Lecture 2

Subspaces

In most applications we will be working with a subset $W$ of a vector space $V$ such that $W$ itself is a vector space.

Question: Do we have to test all the axioms to find out if $W$ is a vector space?

The answer is NO.

Theorem. Let $W \neq \emptyset$ be a subset of a vector space $V$. Then $W$, with the same addition and scalar multiplication as $V$, is a vector space if and only if the following two conditions hold:

1. $u + v \in W$ for all $u, v \in W$ (or $W + W \subseteq W$)
2. $r \cdot u \in W$ for all $r \in \mathbb{R}$ and all $u \in W$ (or $\mathbb{R}W \subseteq W$).

In this case we say that $W$ is a subspace of $V$.

Proof. Assume that $W + W \subseteq W$ and $\mathbb{R}W \subseteq W$.

To show that $W$ is a vector space we have to show that all the 10 axioms of Definition 1.1 hold for $W$. But that follows because the axioms hold for $V$ and $W$ is a subset of $V$:

A1 (Commutativity of addition)

For $u, v \in W$, we have $u + v = v + u$. This is because $u, v$ are also in $V$ and commutativity holds in $V$.  

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A4 (Existence of additive identity)
Take any vector \( u \in W \). Then by assumption \( 0 \cdot u = \vec{0} \in W \). Hence \( \vec{0} \in W \).

A5 (Existence of additive inverse)
If \( u \in W \) then \( -u = (-1) \cdot u \in W \).

One can check that the other axioms follow in the same way.

\[ \square \]

2.1 Examples
Usually the situation is that we are given a vector space \( V \) and a subset of vectors \( W \) satisfying some conditions and we need to see if \( W \) is a subspace of \( V \).

\[ W = \{ v \in V : \text{some conditions on } v \} \]

We will then have to show that

\[
\begin{align*}
    u, v & \in W \\
    r & \in \mathbb{R}
\end{align*}
\]

\[
\begin{align*}
    u + v \\
    r \cdot u
\end{align*}
\]

Satisfy the same conditions.

2.2 Lines through the origin as subspaces of \( \mathbb{R}^2 \)

Example.

\[
\begin{align*}
    V & = \mathbb{R}^2, \\
    W & = \{(x, y) | y = kx\} \quad \text{for a given } k \\
    & = \text{line through (0, 0) with slope } k.
\end{align*}
\]

To see that \( W \) is in fact a subspace of \( \mathbb{R}^2 \):
Let \( u = (x_1, y_1) \), \( v = (x_2, y_2) \in W \). Then \( y_1 = kx_1 \) and \( y_2 = kx_2 \).
2.3. A SUBSET OF \( \mathbb{R}^2 \) THAT IS NOT A SUBSPACE

and

\[
  u + v = (x_1 + x_2, y_1 + y_2) \\
  = (x_1 + x_2, kx_1 + kx_2) \\
  = (x_1 + x_2, k(x_1 + x_2)) \in W
\]

Similarly, \( r \cdot u = (rx_1, ry_1) = (rx_1, krx_1) \in W \)

So what are the subspaces of \( \mathbb{R}^2 \)?

1. \( \{0\} \)

2. Lines. But only those that contain \((0, 0)\). Why?

3. \( \mathbb{R}^2 \)

**Remark** (First test). If \( W \) is a subspace, then \( \vec{0} \in W \).

**Thus**: If \( \vec{0} \not\in W \), then \( W \) is not a subspace.

This is why a line not passing through \((0, 0)\) can not be a subspace of \( \mathbb{R}^2 \).

2.3 A subset of \( \mathbb{R}^2 \) that is not a subspace

**Warning.** We can not conclude from the fact that \( \vec{0} \in W \), that \( W \) is a subspace.

**Example.** Lets consider the following subset of \( \mathbb{R}^2 \):

\[
  W = \{(x, y)|x^2 - y^2 = 0\}
\]

Is \( W \) a subspace of \( \mathbb{R}^2 \)? Why?

The answer is NO.

We have \((1, 1) \in W \) but \((1, 1) + (1, -1) = (2, 0) \not\in W \). i.e., \( W \) is not closed under addition.

Notice that \((0, 0) \in W \) and \( W \) is closed under multiplication by scalars.
2.4 Subspaces of $\mathbb{R}^3$

What are the subspaces of $\mathbb{R}^3$?

1. \{0\} and $\mathbb{R}^3$.

2. Planes: A plane $W \subseteq \mathbb{R}^3$ is given by a normal vector $(a, b, c)$ and its distance from $(0, 0, 0)$ or

$$W = \{(x, y, z)\mid ax + by + cz = p\}$$

condition on $(x, y, z)$

For $W$ to be a subspace, $(0, 0, 0)$ must be in $W$ by the first test. Thus

$$p = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$$

or

$$p = 0$$

2.4.1 Planes containing the origin

A plane containing $(0, 0, 0)$ is indeed a subspace of $\mathbb{R}^3$.

Proof. Let $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2) \in W$. Then

$$ax_1 + by_1 + cz_1 = 0$$

$$ax_2 + by_2 + cz_2 = 0$$

Then we have

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = \underbrace{ax_1 + by_1 + cz_1\,}_{0} + \underbrace{ax_2 + by_2 + cz_2\,}_{0} = 0$$

and

$$a(rx_1) + b(ry_1) + c(rz_1) = r(ax_1 + by_1 + cz_1) = 0 \quad \square$$
2.5 Summary of subspaces of $\mathbb{R}^3$

1. $\{0\}$ and $\mathbb{R}^3$.

2. Planes containing $(0, 0, 0)$.

3. Lines containing $(0, 0, 0)$.
   (Intersection of two planes containing $(0, 0, 0)$)

2.6 Exercises

Determine whether the given subset of $\mathbb{R}^n$ is a subspace or not (Explain):

a) $W = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.

b) $W = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 2y^2 + z = 0\}$.

c) $W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y - z = 0\}$.

d) The set of all vectors $(x_1, x_2, x_3)$ satisfying
   \[ 2x_3 = x_1 - 10x_2. \]

e) The set of all vectors in $\mathbb{R}^4$ satisfying the system of linear equations
   \[
   \begin{align*}
   2x_1 + 3x_2 + 5x_4 &= 0 \\
   x_1 + x_2 - 3x_3 &= 0
   \end{align*}
   \]

f) The set of all points $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ satisfying
   \[ x_1 + 2x_2 + 3x_3 + x_4 = -1. \]