Lecture 7

Gram-Schmidt
Orthogonalization

The "best" basis we can have for a vector space is an orthogonal basis. That is because we can most easily find the coefficients that are needed to express a vector as a linear combination of the basis vectors $v_1, ..., v_n$:

$$v = \frac{(v, v_1)}{\|v_1\|^2} v_1 + \ldots + \frac{(v, v_n)}{\|v_n\|^2} v_n.$$ 

But usually we are not given an orthogonal basis. In this section we will show how to find an orthogonal basis starting from an arbitrary basis.

7.1 Procedure

Let us start with two linear independent vectors $v_1$ and $v_2$ (i.e. not on the same line through zero). Let $u_1 = v_1$. How can we find a vector $u_2$ which is perpendicular to $u_1$ and that the span of $u_1$ and $u_2$ is the same as the span of $v_1$ and $v_2$? We try to find a number $a \in \mathbb{R}$ such that:

$$u_2 = au_1 + v_2, \quad u_2 \perp u_1$$

Take the inner product with $u_1$ to get:

$$0 = (u_2, u_1) = a(u_1, u_1) + (v_2, u_1) = a\|u_1\|^2 + (v_2, u_1)$$
or
\[ a = -\frac{(v_2, u_1)}{\|u_1\|^2} \]

What if we have a third vector \( v_3 \)? Then, after choosing \( u_1, u_2 \) as above, we would look for \( u_3 \) of the form:
\[ u_3 = a_1 u_1 + a_2 u_2 + v_3 \]

Take the inner product with \( u_1 \) to find \( a_1 \):
\[ 0 = (u_3, u_1) = a_1 \|u_1\|^2 + (v_3, u_1) \]
or
\[ a_1 = -\frac{(v_3, u_1)}{\|u_1\|^2} \]

and the inner product with \( u_2 \) to find \( a_2 \):
\[ 0 = (u_3, u_2) = a_2 \|u_2\|^2 + (v_3, u_2) \]
or
\[ a_2 = -\frac{(v_3, u_2)}{\|u_2\|^2} \]

Thus:
\[ u_1 = v_1 \]
\[ u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \]
\[ u_3 = v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} u_2 \]

### 7.2 Examples

**Example.** Let \( v_1 = (1, 1), v_2 = (2, -1) \). Then, we set \( u_1 = (1, 1) \) and
\[ u_2 = (2, -1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \]
\[ = (2, -1) - \frac{2 - 1}{2} (1, 1) \]
\[ = \frac{3}{2} (1, -1) \]
7.2. EXAMPLES

Example. Let \( v_1 = (2, -1), v_2 = (0, 1) \). Then, we set \( u_1 = (2, -1) \) and

\[
\begin{align*}
u_2 &= (0, 1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\
&= (0, 1) - \frac{-1}{5}(2, -1) \\
&= \frac{2}{5}(1, 2)
\end{align*}
\]

Note We could have also started with \( v_2 = (0, 1) \), and get first basis vector to be \((0, 1)\) and second vector to be:

\[
(2, -1) - \frac{(2, -1) \cdot (0, 1)}{||(0, 1)||^2} (0, 1) = (2, 0)
\]

Example. Let \( v_1 = (0, 1, 2), v_2 = (1, 1, 2), v_3 = (1, 0, 1) \). Then, we set \( u_1 = (0, 1, 2) \) and

\[
\begin{align*}
u_2 &= (1, 1, 2) - \frac{(0, 1, 2) \cdot (1, 1, 2)}{||(0, 1, 2)||^2} (0, 1, 2) \\
&= (1, 1, 2) - \frac{5}{2} (0, 1, 2) \\
&= (1, 0, 0)
\end{align*}
\]

\[
u_3 = (1, 0, 1) - \frac{2}{5} (0, 1, 2) - (1, 0, 0) = \frac{1}{5}(0, -2, 1)
\]

Example. Let \( v_0 = 1, v_1 = x, v_2 = x^2 \). Then, \( v_0, v_1, v_2 \) is a basis for the space of polynomials of degree \( \leq 2 \). But they are not orthogonal, so we start with \( u_0 = v_0 \) and \( u_1 = v_1 - \frac{(v_1, u_0)}{\|u_0\|^2} u_0 \). So we need to find:

\[
\begin{align*}(v_1, u_0) &= \int_0^1 x \, dx = \left[ \frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} \\
\|u_0\|^2 &= \int_0^1 1 \, dx = [x]_0^1 = 1
\end{align*}
\]
Hence, \( u_1 = x - \frac{1}{2} \). Then:

\[
 u_1 = v_2 - \frac{(v_2, u_0)}{\|u_0\|^2} u_0 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1.
\]

We also find that:

\[
(v_2, u_0) = \int_0^1 x^2 \, dx = \frac{1}{3},
\]
\[
(v_2, u_1) = \int_0^1 x^2(x - \frac{1}{2}) \, dx = \frac{1}{12},
\]
\[
\|u_1\|^2 = \int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{1}{12}.
\]

Hence, \( u_2 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6} \).

7.3 Theorem

**Theorem.** (Gram-Schmidt Orthogonalization) Let \( V \) be a vector space with inner product \((.,.)\). Let \( v_1, ..., v_k \) be a linearly independent set in \( V \). Then, there exists an orthogonal set \( u_1, ..., u_k \) such that \((v_i, u_i) > 0\) and \( \text{span}\{v_i, ..., v_i\} = \text{span}\{u_1, ..., u_i\} \) for all \( i = 1, ..., k \).

**Proof.** See the book, p.129 – 131. \( \square \)