z \in C([0, T]; H^s(\Omega))$ of (1.1).*

Local well-posedness of (1.1) in Hölder spaces $C^{1,\alpha}(\Omega)$ has been obtained in [1]. Since the velocity u has the same Sobolev regularity as the unknown z, Theorem 1.4 can be proved using standard energy methods.

The paper is organized as follows. Section 2 is devoted to various commutator estimates involving the Leray projector, while the proof of Theorem 1.1 is given in Section 3. Throughout this paper, we denote by ∂_j, $j \in \{1, ..., d\}$ the jth partial derivative and by D^m any partial derivatives of order $m \in \mathbb{N}$.

2 Commutator estimates

2.1 A boundary adapted system of derivatives

For simplicity, we assume from now on that Ω is a C^∞ domain. Let $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ be the distance function. There exists a small number $\kappa > 0$ such that δ is C^∞ in the neighborhood of the region

$$\Omega_{3\kappa} = \{x \in \Omega : \delta(x) \leq 3\kappa\}$$

and $\nabla\delta(x) \neq 0$ for any $x \in \Omega_{3\kappa}$. Note that the unit outward normal $n(x) = -\nabla\delta(x)$ for $x \in \partial\Omega$. We thus can extend n to $\Omega_{3\kappa}$ by setting

$$n(x) = -\frac{\nabla\delta(x)}{|\nabla\delta(x)|}, \quad x \in \Omega_{3\kappa}.$$

For each $x \in \Omega_{3\kappa}$, we can choose $\tau(x) = \{\tau_j(x) : j = 1, ..., d-1\}$ an orthonormal basis of $(n(x))^\perp$ in $\mathbb{R}^d$ such that $\tau_j \in C^\infty(\Omega_{3\kappa})$.

Next we fix a cutoff function $\chi_1 : \overline{\Omega} \to [0, 1]$ satisfying

$$\chi_1 \equiv 1 \quad \text{in a neighborhood of } \Omega_{2\kappa}, \qquad \chi_1 \equiv 0 \quad \text{in } \overline{\Omega} \setminus \Omega_{3\kappa}. \tag{2.1}$$

For a vector field $v : \Omega \to \mathbb{R}^d$ we define its weighted normal and tangential components respectively by

$$v_n(x) = \chi_1(x)v(x) \cdot n(x), \quad v_{\tau_j}(x) = \chi_1(x)v(x) \cdot \tau_j(x), \quad j = 1, ..., d-1$$

for $x \in \Omega$. In particular, $v = v_n n + \sum_{j=1}^{d-1} v_{\tau_j}\tau_j$ in $\Omega_{2\kappa}$. In the special case of gradient vectors ∇f where $f : \Omega \to \mathbb{R}$, we write

$$\partial_n f = (\nabla f)_n, \quad \partial_{\tau_j} f = (\nabla f)_{\tau_j}, \quad j = 1, ..., d-1.$$

Both $\partial_n f$ and $\partial_\tau f$ are defined over Ω and become the usual normal and tangential derivatives when restricted to the boundary. Note in addition that

$$\nabla f = n\partial_n f + \sum_{j=1}^{d-1} \tau_j \partial_{\tau_j} f \quad \text{in } \Omega_{2\kappa}. \tag{2.2}$$

For a vector field $v : \Omega \to \mathbb{R}^d$ we write $\partial_n v = (\nabla v) \cdot n$ and similarly for $\partial_{\tau_j} v$. Then we have

$$|\nabla v|^2 = \sum_{i=1}^{d} |\nabla v_i|^2 = \sum_{i=1}^{d}(|\partial_n v_i|^2 + \sum_{j=1}^{d-1} |\partial_{\tau_j} v_i|^2) = |\partial_n v|^2 + \sum_{j=1}^{d-1} |\partial_{\tau_j} v|^2 \tag{2.3}$$

for $x \in \Omega_{2\kappa}$.

Lemma 2.1. *For $v : \Omega \to \mathbb{R}^d$ and $f : \Omega \to \mathbb{R}$ we have*

$$\partial_n v \cdot n + \sum_{j=1}^{d-1} \partial_{\tau_j} v \cdot \tau_j = \operatorname{div} v \tag{2.4}$$

and

$$\partial_n^2 f + \sum_{j=1}^{d-1} \partial_{\tau_j}^2 f = \Delta f + \nabla f \cdot (n \cdot \nabla)n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla)\tau_j \tag{2.5}$$

at any $x \in \Omega_{2\kappa}$.

Proof. We first notice that since $\chi_1 \equiv 1$ in $\Omega_{2\kappa}$. If R denotes the matrix whose first $d-1$ columns are $\tau_1, ..., \tau_{d-1}$ and whose dth column is n, then R is orthonormal; that is, $RR^T = \text{Id}$. Using this and the above definitions of ∂_n and ∂_{τ_j} we have

$$\begin{aligned}
\partial_n v \cdot n + \sum_{j=1}^{d-1} \partial_{\tau_j} v \cdot \tau_j &= \sum_{k,\ell=1}^{d} \partial_k v_\ell n_k n_\ell + \sum_{j=1}^{d-1} \sum_{k,\ell=1}^{d} \partial_k v_\ell \tau_{j,k} \tau_{j,\ell} \\
&= \sum_{k,\ell=1}^{d} \partial_k u_\ell \sum_{j=1}^{d} R_{k,j} R_{j,\ell}^T \\
&= \sum_{k,\ell=1}^{d} \partial_k u_\ell \delta_{k,\ell} \\
&= \operatorname{div} u.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\partial_n^2 f + \sum_{j=1}^{d-1} \partial_{\tau_j}^2 f &= \sum_{k,\ell=1}^{d} \partial_k \partial_\ell f n_k n_\ell + \sum_{j=1}^{d-1} \sum_{k,\ell=1}^{d} \partial_k \partial_\ell f \tau_{j,k} \tau_{j,\ell} + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \\
&= \sum_{k,\ell=1}^{d} \partial_k \partial_\ell f \delta_{k,\ell} + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j \\
&= \Delta f + \nabla f \cdot (n \cdot \nabla) n + \sum_{j=1}^{d-1} \nabla f \cdot (\tau_j \cdot \nabla) \tau_j .
\end{aligned}$$

□

2.2 Proof of Theorem 1.3

We now give the proof of Theorem 1.3. Fix an integer $s > 1 + \frac{d}{2}$. By definition, we write $z = \mathbb{P}z + \nabla f$, where f solves

$$\begin{cases} \Delta f = \operatorname{div} z & \text{in } \Omega, \\ \frac{\partial f}{\partial n} = z \cdot n & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

In particular, the standard elliptic regularity theory yields $\|f\|_{H^{s+1}(\Omega)} \le C\|z\|_{H^s(\Omega)}$. Similarly, we write $(u \cdot \nabla)z = \mathbb{P}((u \cdot \nabla)z) + \nabla g$, where g solves

$$\begin{cases} \Delta g = \operatorname{div}((u \cdot \nabla) z) & \text{in } \Omega, \\ \frac{\partial g}{\partial n} = (u \cdot \nabla) z \cdot n & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Combining, we have

$$\begin{aligned}
[\mathbb{P}, u \cdot \nabla] z &= \mathbb{P}((u \cdot \nabla) z) - (u \cdot \nabla) \mathbb{P} z \\
&= (u \cdot \nabla) z - \nabla g - (u \cdot \nabla)(z - \nabla f) \\
&= (u \cdot \nabla)(\nabla f) - \nabla g.
\end{aligned} \tag{2.8}$$

We shall bound the H^s norm of $[\mathbb{P}, u \cdot \nabla]z$, using the following elliptic estimate

$$\|h\|_{H^s(\Omega)} \le C\|\operatorname{div} h\|_{H^{s-1}(\Omega)} + C\|\operatorname{curl} h\|_{H^{s-1}(\Omega)} + C\|h \cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)}, \tag{2.9}$$

for $h = [\mathbb{P}, u \cdot \nabla]z$, where the terms on the right hand side are estimated in the following lemmas.

Lemma 2.2. *There exists a positive constant C such that*

$$\|\operatorname{div}([\mathbb{P}, u \cdot \nabla] z)\|_{H^{s-1}(\Omega)} \le C\|u\|_{H^s(\Omega)} \|z\|_{H^s(\Omega)} \tag{2.10}$$

and

$$\|\operatorname{curl}([\mathbb{P}, u \cdot \nabla] z)\|_{H^{s-1}(\Omega)} \le C\|u\|_{H^s(\Omega)} \|z\|_{H^s(\Omega)}. \tag{2.11}$$

Proof. In view of (2.8), we compute, using (2.6),

$$\operatorname{div}((u\cdot\nabla)(\nabla f)) = \nabla u : (\nabla\otimes\nabla)f + u\cdot\nabla \operatorname{div} z.$$

On the other hand, using equation (2.7), we have

$$\operatorname{div}(\nabla g) = \Delta g = \operatorname{div}((u\cdot\nabla)z). = \nabla u : (\nabla z)^T.$$

Combining, we have

$$\operatorname{div}([\mathbb{P}, u\cdot\nabla]z) = \nabla u : [(\nabla\otimes\nabla)f - (\nabla z)^T]. \tag{2.12}$$

The estimate (2.10) thus follows directly from (2.12), upon using the fact that $H^{s-1}(\Omega)$ is an algebra and the elliptic estimates $\|f\|_{H^{s+1}(\Omega)} \le C\|z\|_{H^s(\Omega)}$.

Next, in view of (2.8), we write

$$[\mathbb{P}, u\cdot\nabla]z = \nabla(u\cdot\nabla f - g) - \nabla u_k\partial_k f$$

which gives $\operatorname{curl}([\mathbb{P}, u\cdot\nabla]z) = \nabla u_k \times \partial_k\nabla f$. The estimate (2.11) then follows from elliptic estimates as before. □

Lemma 2.3. *There exists a positive constant C such that*

$$\|[\mathbb{P}, u\cdot\nabla]z\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^s(\Omega)}\|z\|_{H^s(\Omega)}. \tag{2.13}$$

Proof. We use the decomposition $u = u_n n + \sum_{j=1}^{d-1} u_{\tau_j}\tau_j$ in $\Omega_{2\kappa}$. Then we compute

$$\begin{aligned}(u\cdot\nabla)z\cdot n &= (u\cdot\nabla)(z\cdot n) - (u\otimes z) : \nabla n\\ &= u_n(n\cdot\nabla)(z\cdot n) + \sum_{j=1}^{d-1} u_{\tau_j}(\tau_j\cdot\nabla)(z\cdot n) - (u\otimes z) : \nabla n\end{aligned}$$

in $\Omega_{2\kappa}$. Since $u\cdot n = 0$ on $\partial\Omega$, we have $u_n = 0$ on $\partial\Omega$. Taking the trace of the above equation on $\partial\Omega$ and recalling (2.7), we get

$$\nabla g\cdot n = (u\cdot\nabla)z\cdot n = \sum_{j=1}^{d-1} u_{\tau_j}\partial_{\tau_j}(z\cdot n) - (u\otimes z) : \nabla n \quad \text{on } \partial\Omega.$$

Similarly, on $\partial\Omega$, we have

$$(u\cdot\nabla)(\nabla f)\cdot n = \sum_{j=1}^{d-1} u_{\tau_j}\partial_{\tau_j}(\nabla f\cdot n) - (u\otimes\nabla f) : \nabla n.$$

Recalling (2.8) and using the boundary condition (2.6), which gives $\partial_{\tau_j}(\nabla f\cdot n) = \partial_{\tau_j}(z\cdot n)$ on $\partial\Omega$, we obtain

$$[\mathbb{P}, u\cdot\nabla]z\cdot n = (u\cdot\nabla)(\nabla f)\cdot n - \nabla g\cdot n = [u\otimes(z-\nabla f)] : \nabla n$$

on $\partial\Omega$. Using the trace inequality, we bound

$$\begin{aligned}\|[\mathbb{P}, u\cdot\nabla]z\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &\le C\|u\otimes(z-\nabla f)\|_{H^s(\Omega)}\\ &\le C\|u\|_{H^s(\Omega)}\big(\|z\|_{H^s(\Omega)}+\|f\|_{H^{s+1}(\Omega)}\big)\end{aligned}$$

which gives (2.13), upon recalling the elliptic estimates $\|f\|_{H^{s+1}} \le C\|z\|_{H^s}$. □

2.3 Commutators between the Leray projector and tangential derivatives

Proposition 2.4. *Let $m \ge 2$ be an integer. There exists a constant $C > 0$ such that*

$$\|[P, \mathbb{P}]u\|_{L^2(\Omega)} \le C\|u\|_{H^{m-1}(\Omega)}$$

for any $P \in \{\Pi_{j=1}^m \partial_{\sigma_j} : \sigma_j \in \{\tau_1, ..., \tau_{d-1}\}\}$ and any vector field $u \in H^{m-1}(\Omega)$.

Proof. Without loss of generality we consider $P = \partial_{\tau_1}^m$. In view of the identity

$$[\partial_{\tau_1}^{q+1}, \mathbb{P}]u = [\partial_{\tau_1}^q, \mathbb{P}]\partial_{\tau_1}u + \partial_{\tau_1}^q[\partial_{\tau_1}, \mathbb{P}]u, \quad q \ge 1$$

and by induction in m, it suffices to prove that

$$\|[\partial_{\tau_1}, \mathbb{P}]\partial_{\tau_1}u\|_{L^2(\Omega)} \le C\|u\|_{H^1(\Omega)} \tag{2.14}$$

and

$$\|[\partial_{\tau_1}, \mathbb{P}]u\|_{H^j(\Omega)} \le C\|u\|_{H^j(\Omega)} \quad \forall j \ge 1. \tag{2.15}$$

To this end, for any vector field v, we write $\mathbb{P}v = v - \nabla f$ and $\mathbb{P}(\partial_{\tau_1}v) = \partial_{\tau_1}v - \nabla g$ where f and g solve

$$\begin{cases}\Delta f = \operatorname{div} v & \text{in } \Omega,\\ \frac{\partial f}{\partial n} = v\cdot n & \text{on } \partial\Omega\end{cases}$$

and

$$\begin{cases}\Delta g = \operatorname{div}(\partial_{\tau_1}v) & \text{in } \Omega,\\ \frac{\partial g}{\partial n} = (\partial_{\tau_1}v)\cdot n & \text{on } \partial\Omega\end{cases}$$

respectively. Then

$$[\partial_{\tau_1}, \mathbb{P}]v = \nabla g - \partial_{\tau_1}\nabla f = \nabla(g - \partial_{\tau_1}f) - [\partial_{\tau_1}, \nabla]f. \tag{2.16}$$

We compute

$$\begin{aligned}&\Delta\partial_{\tau_1}f = \partial_{\tau_1}\Delta f + \Delta(\chi_1\tau_1)\cdot\nabla f + 2\nabla(\chi_1\tau_1) : \nabla\nabla f,\\ &\operatorname{div}(\partial_{\tau_1}v) = \partial_{\tau_1}\operatorname{div} v + \nabla v : (\nabla(\chi_1\tau_1))^T,\end{aligned}$$

where χ_1 is defined as in (2.1). As a consequence, $h := g - \partial_{\tau_1}f$ satisfies

$$\Delta h = \nabla v : (\nabla(\chi_1\tau_1))^T - \Delta(\chi_1\tau_1)\cdot\nabla f - 2\nabla(\chi_1\tau_1) : \nabla\nabla f \quad \text{in } \Omega. \tag{2.17}$$

Regarding the boundary condition, we have

$$\begin{aligned}
\partial_{\tau_1}(v\cdot n) &= \partial_{\tau_1}v\cdot n + \nabla n : (v\otimes\tau_1),\\
\partial_n(\partial_{\tau_1}f) &= \nabla\nabla f : (n\otimes\tau_1) + \nabla\tau_1 : (\nabla f\otimes n),\\
\partial_{\tau_1}(\partial_n f) &= \nabla\nabla f : (n\otimes\tau_1) + \nabla n : (\nabla f\otimes\tau_1)
\end{aligned}$$

in Ω. This yields

$$\begin{aligned}
\partial_n h &= \partial_n g - \partial_{\tau_1}\partial_n f - \nabla\tau_1 : (\nabla f\otimes n) + \nabla n : (\nabla f\otimes\tau_1)\\
&= (\partial_{\tau_1}v)\cdot n - \partial_{\tau_1}(v\cdot n) - \nabla\tau_1 : (\nabla f\otimes n) + \nabla n : (\nabla f\otimes\tau_1)\\
&= -\nabla n : (v\otimes\tau_1) - \nabla\tau_1 : (\nabla f\otimes n) + \nabla n : (\nabla f\otimes\tau_1) \quad \text{on } \partial\Omega.
\end{aligned} \tag{2.18}$$

In addition, elliptic estimates combined with trace inequalities

$$\|v\|_{H^{\ell-\frac{3}{2}}(\partial\Omega)} \le C\|v\|_{H^{\ell-1}(\Omega)} \quad \forall \ell\ge 2$$

yield

$$\|f\|_{H^\ell(\Omega)} \le C\|v\|_{H^{\ell-1}(\Omega)} \quad \forall \ell \ge 2. \tag{2.19}$$

Proof of (2.15). In view of (2.17), (2.18) we deduce using elliptic estimates, trace inequalities and (2.19) that for any $\ell\ge 2$,

$$\begin{aligned}
\|h\|_{H^\ell(\Omega)} &\le C\|\nabla v : (\nabla(\chi_1\tau_1))^T - \Delta(\chi_1\tau_1)\cdot\nabla f - 2\nabla(\chi_1\tau_1) : \nabla\nabla f\|_{H^{\ell-2}(\Omega)}\\
&\quad + C\| - \nabla n : (v\otimes\tau_1) - \nabla\tau_1 : (\nabla f\otimes n) + \nabla n : (\nabla f\otimes\tau_1)\|_{H^{\ell-\frac{3}{2}}(\partial\Omega)}\\
&\le C\|\nabla v : (\nabla(\chi_1\tau_1))^T - \Delta(\chi_1\tau_1)\cdot\nabla f - 2\nabla(\chi_1\tau_1) : \nabla\nabla f\|_{H^{\ell-2}(\Omega)}\\
&\quad + C\| - \nabla n : (v\otimes\tau_1) - \nabla\tau_1 : (\nabla f\otimes n) + \nabla n : (\nabla f\otimes\tau_1)\|_{H^{\ell-1}(\Omega)}\\
&\le C\|v\|_{H^{\ell-1}(\Omega)} + C\|f\|_{H^\ell(\Omega)}\\
&\le C'\|v\|_{H^{\ell-1}(\Omega)}.
\end{aligned} \tag{2.20}$$

Note that the trace inequality used in the second inequality in (2.20) does not hold when $\ell = 1$. Now for any $j\ge 1$ using (2.16), (2.19) and (2.20) with $\ell = j+1\ge 2$ together with the estimate

$$\|[\partial_{\tau_1},\nabla]f\|_{H^j(\Omega)} \le C\|f\|_{H^{j+1}(\Omega)}$$

we obtain

$$\begin{aligned}
\|[\partial_{\tau_1},\mathbb{P}]v\|_{H^j(\Omega)} &\le \|h\|_{H^{j+1}(\Omega)} + \|[\partial_{\tau_1},\nabla]f\|_{H^j(\Omega)}\\
&\le C\|v\|_{H^j(\Omega)} + C\|f\|_{H^{j+1}(\Omega)}\\
&\le C'\|v\|_{H^j(\Omega)}
\end{aligned}$$

which is the desired estimate (2.15) if we set $v = u$.

Proof of (2.14). Again, we use the equations (2.17), (2.18) with $v = \partial_{\tau_1} u$ and H^1 elliptic estimate for the Neumann problem to have

$$\begin{aligned}
\|h\|_{H^1(\Omega)} &\le C\|\nabla v : (\nabla(\chi_1\tau_1))^T - \Delta(\chi_1\tau_1)\cdot\nabla f - 2\nabla(\chi_1\tau_1) : \nabla\nabla f\|_{H^{-1}(\Omega)} \\
&\quad + C\| - \nabla n : (v\otimes\tau_1) - \nabla\tau_1 : (\nabla f\otimes n) + \nabla n : (\nabla f\otimes\tau_1)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\
&\le C'\|v\|_{L^2(\Omega)} + C'\|f\|_{H^1(\Omega)} + C'\|v\|_{H^{-\frac{1}{2}}(\partial\Omega)} + C'\|\nabla f\|_{H^{-\frac{1}{2}}(\partial\Omega)}.
\end{aligned}$$

Since $v = \partial_{\tau_1} u$ we have $\|v\|_{L^2(\Omega)} \le C\|u\|_{H^1(\Omega)}$ and

$$\|v\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C'\|u\|_{H^1(\Omega)}.$$

Moreover, using (2.2) and the Neumann boundary condition for f we can write

$$\nabla f = \sum_{j=1}^{d-1} \tau_j \partial_{\tau_j} f + n\partial_n f = \sum_{j=1}^{d-1} \tau_j \partial_{\tau_j} f + n(\partial_{\tau_1} u\cdot n) \quad \text{on } \partial\Omega.$$

This implies

$$\begin{aligned}
\|\nabla f\|_{H^{-\frac{1}{2}}(\partial\Omega)} &\le C\sum_{j=1}^{d-1} \|\partial_{\tau_j} f\|_{H^{-\frac{1}{2}}(\partial\Omega)} + C\|\partial_{\tau_1} u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\
&\le C'\|f\|_{H^{\frac{1}{2}}(\partial\Omega)} + C\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le C''\|f\|_{H^1(\Omega)} + C\|u\|_{H^1(\partial\Omega)}.
\end{aligned}$$

Thus, we obtain

$$\|h\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)} + C\|f\|_{H^1(\Omega)}.$$

The H^1 elliptic estimate for f gives

$$\begin{aligned}
\|f\|_{H^1(\Omega)} &\le C\|\partial_{\tau_1} u\|_{L^2(\Omega)} + C\|\partial_{\tau_1} u\cdot n\|_{H^{-\frac{1}{2}}(\partial\Omega)} \\
&\le C\|\partial_{\tau_1} u\|_{L^2(\Omega)} + C'\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le C''\|u\|_{H^1(\Omega)}.
\end{aligned}$$

Consequently

$$\|h\|_{H^1(\Omega)} \le C\|u\|_{H^1(\Omega)}$$

which combined with the commutator estimate

$$\|[\partial_{\tau_1}, \nabla] f\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)} \le C\|f\|_{H^1(\Omega)}$$

completes the proof of (2.14). □

Next we fix a cutoff function $\chi_2 : \overline{\Omega} \to [0, 1]$ satisfying

$$\chi_2 \equiv 0 \quad \text{in } \Omega_\kappa, \qquad \chi_2 \equiv 1 \quad \text{in } \overline{\Omega}\setminus\Omega_{2\kappa}. \tag{2.21}$$

Proposition 2.5. *Let $m \geq 1$ be an integer. There exists a constant $C > 0$ such that*

$$\|[\chi_2 D^m, \mathbb{P}]u\|_{L^2(\Omega)} \leq C\|u\|_{H^{m-1}(\Omega)}$$

for any vector field $u \in H^{m-1}(\Omega)$.

Proof. Without loss of generality we consider $D^m = \partial_1^m$. We have

$$\begin{aligned} u &= \chi_2\partial_1^m(u - \nabla f) - [\chi_2\partial_1^m u - \nabla g] \\ &= \chi_2\partial_1^m \nabla f - \nabla g \\ &= \nabla(\chi_2\partial_1^m f - g) - \nabla\chi_2\partial_1^m f. \end{aligned}$$

where f and g solve

$$\begin{cases} \Delta f = \operatorname{div} u & \text{in } \Omega, \\ \frac{\partial f}{\partial n} = u \cdot n & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta g = \operatorname{div}(\chi_2\partial_{\tau_1}^m u) & \text{in } \Omega, \\ \frac{\partial g}{\partial n} = (\chi_2\partial_{\tau_1^m} u) \cdot n & \text{on } \partial\Omega \end{cases}$$

respectively. By elliptic estimates for f we have

$$\|\nabla\chi_2\partial_1^m f\|_{L^2(\Omega)} \leq C\|u\|_{H^{m-1}(\Omega)}. \tag{2.22}$$

Setting $h = \chi_2\partial_1^m \nabla f - g$ we compute

$$\begin{aligned} \Delta h &= \Delta\chi_2\partial_1^m f + 2\nabla\chi_2 \cdot \nabla\partial_1^m f - \nabla\chi_2 \cdot \partial_1^m u \\ &= \Delta\chi_2\partial_1^m f + 2\operatorname{div}(\nabla\chi_2\partial_1^m f) - 2\Delta\chi_2\partial_1^m f - \partial_j(\nabla\chi_2 \cdot \partial_1^{m-1}u) + \nabla\partial_1\chi_2 \cdot \partial_1^{m-1}u. \end{aligned}$$

On the other hand, since $\chi_2 \equiv 0$ near $\partial\Omega$, $h \equiv 0$ near $\partial\Omega$. Thus, standard elliptic estimates give

$$\|h\|_{H^1(\Omega)} \leq C\|u\|_{H^{m-1}(\Omega)}. \tag{2.23}$$

A combination of (2.22) and (2.23) concludes the proof. □

3 Proof of Theorem 1.1

Let us start with a priori estimates for the perturbation $z = y - y_*$ which solves

$$\partial_t z + u \cdot \nabla y_* + u \cdot \nabla z = 0, \qquad u = \mathbb{P}z. \tag{3.1}$$

In what follows, we fix an integer $s > 1 + \frac{d}{2}$.

Lemma 3.1. *There exists $C_1 > 0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^s(\Omega)} \leq C_1(1 + \|z(t)\|_{H^s(\Omega)})\|u(t)\|_{H^s(\Omega)}\|z(t)\|_{H^s(\Omega)}. \tag{3.2}$$

Proof. First of all, an L^2 estimate for (3.1) gives

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{L^2(\Omega)} = -\int_\Omega z\cdot(u\cdot\nabla y_*)dx \le \|\nabla y_*\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|z\|_{L^2(\Omega)}, \tag{3.3}$$

where we used the fact that u is divergence-free and tangent to the boundary to have $\int_\Omega z\cdot(u\cdot\nabla z) = 0$ upon integration by parts. Recall that D^s denotes any partial derivatives of order s. Applying D^s to equation (3.1), then multiplying the resulting equation by $D^s z$ and integrating in space we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |D^s z|^2\,dx &= -\int_\Omega \Big[D^s z\cdot D^s(u\cdot\nabla z) + D^s(u\cdot\nabla y_*)\cdot D^s z\Big]\,dx\\ &= -\int_\Omega \Big[D^s z\cdot\big([D^s,u]\cdot\nabla z\big)dx + D^s z\cdot\big(u\cdot\nabla D^s z\big) + D^s(u\cdot\nabla y_*)\cdot D^s z\Big]\,dx\\ &= -\int_\Omega \Big[D^s z\cdot\big([D^s,u]\cdot\nabla z\big) + \frac{1}{2}u\cdot\nabla|D^s z|^2 dx + D^s(u\cdot\nabla y_*)\cdot D^s z\Big]\,dx\\ &= -\int_\Omega \Big[D^s z\cdot\big([D^s,u]\cdot\nabla z\big) + D^s(u\cdot\nabla y_*)\cdot D^s z\Big]\,dx\end{aligned}$$

where we used again the fact that u is divergence-free and tangent to the boundary to cancel out the term $\int_\Omega u\cdot\nabla|D^s z|^2 dx$. To bound the first term on the right-hand side we appeal to the commutator estimate (see [9] page 129)

$$\|[D^s,f]g\|_{L^2(\Omega)} \le C\|\nabla f\|_{L^\infty(\Omega)}\|g\|_{H^{s-1}(\Omega)} + C\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)}. \tag{3.4}$$

This combined with the embedding $H^s(\Omega)\subset W^{1,\infty}(\Omega)$ yields

$$\left|\int_\Omega \Big[D^s z\cdot\big([D^s,u]\cdot\nabla z\big)dx\right| \le C\|z\|^2_{H^s(\Omega)}\|u\|_{H^s(\Omega)}.$$

On the other hand, using the product rule gives

$$\left|\int_\Omega D^s(u\cdot\nabla y_*)\cdot D^s z\Big]\,dx\right| \le C\|y_*\|_{H^{s+1}(\Omega)}\|z\|_{H^s(\Omega)}\|u\|_{H^s(\Omega)}.$$

We thus obtain

$$\frac{1}{2}\frac{d}{dt}\|D^s z(t)\|^2_{L^2(\Omega)} \le C(1+\|z(t)\|_{H^s(\Omega)})\|u(t)\|_{H^s(\Omega)}\|z(t)\|_{H^s(\Omega)} \tag{3.5}$$

for some $C>0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$. Combining (3.3) and (3.5) leads to the estimate (3.2). □

As explained in the introduction, we expect that the divergence-free part u of z decays. To this end, let us take the Leray projection of (3.1):

$$\partial_t u + \mathbb{P}(u\cdot\nabla y_*) + \mathbb{P}(u\cdot\nabla z) = 0. \tag{3.6}$$

First, we prove an L^2 decay estimate for u.

Lemma 3.2. *There exists $C_2 > 0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \theta_0\|u(t)\|^2_{L^2(\Omega)} \leq C_2\|u(t)\|^2_{L^2(\Omega)}\|z(t)\|_{H^s(\Omega)}. \tag{3.7}$$

Proof. We multiply equation (3.6) by u, then integrate over Ω and use the fact that $\mathbb{P}$ is self-adjoint, giving

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx &= -\int_\Omega \mathbb{P}(u\cdot\nabla y_*)\cdot u dx - \int_\Omega \mathbb{P}(u\cdot\nabla z)\cdot u dx \\ &= -\int_\Omega (u\cdot\nabla y_*)\cdot u dx - \int_\Omega (u\cdot\nabla z)\cdot u dx.\end{aligned} \tag{3.8}$$

It is readily seen that

$$\left|\int_\Omega (u\cdot\nabla z)\cdot u dx\right| \leq C\|u\|^2_{L^2(\Omega)}\|z\|_{H^s(\Omega)}.$$

On the other hand, the convexity condition (1.5) implies that

$$-\int_\Omega (u\cdot\nabla y_*)\cdot u dx \leq -\theta_0\|u\|^2_{L^2(\Omega)}.$$

The lemma then follows. □

We observe that the L^2 decay estimate (3.7) was obtained using only the strict convexity of the potential p_* and the fact that $\mathbb{P}$ is self-adjoint. We will need however decay of the H^s norm of u in order to close the nonlinear iteration. The proof of (3.7) does not carry over to H^s decay since the Leray projector does not commute with D^s. It turns out that the commutator $[D^s, \mathbb{P}]$ does not gain derivative in general, leading to terms of the of same order as the damping term. To treat the boundary and the nonlocality of $\mathbb{P}$, we use the derivatives ∂_{τ_j} and ∂_n introduced in Section 2.3. These derivatives are defined everywhere in Ω and become the usual tangential and normal derivative when restricted to the boundary. The trade-off is that ∂_{τ_j} and ∂_n do not commute with usual partial derivatives, leading to commutators that are of lower order.

For $k \in \{0, 1, .., s\}$ we set

$$\mathcal{D}^s_k = \left\{\Pi^s_{j=1}\partial_{\sigma_j} : \sigma_j \in \{\tau_1, ..., \tau_{d-1}, n\} \text{ and } \#\{j : \sigma_j = n\} = k\right\}.$$

In other words, each derivative in $\mathcal{D}^s_k$ has exactly k normal derivatives and $s-k$ tangential derivatives. We also define the norms

$$\|v\|_{s,k} = \Big(\sum_{j=0}^{k}\sum_{P\in\mathcal{D}^s_j}\|Pv\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}$$

for $v : \Omega \to \mathbb{R}^d$.

Due to the presence of χ_1 in ∂_n and ∂_{τ_j}, the norms $\|u\|_{s,k}$ control u near the boundary.

3.1 Interior estimates for u

We commute $\mathbb{P}$ with $u\cdot\nabla$ in the last term of equation (3.6) to have

$$\partial_t u + \mathbb{P}(u\cdot\nabla y_*) + u\cdot\nabla u + [\mathbb{P}, u\cdot\nabla]z = 0. \tag{3.9}$$

This makes appear the good convection term $u\cdot\nabla u$ but at the same time generates the commutator $[\mathbb{P}, u\cdot\nabla]z$, which will be controlled by virtue of Theorem 1.3.

The next lemma provides a control of u in the interior.

Lemma 3.3. *There exists $C>0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2}\frac{d}{dt}\|\chi_2 u\|_{H^s}^2 + \theta_0\|\chi_2 u\|_{H^s}^2 \le C\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + C\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} \tag{3.10}$$

where χ_2 is defined in (2.21).

Proof. Set $P = \chi_2\partial_1^s$. Applying P to (3.9), then multiplying the resulting equation by Pu and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |Pu|^2\,dx + \int_\Omega Pu\cdot P\mathbb{P}(u\cdot\nabla y_*)\,dx\\ &\quad = -\int_\Omega Pu\cdot\big([P,u]\cdot\nabla u\big)dx - \int_\Omega Pu\cdot\big(u\cdot[P,\nabla]u\big)dx - \int_\Omega Pu\cdot P\big([\mathbb{P},u\cdot\nabla]z\big)dx.\end{aligned} \tag{3.11}$$

where we used the fact that

$$\int_\Omega Pu\cdot\big(u\cdot Pu\big)dx = \frac{1}{2}\int_\Omega u\cdot\nabla|Pu|^2dx = 0$$

since $\nabla\cdot u = 0$ in Ω and $u\cdot n|_{\partial\Omega} = 0$. We now treat each term on the right-hand side of (3.11). It is readily seen that

$$\begin{aligned}&\|[P,u]\cdot\nabla u\|_{L^2(\Omega)} \le C\|u\|_{H^s(\Omega)}^2,\\ &\|u\cdot[P,\nabla]u\|_{L^2(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}^2.\end{aligned} \tag{3.12}$$

In addition, Theorem 1.3 applied to Ω gives

$$\|P\big([\mathbb{P},u\cdot\nabla]z\big)\|_{L^2(\Omega)} \le C\|[\mathbb{P},u\cdot\nabla]z\|_{H^s(\Omega)} \le C\|u\|_{H^s(\Omega)}\|z\|_{H^s(\Omega)}. \tag{3.13}$$

Putting together (3.11), (3.12), (3.13) and using the estimate $\|u\|_{H^s}\le C\|z\|_{H^s}$ we obtain

$$\frac{1}{2}\quad\frac{d}{dt}\int_\Omega |Pu|^2\,dx + \int_\Omega Pu\cdot P\mathbb{P}(u\cdot\nabla y_*)\,dx \le C\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)}. \tag{3.14}$$

As for the second term on the left-hand side of (3.14), we commute P with $\mathbb{P}$ and then with ∇y_* to have

$$\begin{aligned}\int_\Omega Pu \cdot P\mathbb{P}(u\cdot\nabla y_*)\,dx &= \int_\Omega Pu\cdot[P,\mathbb{P}](u\cdot\nabla y_*)\,dx + \int_\Omega Pu\cdot\mathbb{P}([P,\nabla y_*\cdot]u)\,dx \\ &\quad + \int_\Omega Pu\cdot\mathbb{P}(\nabla y_*\cdot Pu)\,dx \\ &= \int_\Omega Pu\cdot[P,\mathbb{P}](u\cdot\nabla y_*)\,dx + \int_\Omega Pu\cdot\mathbb{P}([P,\nabla y_*\cdot]u)\,dx \\ &\quad + \int_\Omega [\mathbb{P},P]u\cdot(\nabla y_*\cdot Pu)\,dx + \int_\Omega Pu\cdot(\nabla y_*\cdot Pu)\,dx.\end{aligned} \tag{3.15}$$

By virtue of Proposition 2.5,

$$\|[\mathbb{P},P]u\|_{H^s(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}$$

and

$$\|[P,\mathbb{P}](u\cdot\nabla y_*)\|_{L^2(\Omega)} \le C\|u\cdot\nabla y_*\|_{H^{s-1}(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}\|y_*\|_{H^s(\Omega)}.$$

The local commutator $[P,\nabla y_*\cdot]u$ can be bounded as

$$\|[P,\nabla y_*\cdot]u\|_{L^2(\Omega)} \le C\|y_*\|_{H^{s+1}(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \tag{3.16}$$

On the other hand, the convexity condition (1.5) yields

$$\int_\Omega Pu\cdot(\nabla y_*\cdot Pu)\,dx \ge \theta_0\|Pu\|^2_{L^2(\Omega)}.$$

We then deduce from (3.15) that

$$\int_\Omega Pu\cdot P\mathbb{P}(u\cdot\nabla y_*)\,dx \ge \theta_0\|Pu\|^2_{L^2(\Omega)} - C\|y_*\|_{H^{s+1}(\Omega)}\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} \tag{3.17}$$

which combined with (3.14) leads to (3.18). The same estimates hold for mixed derivatives $\chi_2 D^s$ where D^s is any partial derivative of order s. □

3.2 Estimates for tangential derivatives of u

Lemma 3.4. *There exists $C > 0$ depending only on $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{s,0} + \theta_0\|u\|^2_{s,0} \le C\|u\|^2_{H^s(\Omega)}\|z\|_{H^s(\Omega)} + C\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)}. \tag{3.18}$$

Proof. The proof follows along the same lines as in Lemma 3.3 upon taking $P \in \mathcal{D}^s_0$ and using Proposition 2.4 in place of Proposition 2.5. □

3.3 Estimates for mixed derivatives of u

The next lemma concerns $\|u\|_{s,1}$.

Lemma 3.5. *There exists $M_1 > 0$ depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2}\frac{d}{dt}\|u\|_{s,1}^2 + \frac{\theta_0}{2}\|u\|_{s,1}^2 \leq M_1\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + M_1\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} + M_1\|u\|_{s,1}\|u\|_{s,0}. \tag{3.19}$$

Proof. Let $P \in \mathcal{D}_1^s$. Assume without loss of generality that $P = \partial_{\tau_1}^{s-1}\partial_n$. Commuting equation (3.9) with P gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |Pu|^2\,dx + \int_\Omega Pu\cdot P\mathbb{P}(u\cdot\nabla y_*)\,dx \\
&\quad = -\int_\Omega Pu\cdot\big([P,u]\cdot\nabla u\big)dx - \int_\Omega Pu\cdot\big(u\cdot[P,\nabla]u\big)dx - \int_\Omega Pu\cdot P\big([\mathbb{P},u\cdot\nabla]z\big)dx.
\end{aligned} \tag{3.20}$$

Arguing as in the proof of Lemma 3.4, we find that the right-hand side is bounded by $C\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)}$. Now we write using the definition of $\mathbb{P}$ that

$$\int_\Omega Pu\cdot P\mathbb{P}(u\cdot\nabla y_*)\,dx = \int_\Omega Pu\cdot P(u\cdot\nabla y_*)\,dx - \int_\Omega Pu\cdot P\nabla f\,dx$$

where f solves

$$\begin{cases} \Delta f = \operatorname{div}(u\cdot\nabla y_*) & \text{in } \Omega, \\ \partial_n f = (u\cdot\nabla y_*)\cdot n & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

Commuting P with ∇y_* gives

$$\int_\Omega Pu\cdot P(u\cdot\nabla y_*)\,dx = \int_\Omega Pu\cdot(Pu\cdot\nabla y_*)\,dx + \int_\Omega Pu\cdot[P,\nabla y_*\cdot]u\,dx$$

where the local commutator $[P,\nabla y_*\cdot]u$ satisfies

$$\|[P,\nabla y_*]\cdot u\|_{L^2(\Omega)} \leq C\|y_*\|_{H^{s+1}(\Omega)}\|u\|_{H^{s-1}(\Omega)}$$

and by the convexity assumption (1.5),

$$\int_\Omega Pu\cdot(Pu\cdot\nabla y_*)\,dx \geq \theta_0\|Pu\|_{L^2(\Omega)}^2.$$

The rest of this proof is devoted to the control of $\int_\Omega Pu\cdot P\nabla f\,dx$. First, since $\chi_1 \equiv 1$ in $\Omega_{2\kappa} \supset \operatorname{supp}(1-\chi_2)$, in view of (2.2), the decomposition

$$\nabla g = (1-\chi_2)n\partial_n g + (1-\chi_2)\tau_j\partial_{\tau_j}g + \chi_2\nabla g \tag{3.22}$$

holds in Ω for any scalar $g : \Omega \to \mathbb{R}$. Using this with $g = Pf$, we write

$$\begin{aligned}\int_\Omega Pu \cdot P\nabla f dx &= \int_\Omega Pu \cdot \nabla Pf dx + \int_\Omega Pu \cdot [P, \nabla] f dx \\ &= \int_\Omega (1-\chi_2)(Pu\cdot n)\partial_n Pf dx + \int_\Omega (1-\chi_2)(Pu\cdot \tau_j)\partial_{\tau_j} Pf dx \\ &\quad + \int_\Omega \chi_2 Pu \cdot \nabla Pf dx + \int_\Omega Pu\cdot [P,\nabla] f dx \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.23}$$

Due to the presence of the local commutator $[P, \nabla]f$, it is readily seen that

$$|I_4| \le C\|u\|_{H^s}\|f\|_{H^s(\Omega)} \le C'\|u\|_{H^s}\|u\|_{H^{s-1}(\Omega)}. \tag{3.24}$$

As for I_3, we integrate by parts noticing that $\operatorname{div} u = 0$ in Ω and $\chi_2 \equiv 0$ near $\partial\Omega$ to obtain

$$I_3 = \int_\Omega \chi_2 Pu \cdot \nabla Pf dx = -\int_\Omega [\operatorname{div}, \chi_2 P] u Pf \tag{3.25}$$

which implies

$$|I_3| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \tag{3.26}$$

Estimate for I_1. We first note that

$$\begin{aligned}Pu\cdot n &= \partial_{\tau_1}^{s-1}(\partial_n u \cdot n) - [\partial_{\tau_1}^{s-1}, n\cdot]\partial_n u \\ &= -\partial_{\tau_1}^{s-1}(\partial_{\tau_j} u\cdot \tau_j) - [\partial_{\tau_1}^{s-1}, n\cdot]\partial_n u \\ &= -(\partial_{\tau_1}^{s-1}\partial_{\tau_j} u)\cdot \tau_j - [\partial_{\tau_1}^{s-1}, \tau_j]\partial_{\tau_j} u - [\partial_{\tau_1}^{s-1}, n\cdot]\partial_n u\end{aligned}$$

This implies

$$\|Pu\cdot n\|_{L^2(\Omega)} \le C\|u\|_{s,0} + C\|u\|_{H^{s-1}(\Omega)}. \tag{3.27}$$

On the other hand, it follows from (3.21) that

$$\begin{cases}\Delta\partial_{\tau_1}^{s-1} f = [\Delta, \partial_{\tau_1}^{s-1}]f + [\partial_{\tau_1}^{s-1}, \operatorname{div}](u\cdot\nabla y_*) + \operatorname{div}\partial_{\tau_1}^{s-1}(u\cdot\nabla y_*) := g_1 \quad \text{in } \Omega, \\ \partial_n\partial_{\tau_1}^{s-1} f = [\partial_n, \partial_{\tau_1}^{s-1}]f + \partial_{\tau_1}^{s-1}\{(u\cdot\nabla y_*)\cdot n\} := g_2 \quad \text{on } \partial\Omega.\end{cases} \tag{3.28}$$

It is easy to see that

$$\|[\Delta, \partial_{\tau_1}^{s-1}]f\|_{L^2(\Omega)} + [\partial_{\tau_1}^{s-1}, \operatorname{div}](u\cdot\nabla y_*)\|_{L^2(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}.$$

In addition, (2.3) gives

$$\begin{aligned}&\|\operatorname{div}\partial_{\tau_1}^{s-1}(u\cdot\nabla y_*)\|_{L^2(\Omega)} \\ &\le C\|\nabla\partial_{\tau_1}^{s-1}(u\cdot\nabla y_*)\|_{L^2(\Omega)} \\ &\le C\|\partial_n\partial_{\tau_1}^{s-1}(u\cdot\nabla y_*)\|_{L^2(\Omega)} + C\|\partial_{\tau_j}\partial_{\tau_1}^{s-1}(u\cdot\nabla y_*)\|_{L^2(\Omega)} \\ &\le C\|u\|_{s,1} + C\|u\|_{s,0}.\end{aligned}$$

Consequently

$$\|g_1\|_{L^2(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,0} + C\|u\|_{s,1}.$$

Using the trace inequality and arguing as above we obtain that

$$\|g_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,0} + C\|u\|_{s,1}.$$

Then the H^2 elliptic estimate for (3.28) leads to

$$\|\partial_{\tau_1}^{s-1} f\|_{H^2(\Omega)} \le C\|g_1\|_{L^2(\Omega)} + C\|g_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,0} + C\|u\|_{s,1}. \tag{3.29}$$

Next we write

$$\begin{aligned}
I_1 &= \int_\Omega (1-\chi_2)(Pu\cdot n)\partial_n P f dx \\
&= \int_\Omega (1-\chi_2)(Pu\cdot n)P\partial_n f dx + \int_\Omega (1-\chi_2)(Pu\cdot n)[\partial_n, P] f dx \\
&= \int_\Omega (1-\chi_2)(Pu\cdot n)\partial_{\tau_1}^{s-1}\partial_n^2 f dx + \int_\Omega (1-\chi_2)(Pu\cdot n)[\partial_n, P] f dx \\
&= \int_\Omega (1-\chi_2)(Pu\cdot n)\partial_n^2\partial_{\tau_1}^{s-1} f dx + \int_\Omega (1-\chi_2)(Pu\cdot n)[\partial_{\tau_1}^{s-1}, \partial_n^2] f dx \\
&\quad + \int_\Omega (1-\chi_2)(Pu\cdot n)[\partial_n, P] f dx.
\end{aligned}$$

In view of (3.27) and (3.29) we deduce that

$$|I_1| \le C\|u\|^2_{H^{s-1}(\Omega)} + C\|u\|^2_{s,0} + C\|u\|_{s,1}\|u\|_{s,0}. \tag{3.30}$$

Estimate for I_2. We first write

$$\begin{aligned}
I_2 &= \int_\Omega (1-\chi_2)(Pu\cdot\tau_j)\partial_{\tau_j}\partial_{\tau_1}^{s-1}\partial_n f dx \\
&= \int_\Omega (1-\chi_2)(Pu\cdot\tau_j)\partial_n\partial_{\tau_j}\partial_{\tau_1}^{s-1} f dx + \int_\Omega (1-\chi_2)(Pu\cdot\tau_j)[\partial_{\tau_j}\partial_{\tau_1}^{s-1}, \partial_n] f dx
\end{aligned}$$

where

$$\left|\int_\Omega (1-\chi_2)(Pu\cdot\tau_j)[\partial_{\tau_j}\partial_{\tau_1}^{s-1}, \partial_n] f dx\right| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \tag{3.31}$$

On the other hand, by Hölder's and Young's inequality,

$$\begin{aligned}
\left|\int_\Omega (1-\chi_2)(Pu\cdot\tau_j)\partial_n\partial_{\tau_j}\partial_{\tau_1}^{s-1} f dx\right| &\le C\|Pu\cdot\tau_j\|_{L^2(\Omega)}\|\partial_n\partial_{\tau_j}\partial_{\tau_1}^{s-1} f\|_{L^2(\Omega)} \\
&\le C\|Pu\|_{L^2(\Omega)}\|\partial_{\tau_j}\partial_{\tau_1}^{s-1} f\|_{H^1(\Omega)} \\
&\le \frac{\theta_0}{2}\|Pu\|^2_{L^2(\Omega)} + C'\|\partial_{\tau_j}\partial_{\tau_1}^{s-1} f\|^2_{H^1(\Omega)}.
\end{aligned} \tag{3.32}$$

Using again equation (3.21) we find

$$\begin{cases}\Delta\partial_{\tau_j}\partial_{\tau_1}^{s-1}f = [\Delta,\partial_{\tau_j}\partial_{\tau_1}^{s-1}]f + [\partial_{\tau_j}\partial_{\tau_1}^{s-1},\mathrm{div}](u\cdot\nabla y_*) + \mathrm{div}\,\partial_{\tau_j}\partial_{\tau_1}^{s-1}(u\cdot\nabla y_*) \quad \text{in } \Omega,\\ \partial_n\partial_{\tau_j}\partial_{\tau_1}^{s-1}f = [\partial_n,\partial_{\tau_j}\partial_{\tau_1}^{s-1}]f + \partial_{\tau_j}\partial_{\tau_1}^{s-1}\{(u\cdot\nabla y_*)\cdot n\} \quad \text{on } \partial\Omega.\end{cases} \tag{3.33}$$

Multiplying the first equation by $\partial_{\tau_j}\partial_{\tau_1}^{s-1}f$ then integrating over Ω and using the second equation to cancel out the leading boundary term, we deduce that $h = \partial_{\tau_j}\partial_{\tau_1}^{s-1}f$ satisfies

$$\begin{aligned}\int_\Omega |\nabla h|^2 dx &= -\int_\Omega h\Big\{[\Delta,\partial_{\tau_j}\partial_{\tau_1}^{s-1}]f + [\partial_{\tau_j}\partial_{\tau_1}^{s-1},\mathrm{div}](u\cdot\nabla y_*)\Big\}dx + \int_{\partial\Omega} h[\partial_n,\partial_{\tau_j}\partial_{\tau_1}^{s-1}]f dS\\ &= I_{2a} + I_{2b}.\end{aligned}$$

We observe that $\|h\|_{L^2(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}$ and

$$\|[\Delta,\partial_{\tau_j}\partial_{\tau_1}^{s-1}]f\|_{L^2(\Omega)} + [\partial_{\tau_j}\partial_{\tau_1}^{s-1},\mathrm{div}](u\cdot\nabla y_*)\|_{L^2(\Omega)} \le C\|u\|_{H^s(\Omega)},$$

hence

$$|I_{2a}| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}.$$

On the other hand, by virtue of the trace inequality and interpolation, the surface integral is controlled as

$$|I_{2b}| \le C\|f\|^2_{H^s(\partial\Omega)} \le C'\|f\|^2_{H^{s+\frac{1}{2}}(\Omega)} \le C''\|f\|_{H^s(\Omega)}\|f\|_{H^{s+1}(\Omega)} \le C'''\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)}.$$

It follows that

$$\|h\|^2_{H^1(\Omega)} \le \|h\|^2_{L^2(\Omega)} + \|\nabla h\|^2_{L^2(\Omega)} \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}.$$

Plugging this into (3.32) and recalling (3.31) we deduce that

$$|I_2| \le \frac{\theta_0}{2}\|Pu\|^2_{L^2(\Omega)} + C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \tag{3.34}$$

Putting together the above considerations we arrive at

$$\frac{1}{2}\frac{d}{dt}\|Pu\|^2_{L^2} + \frac{\theta_0}{2}\|Pu\|_{L^2} \le C\|u\|^2_{H^s(\Omega)}\|z\|_{H^s(\Omega)} + C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}.$$

Then summing over all $P \in \mathcal{D}_1^s$ yields

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{s,1} + \frac{\theta_0}{2}\|u\|^2_{s,1} \le M_1\|u\|^2_{H^s(\Omega)}\|z\|_{H^s(\Omega)} + M_1\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}$$

which combined with (3.18) for tangential derivatives leads to the desired estimate (3.19).
□

Lemma 3.6. *For each $k \in \{1, 2, ..., s\}$ there exists $M_k > 0$ depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$ such that*

$$\frac{1}{2}\frac{d}{dt}\|u\|_{s,k}^2 + \frac{\theta_0}{2}\|u\|_{s,k}^2 \le M_k\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + M_k\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} + M_k\|u\|_{s,k}\|u\|_{s,k-1}. \tag{3.35}$$

Proof. The base case $k = 1$ has been proved in Lemma 3.5. Assume (3.35) for some $k \in \{1, 2, ..., s-1\}$ we prove it for $k+1$ in place of k. Let $P \in \mathcal{D}_{k+1}^s$. We assume without loss of generality that $P = \partial_{\tau_1}^{s-k-1}\partial_n^{k+1}$. Commuting equation (3.9) with P gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |Pu|^2\, dx + \int_\Omega Pu \cdot P\mathbb{P}(u\cdot\nabla y_*)\, dx \\ &\quad = -\int_\Omega Pu\cdot\big([P,u]\cdot\nabla u\big)dx - \int_\Omega Pu\cdot\big(u\cdot[P,\nabla]u\big)dx - \int_\Omega Pu\cdot P\big([\mathbb{P}, u\cdot\nabla]z\big)dx.\end{aligned} \tag{3.36}$$

As in the proof of Lemma 3.5 it suffices to treat the damping term

$$\int_\Omega Pu\cdot P\mathbb{P}(u\cdot\nabla y_*)\, dx = \int_\Omega Pu\cdot P(u\cdot\nabla y_*)\, dx - \int_\Omega Pu\cdot P\nabla f\, dx$$

where f solves (3.21):

$$\begin{cases}\Delta f = \operatorname{div}(u\cdot\nabla y_*) & \text{in } \Omega,\\ \partial_n f = (u\cdot\nabla y_*)\cdot n & \text{on } \partial\Omega.\end{cases} \tag{3.37}$$

Commuting P with ∇y_* gives

$$\int_\Omega Pu\cdot P(u\cdot\nabla y_*)\, dx = \int_\Omega Pu\cdot(Pu\cdot\nabla y_*)\, dx + \int_\Omega Pu\cdot[P,\nabla y_*]\cdot u\, dx$$

where the local commutator $[P, \nabla y_*]\cdot u$ satisfies

$$\|[P,\nabla y_*]\cdot u\|_{L^2(\Omega)} \le C\|y_*\|_{H^{s+1}(\Omega)}\|u\|_{H^{s-1}(\Omega)},$$

and by the convexity assumption (1.5),

$$\int_\Omega Pu\cdot(Pu\cdot\nabla y_*)\, dx \ge \theta_0\|Pu\|_{L^2(\Omega)}^2.$$

Then it remains to prove that

$$\int_\Omega Pu\cdot P\nabla f\, dx \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k+1}\|u\|_{s,k}. \tag{3.38}$$

To this end, let us write using the decomposition (3.22) that for $k \ge 1$,

$$\begin{aligned}P\nabla f &= \nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_n^2 f + [\partial_{\tau_1}^{s-k-1}\partial_n^{k+1}, \nabla]f\\ &= (1-\chi_2)\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_n^2 f + \chi_2\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_n^2 f + [\partial_{\tau_1}^{s-k-1}\partial_n^{k+1}, \nabla]f.\end{aligned}$$

The commutator is a lower order term in the sense that

$$\|[\partial_{\tau_1}^{s-k-1}\partial_n^{k+1},\nabla]f\|_{L^2(\Omega)} \le C\|f\|_{H^s(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)},$$

leading to the bound

$$\int_\Omega Pu\cdot[\partial_{\tau_1}^{s-k-1}\partial_n^{k+1},\nabla]f dx \le C\|Pu\|_{L^2(\Omega)}\|u\|_{H^{s-1}(\Omega)} \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}.$$

Integration by parts as in (3.26) yields

$$\left|\int_\Omega Pu\cdot\chi_2\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_n^2 f\right| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}.$$

In the main term $(1-\chi_2)\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_n^2 f$, since the support of $(1-\chi_2)$ is contained in $\Omega_{2\kappa}$, we can use (2.5) and (3.37) to write

$$\begin{aligned}
\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_n^2 f &= -\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\partial_{\tau_j}^2 f + \nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\operatorname{div}(u\cdot\nabla y_*)\\
&\quad + \nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\Big[\nabla f\cdot(n\cdot\nabla)n + \nabla f\cdot(\tau_j\cdot\nabla)\tau_j\Big]\\
&= -\nabla\partial_n^{k-1}\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}^2 f - \nabla[\partial_{\tau_1}^{s-k-1},\partial_n^{k-1}]\partial_{\tau_j}^2 f + \nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\operatorname{div}(u\cdot\nabla y_*)\\
&\quad + \nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\Big[\nabla f\cdot(n\cdot\nabla)n + \nabla f\cdot(\tau_j\cdot\nabla)\tau_j\Big]
\end{aligned}$$

in $\Omega_{2\kappa}$, where the sums over j were taken. Since the commutator $[\partial_{\tau_1}^{s-k-1},\partial_n^{k-1}]\partial_{\tau_j}^2 f$ is bounded in $H^1(\Omega)$ by $C\|f\|_{H^s(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}$ we obtain

$$\left|\int_\Omega(1-\chi_2)Pu\cdot\nabla[\partial_{\tau_1}^{s-k-1},\partial_n^{k-1}]\partial_{\tau_j}^2 f dx\right| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \tag{3.39}$$

In addition, we have

$$\left|\int_\Omega(1-\chi_2)Pu\cdot\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\Big[\nabla f\cdot(n\cdot\nabla)n + \nabla f\cdot(\tau_j\cdot\nabla)\tau_j\Big]dx\right| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)}. \tag{3.40}$$

Thus, we are left with the two integrals

$$\begin{aligned}
I_1 &= \int_\Omega(1-\chi_2)Pu\cdot\nabla\partial_n^{k-1}\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}^2 f dx,\\
I_2 &= \int_\Omega(1-\chi_2)Pu\cdot\nabla\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\operatorname{div}(u\cdot\nabla y_*)dx.
\end{aligned}$$

Estimate for I_1. We claim that

$$\|\nabla\partial_n^{k-1}\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}^2 f\|_{L^2(\Omega)} \le \|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k}. \tag{3.41}$$

First, taking $\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}$ of (3.37) gives

$$\begin{cases} \Delta\partial_{\tau_1}^{s-k-1}\partial_{\tau_j} f = [\Delta, \partial_{\tau_1}^{s-k-1}\partial_{\tau_j}]f + [\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}, \operatorname{div}](u\cdot\nabla y_*) + \operatorname{div}\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*) \\ \qquad\qquad := g_1 \quad \text{in } \Omega, \\ \partial_n\partial_{\tau_1}^{s-k-1}\partial_{\tau_j} f = [\partial_n, \partial_{\tau_1}^{s-k-1}\partial_{\tau_j}]f + \partial_{\tau_1}^{s-k-1}\partial_{\tau_j}\{(u\cdot\nabla y_*)\cdot n\} := g_2 \quad \text{on } \partial\Omega. \end{cases} \tag{3.42}$$

In view of the bound

$$\begin{aligned} \|\nabla\partial_n^{k-1}\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}^2 f\|_{L^2(\Omega)} &\le \|(\nabla\partial_n^{k-1}\partial_{\tau_j})\partial_{\tau_1}^{s-k-1}\partial_{\tau_j} f\|_{L^2(\Omega)} + \|\nabla\partial_n^{k-1}[\partial_{\tau_1}^{s-k-1}, \partial_{\tau_j}]\partial_{\tau_j} f\|_{L^2(\Omega)} \\ &\le C\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j} f\|_{H^{k+1}(\Omega)} + C\|f\|_{H^s(\Omega)} \\ &\le C\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j} f\|_{H^{k+1}(\Omega)} + C'\|u\|_{H^{s-1}(\Omega)} \end{aligned}$$

and elliptic estimates for (3.42) we have

$$\|\nabla\partial_n^{k-1}\partial_{\tau_1}^{s-k+1} f\|_{L^2(\Omega)} \le C\|g_1\|_{H^{k-1}(\Omega)} + C\|g_2\|_{H^{k-\frac{1}{2}}(\partial\Omega)} + C\|u\|_{H^{s-1}(\Omega)}. \tag{3.43}$$

The H^{k-1} norm of g_1 is bounded as

$$\begin{aligned} \|g_1\|_{H^{k-1}(\Omega)} &\le \|[\Delta, \partial_{\tau_1}^{s-k-1}\partial_{\tau_j}]f\|_{H^{k-1}(\Omega)} + \|[\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}, \operatorname{div}](u\cdot\nabla y_*)\|_{H^{k-1}(\Omega)} \\ &\quad + \|\operatorname{div}\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*)\|_{H^{k-1}(\Omega)} \\ &\le C\|f\|_{H^s(\Omega)} + C\|u\|_{H^{s-1}(\Omega)} + C\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*)\|_{H^k(\Omega)} \\ &\le C'\|u\|_{H^{s-1}(\Omega)} + C\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*)\|_{H^k(\Omega)}. \end{aligned}$$

We observe that there are at most k normal derivatives appearing when measure $\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*)$ in $H^{k-1}(\Omega)$, hence

$$\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*)\|_{H^k(\Omega)} \le C\|u\|_{s,k}.$$

Consequently

$$\|g_1\|_{H^{k-1}(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k}. \tag{3.44}$$

As for g_2 we first use the trace theorem to have

$$\|[\partial_n, \partial_{\tau_1}^{s-k-1}\partial_{\tau_j}]f\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \le C\|[\partial_n, \partial_{\tau_1}^{s-k-1}\partial_{\tau_j}]f\|_{H^k(\Omega)} \le C\|f\|_{H^s(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}.$$

The fact that $k \ge 1$ was used in the first inequality. Then we write

$$\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}(u\cdot\nabla y_*) = (\partial_{\tau_1}^{s-k-1}\partial_{\tau_j} u)\cdot\nabla y_* + [\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}, \nabla y_*\cdot]u$$

where the commutator can be bounded using the trace theorem as follows

$$\|[\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}, \nabla y_*\cdot]u\|_{H^{k-\frac{1}{2}}(\partial\Omega)} \le C\|[\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}, \nabla y_*\cdot]u\|_{H^k(\Omega)} \le C\|u\|_{H^{s-1}(\Omega)}.$$

In addition,

$$\|(\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}u)\cdot\nabla y_*\|_{H^{k-\frac12}(\partial\Omega)} \le C\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}u\|_{H^{k-\frac12}(\partial\Omega)} \le C\|\partial_{\tau_1}^{s-k-1}\partial_{\tau_j}u\|_{H^k(\Omega)} \le C\|u\|_{s,k}.$$

Thus,

$$\|g_2\|_{H^{k-\frac12}(\partial\Omega)} \le \|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k}. \tag{3.45}$$

Combining (3.43), (3.44) and (3.45) leads to the bound (3.41) which implies that

$$\begin{aligned} I_1 &\le C\|Pu\|_{L^2(\Omega)}\|u\|_{H^{s-1}(\Omega)} + C\|Pu\|_{L^2(\Omega)}\|u\|_{s,k} \\ &\le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k+1}\|u\|_{s,k}. \end{aligned} \tag{3.46}$$

Estimate for I_2. Decomposing $\nabla = \tau_j\partial_{\tau_j} + n\partial_n$ in $\Omega_{2\kappa}\supset \mathrm{supp}(1-\chi_2)$ gives $I_2 = I_{2a}+I_{2b}$ where

$$I_{2a} = \int_\Omega (1-\chi_2)\big\{(\partial_{\tau_1}^{s-k-1}\partial_n^{k+1}u)\cdot\tau_j\big\}\big\{\partial_{\tau_j}\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\,\mathrm{div}(u\cdot\nabla y_*)\big\}dx,$$
$$I_{2b} = \int_\Omega (1-\chi_2)\big\{(\partial_{\tau_1}^{s-k-1}\partial_n^{k+1}u)\cdot n\big\}\big\{\partial_n\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\,\mathrm{div}(u\cdot\nabla y_*)\big\}dx.$$

We notice that there are at most k normal derivatives in $\partial_{\tau_j}\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\,\mathrm{div}(u\cdot\nabla y_*)$, hence

$$|I_{2a}| \le C\|u\|_{s,k+1}\|u\|_{s,k}.$$

As for I_{2b} we write using (2.4) that

$$\begin{aligned}(\partial_{\tau_1}^{s-k-1}\partial_n^{k+1}u)\cdot n &= \partial_{\tau_1}^{s-k-1}\partial_n^{k}(\partial_n u\cdot n) + [\partial_{\tau_1}^{s-k-1}\partial_n^{k}, n\cdot]\partial_n u \\ &= -\partial_{\tau_1}^{s-k-1}\partial_n^{k}(\partial_{\tau_j} u\cdot \tau_j) + [\partial_{\tau_1}^{s-k-1}\partial_n^{k}, n\cdot]\partial_n u.\end{aligned}$$

It is readily seen that

$$\left|\int_\Omega (1-\chi_2)\big\{[\partial_{\tau_1}^{s-k-1}\partial_n^{k}, n\cdot]\partial_n u\big\}\big\{\partial_n\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\,\mathrm{div}(u\cdot\nabla y_*)\big\}dx\right| \le C\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)}.$$

On the other hand, there are at most k normal derivatives in $\partial_{\tau_1}^{s-k-1}\partial_n^{k}(\partial_{\tau_j}u\cdot\tau_j)$, and thus

$$\left|\int_\Omega (1-\chi_2)\big\{\partial_{\tau_1}^{s-k-1}\partial_n^{k}(\partial_{\tau_j} u\cdot \tau_j)\big\}\big\{\partial_n\partial_{\tau_1}^{s-k-1}\partial_n^{k-1}\,\mathrm{div}(u\cdot\nabla y_*)\big\}dx\right| \le C\|u\|_{s,k}\|u\|_{s,k+1}.$$

All together we have prove that

$$|I_2| \le C\|u\|_{H^s(\Omega)}\|u\|_{H^{s-1}(\Omega)} + C\|u\|_{s,k+1}\|u\|_{s,k} \tag{3.47}$$

In view of (3.46) and (3.47) we finish the proof of (3.38), and hence the proof of (3.35) with $k+1$ in place of k. □

3.4 H^s estimate for u

We have proved in Lemmas 3.4, 3.5 and 3.6 that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{s,0}^2 + \theta_0\|u\|_{s,0}^2 \le M_0\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + M_0\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} \tag{3.48}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{s,k}^2 + \frac{\theta_0}{2}\|u\|_{s,k}^2 \le{}& M_k\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + M_k\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} \\ &+ M_k\|u\|_{s,k}\|u\|_{s,k-1}\end{aligned} \tag{3.49}$$

for all $k \in \{1, 2, ..., s\}$. Applying Young's inequality yields

$$M_k\|u\|_{s,j}\|u\|_{s,j-1} \le \frac{\theta_0}{4}\|u\|_{s,j-1}^2 + M_j'\|u\|_{s,j-1}^2, \quad 1 \le j \le s.$$

It follows from this and (3.49) with $k = s$ and $k = s-1$ that

$$\frac{d}{dt}\|u\|_{s,s}^2 + \frac{\theta_0}{2}\|u\|_{s,s}^2 \le 2M_s\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + 2M_s\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} + 2M_s'\|u\|_{s,s-1}^2 \tag{3.50}$$

and

$$\begin{aligned}\frac{d}{dt}\|u\|_{s,s-1}^2 + \frac{\theta_0}{2}\|u\|_{s,s-1}^2 \le{}& 2M_{s-1}\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + 2M_{s-1}\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} \\ &+ 2M_{s-1}'\|u\|_{s,s-2}^2.\end{aligned} \tag{3.51}$$

Let $N_{s-1} > 0$ be such that $\frac{\theta_0}{2}N_{s-1} - 2M_{s-1}' = \frac{\theta_0}{2}$. Multiplying (3.51) by N_{s-1} then adding the resulting inequality to (3.50) we obtain

$$\begin{aligned}\frac{d}{dt}\big(\|u\|_{s,s}^2 + N_{s-1}\|u\|_{s,s-1}^2\big) + \frac{\theta_0}{2}\big(\|u\|_{s,s}^2 + \|u\|_{s,s-1}^2\big) \le{}& N_{s-1}'\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} \\ &+ N_{s-1}'\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)} + N_{s-1}'\|u\|_{s,s-2}^2\end{aligned}$$

for some $N_{s-1}' > 0$. Continuing this process, one can find $s+1$ positive constants B and N_j, $0 \le j \le s-1$ such that

$$\frac{d}{dt}\Big(\|u\|_{s,s}^2 + \sum_{j=0}^{s-1} N_j\|u\|_{s,j}^2\Big) + \frac{\theta_0}{2}\sum_{j=0}^{s}\|u\|_{s,j}^2 \le B\|u\|_{H^s(\Omega)}^2\|z\|_{H^s(\Omega)} + B\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)}.$$

Setting

$$Z^2(u) = \|u\|_{s,s}^2 + \sum_{j=0}^{s-1} N_j\|u\|_{s,j}^2$$

and

$$2\theta_1 = \frac{\theta_0}{2\max_{0\le j\le s-1}\{1, N_j\}},$$

we arrive at

$$\frac{d}{dt}Z^2(u) + 2\theta_1 Z^2(u) \le B\|u(t)\|^2_{H^s(\Omega)}\|z(t)\|_{H^s(\Omega)} + B\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)}. \tag{3.52}$$

Set

$$W^2(u) = Z^2(u) + \|\chi_2 u\|^2_{H^s(\Omega)} \tag{3.53}$$

where χ_2 is given by (2.21). Combining (3.52) with (3.10) one can find a constant $C > 0$ such that

$$\frac{d}{dt}W^2(u) + 2\theta_1 W^2(u) \le C\|u(t)\|^2_{H^s(\Omega)}\|z(t)\|_{H^s(\Omega)} + C\|u\|_{H^{s-1}(\Omega)}\|u\|_{H^s(\Omega)}. \tag{3.54}$$

To recover the H^s estimate for u from the preceding estimate on $W(u)$, we prove the next lemma.

Lemma 3.7. *There exists $A > 0$ depending only on s such that*

$$\frac{1}{A}W^2(u) \le \|u\|^2_{H^s(\Omega)} \le AW^2(u) + A\|u\|^2_{L^2(\Omega)} \tag{3.55}$$

for any H^s vector field $u : \Omega \to \mathbb{R}^d$.

Proof. First, the inequality

$$Z^2(u) \le A\|u\|^2_{H^s(\Omega)} + A\|\chi_2 u\|^2_{H^s(\Omega)}$$

is obvious if A is sufficiently large.

Next recall from (3.22) and (2.3) that for any $w : \Omega \to \mathbb{R}^2$ it holds that

$$|\nabla w|^2 \le |\partial_n w|^2 + \sum_{j=1}^{d} |\partial_{\tau_j} w|^2 + \|\chi_2 \nabla w\|^2_{L^2(\Omega)}. \tag{3.56}$$

In the rest of this proof, the sum over $j \in \{1, ..., d-1\}$ will be omitted. Let D^{s-1} be an arbitrary partial derivative of order $s-1$. Without loss of generality, assume $D^{s-1} = \partial_1 D^{s-2}$ for some partial derivative D^{s-2} of order $s-2$. Applying (3.56) with $w = D^{s-1}u$ gives

$$\|\nabla D^{s-1}u\|^2_{L^2(\Omega)} \le \|\partial_n \partial_1 D^{s-2}u\|^2_{L^2(\Omega)} + \|\partial_{\tau_j}\partial_1 D^{s-2}u\|^2_{L^2(\Omega)} + \|\chi_2 \nabla w\|^2_{L^2(\Omega)}.$$

We thus have replaced one partial derivative with one normal and one tangential derivative. To continue, we commute ∂_n with ∂_1 to have

$$\begin{aligned}\|\partial_n\partial_1 D^{s-2}u\|^2_{L^2(\Omega)} &\le 2\|\partial_1\partial_n D^{s-2}u\|^2_{L^2(\Omega)} + 2\|[\partial_n, \partial_1]D^{s-2}u\|^2_{L^2(\Omega)} \\ &\le 2\|\partial_1\partial_n D^{s-2}u\|^2_{L^2(\Omega)} + C\|u\|^2_{H^{s-1}(\Omega)}.\end{aligned}$$

Similarly for $\|\partial_{\tau_j}\partial_1 D^{s-2}u\|^2_{L^2(\Omega)}$ we obtain

$$\|\nabla D^{s-1}u\|^2_{L^2(\Omega)} \le 2\|\nabla\partial_n D^{s-2}u\|^2_{L^2(\Omega)} + 2\|\nabla\partial_{\tau_j} D^{s-2}u\|^2_{L^2(\Omega)} + C\|u\|^2_{H^{s-1}(\Omega)} + \|\chi_2 u\|^2_{H^s(\Omega)}.$$

Now applying (3.56) with $w = \partial_n D^{s-2}u$ and $w = \partial_\tau D^{s-2}u$ leads to

$$\|\nabla D^{s-1}u\|^2_{L^2(\Omega)} \le 2\|\partial_n\partial_n D^{s-2}u\|^2_{L^2(\Omega)} + 2\|\partial_{\tau_j}\partial_n D^{s-2}u\|^2_{L^2(\Omega)} + 2\|\partial_n\partial_{\tau_j} D^{s-2}u\|^2_{L^2(\Omega)} + 2\|\partial_{\tau_j}\partial_{\tau_j} D^{s-2}u\|^2_{L^2(\Omega)} + C\|u\|^2_{H^{s-1}(\Omega)} + \|\chi_2 u\|^2_{H^s(\Omega)}.$$

Next we write $D^{s-2} = \partial_j D^{s-3}$ with $j \in \{1, ..., d\}$ and continue the process until no partial derivatives are left on the right-hand side, yielding

$$\|\nabla D^{s-1}u\|^2_{L^2(\Omega)} \le CZ^2(u) + C\|u\|^2_{H^{s-1}(\Omega)} + \|\chi_2 u\|^2_{H^s(\Omega)}.$$

This combined with the interpolation inequality $\|u\|_{H^{s-1}} \le C\|u\|^\alpha_{H^s}\|u\|^{1-\alpha}_{L^2}$, $\alpha \in (0,1)$ and a Young inequality implies the desired estimate (3.55). □

By interpolation and Young's inequality, the last term on the right-hand side of (3.54) is bounded as

$$C\|u(t)\|_{H^s(\Omega)}\|u(t)\|_{H^{s-1}(\Omega)} \le \gamma\|u(t)\|^2_{H^s(\Omega)} + C_\gamma\|u(t)\|^2_{L^2(\Omega)}$$

for any $\gamma > 0$. Using (3.55) and choosing γ sufficiently small so that $A\gamma < \theta_1$, we deduce from (3.54) that

$$\frac{d}{dt}W^2(u) + \theta_1 W^2(u) \le C_2\|u(t)\|^2_{H^s(\Omega)}\|z(t)\|_{H^s(\Omega)} + C_2\|u(t)\|^2_{L^2(\Omega)} \tag{3.57}$$

for some $C_2 > 0$. We recall that θ_1 depends only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$.

3.5 Proof of Theorem 1.1

For smooth solutions (u, z) defined on the maximal interval $[0, T^*)$, which is guaranteed by Theorem 1.4, we define the bootstrap norm by

$$\mathcal{N}(t) := \sup_{0\le\tau\le t}\Big(\|z(\tau)\|^2_{H^s(\Omega)} + M^2 e^{\frac{\theta_1}{2}\tau}\|u(\tau)\|^2_{L^2(\Omega)} + Me^{\frac{\theta_1}{2}\tau}\|u(\tau)\|^2_{H^s(\Omega)}\Big) \tag{3.58}$$

for $t < T^*$ and for some large $M > 0$ to be fixed.

Proposition 3.8. *There exist positive constants ε, C_*, depending only on θ_0 and $\|y_*\|_{H^{s+1}(\Omega)}$, such that whenever $\mathcal{N}(0) < \varepsilon$, we have $\mathcal{N}(t) \le C_*\mathcal{N}(0)$ for all $t < T^*$.*

Proof. We shall prove that

$$\mathcal{N}(t) \le C_0\mathcal{N}(0) + C_0\mathcal{N}(t)^{3/2} \tag{3.59}$$

for all $t < T^*$. The proposition then follows directly from the standard continuous induction.

As for the claim (3.59), we integrate (3.2) in time and use the definition of $\mathcal{N}(t)$, yielding

$$\begin{aligned}\|z(t)\|_{H^s}^2 &\leq \|z(0)\|_{H^s}^2 + C_1 \int_0^t (1 + \|z(\tau)\|_{H^s}) \|u(\tau)\|_{H^s} \|z(\tau)\|_{H^s} \, d\tau \\ &\leq \|z(0)\|_{H^s}^2 + C_1 M^{-1/2} (1 + \mathcal{N}(t)^{1/2}) \mathcal{N}(t) \int_0^t e^{-\theta_1 \tau/4} \, d\tau \\ &\leq \|z(0)\|_{H^s}^2 + C_3 M^{-1/2} (1 + \mathcal{N}(t)^{1/2}) \mathcal{N}(t).\end{aligned} \tag{3.60}$$

Next taking M sufficiently large so that $\theta_0 M > C_2$, we obtain from (3.7) and (3.57) that

$$\frac{d}{dt}(M\|u(t)\|_{L^2}^2 + W^2(u)) + \theta_1 (M\|u(t)\|_{L^2}^2 + W^2(u)) \leq C_4 M \|u(t)\|_{H^s}^2 \|z(t)\|_{H^s}$$

where $C_4 = C_2(2 + M)$. This yields

$$\begin{aligned}&M^2\|u(t)\|_{L^2}^2 + MW^2(u) \\ &\quad\leq e^{-\theta_1 t}\Big[M^2\|u(0)\|_{L^2}^2 + MW^2(u(0))\Big] + C_4 M \int_1^t e^{-\theta_1(t-\tau)} \|u(\tau)\|_{H^s}^2 \|z(\tau)\|_{H^s} \, d\tau \\ &\quad\leq e^{-\theta_1 t}\Big[M^2\|u(0)\|_{L^2}^2 + MW^2(u(0))\Big] + C_4 \mathcal{N}(t)^{3/2} \int_0^t e^{-\theta_1(t-\tau)} e^{-\theta_1 \tau/2} \, d\tau \\ &\quad\leq e^{-\theta_1 t/2}\Big[M^2\|u(0)\|_{L^2}^2 + MW^2(u(0))\Big] + C_5 \mathcal{N}(t)^{3/2} e^{-\theta_1 t/2}.\end{aligned} \tag{3.61}$$

Consequently,

$$M^2 e^{\theta_1 t/2}\|u(t)\|_{L^2}^2 + M e^{\theta_1 t/2} W^2(u) \leq \Big[M^2\|u(0)\|_{L^2}^2 + MW^2(u(0))\Big] + C_5 \mathcal{N}(t)^{3/2}. \tag{3.62}$$

Finally, (3.59) follows from (3.60), (3.62) and (3.55). □

With the ε and C_* given in Proposition 3.8, we have proved that

$$\|z(t)\|_{H^s(\Omega)} \leq C_* \mathcal{N}(0) \leq C_* \varepsilon \tag{3.63}$$

and

$$\|u(t)\|_{H^s(\Omega)} \leq C \|y_0 - y_*\|_{H^s(\Omega)} e^{-\frac{\theta_1}{4} t} \tag{3.64}$$

for all time $t < T^*$. As a consequence of this and the local well-posedness in Theorem 1.4, the solution z is global in time and enjoys the same bounds for all $t > 0$. Using equation (3.1) and the estimates (3.63), (3.64) we deduce that $\partial_t z \in L^1(0, \infty; H^{s-1}(\Omega))$. This yields

$$\lim_{t\to\infty} z(x,t) = z_0(x) + \int_0^\infty \partial_t z(x,\tau) d\tau := z_\infty(x) \quad \text{in } H^{s-1}(\Omega),$$

and thus

$$\lim_{t\to\infty} y(x,t) = y_\infty(x) := z_\infty(x) + y_*(x) \quad \text{in } H^{s-1}(\Omega).$$

Furthermore,

$$\|y(t)-y_\infty\|_{H^{s-1}(\Omega)} = \|z(t)-z_\infty\|_{H^{s-1}(\Omega)} = \|\int_t^\infty u\cdot\nabla z(\tau)d\tau\|_{H^{s-1}(\Omega)} \le Ce^{-\frac{\theta_1 t}{4}} \quad (3.65)$$

for all $t \ge 0$. Using the Leray projection we write

$$y(x,t) = u(x,t) + \nabla p(x,t), \quad y_\infty(x) = u_\infty(x) + \nabla p_\infty(x)$$

where $u_\infty = \mathbb{P}y_\infty$. In view of the Pythagorean identity

$$\|y(t)-y_\infty\|^2_{L^2(\Omega)} = \|u(t)-u_\infty\|^2_{L^2(\Omega)} + \|\nabla p(t) - \nabla p_\infty\|^2_{L^2(\Omega)}$$

we find that each term on the right hand side converges to 0 as $t \to \infty$. This, together with the fact that $u(t) \to 0$ in $H^s(\Omega)$, implies that $u_\infty \equiv 0$. Thus, $y_\infty = \nabla p_\infty$ is a gradient and in view of (3.65) we have

$$\|y(t) - \nabla p_\infty\|_{L^2(\Omega)} \le Ce^{-\frac{\theta_1 t}{4}}$$

for all $t \ge 0$. As a consequence of this and the bound (3.63), if ε is sufficiently small then $\nabla^2 p_\infty > 0$. Thus, p_∞ is (strictly) convex and ∇p_∞ is the optimal rearrangement of y_0 by virtue of Brenier's theorem ([3]). This ends the proof of Theorem 1.1.

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