Mass Problems

Stephen G. Simpson
Department of Mathematics
Pennsylvania State University
http://www.math.psu.edu/simpson/

May 24, 2004

Abstract

A mass problem is a set of Turing oracles. If $P$ and $Q$ are mass problems, we say that $P$ is weakly reducible to $Q$ if every member of $Q$ Turing computes a member of $P$. Two mass problems are said to be weakly equivalent if each is weakly reducible to the other. A weak degree is an equivalence class under weak reducibility. The weak degrees are partially ordered in the obvious way, by weak reducibility. This partial ordering is easily seen to be a complete distributive lattice. We focus on the countable sublattice obtained by restricting to mass problems of the form $P = \{\text{all paths through } T\}$, where $T$ is an infinite computable subtree of the full binary tree. We present natural examples of such mass problems arising from mathematical logic, Martin-Löf randomness, effective immunity, and the Arslanov Completeness Criterion. We also present artificial examples constructed by means of priority arguments.

This is the lecture notes that I prepared for my series of five lectures at the Summer School and Workshop on Proof Theory, Computation and Complexity, held at the Technical University of Dresden, June 23–July 4, 2003. See http://www.ki.inf.tu-dresden.de/~guglielm/WPT2/.
1 Turing Degrees

Definition 1.1. We use \( \omega \) to denote the set of natural numbers:
\[
\omega = \{0, 1, 2, \ldots \}.
\]

We use \( \omega^\omega \) to denote the set of total functions from \( \omega \) into \( \omega \):
\[
\omega^\omega = \{ f \mid f : \omega \to \omega \}.
\]

We use \( 2^\omega \) to denote the set of total functions from \( \omega \) into \( \{0, 1\} \):
\[
2^\omega = \{ X \mid X : \omega \to \{0, 1\} \}.
\]

The space \( \omega^\omega \) with the obvious product topology is called Baire space. Its subspace \( 2^\omega \) is called Cantor space. We use \( f, g, h, \ldots \) to denote points of the Baire space, and \( X, Y, Z, \ldots \) to denote points of the Cantor space.

Remark 1.2. We sometimes identify subsets of \( \omega \) with their characteristic functions in \( 2^\omega \). The characteristic function of \( A \subseteq \omega \) is \( \chi_A \in 2^\omega \) given by
\[
\chi_A(n) = \begin{cases} 
1 & \text{if } n \in A, \\
0 & \text{if } n \notin A.
\end{cases}
\]
Definition 1.3. For $e, m, n \in \omega$ and $g \in \omega^\omega$, we write
\[
\{e\}^g(m) = n
\]
to mean that $e$ is the Gödel number of a Turing machine which, if started with input $m$ (on the input tape) and oracle $g$ (on the auxiliary or oracle tape), eventually halts with output $n$ (on the output tape).

Remark 1.4. The idea of using an arbitrary, possibly non-computable member of $\omega^\omega$ as an oracle is due to Turing [44]. See also Rogers [30, Section 9.2].

Definition 1.5. According to Definition 1.3, each $e \in \omega$ gives rise to a partial recursive functional $\Phi$ from $\omega^\omega \times \omega$ to $\omega$, given by $\Phi(g, m) \simeq \{e\}^g(m)$. We may also view $\Phi$ as a partial recursive functional from $\omega^\omega$ to $\omega^\omega$, given by $\Phi(g)(m) \simeq \Phi(g, m) \simeq \{e\}^g(m)$. In either case we say that $e$ is an index of $\Phi$.

Remark 1.6. Here $\simeq$ denotes strong equality for expressions which may be undefined. Thus $E_1 \simeq E_2$ if and only if $E_1, E_2$ are both undefined, or both defined and equal. We write $E \downarrow$ to mean that $E$ is defined. We write $E \uparrow$ to mean that $E$ is undefined.

Definition 1.7. For $f, g \in \omega^\omega$, we say that $g$ Turing computes $f$, or $f$ is recursive in $g$, or $f$ is Turing reducible to $g$, abbreviated $f \leq_T g$, if $\Phi(g) = f$ for some partial recursive functional $\Phi$. Thus $f \leq_T g$ if and only if there exists $e \in \omega$ such that $\{e\}^g(m) = f(m)$ for all $m \in \omega$. Clearly $\leq_T$ is a reflexive, transitive relation on $\omega^\omega$.

We say that $f$ is Turing equivalent to $g$, abbreviated $f \equiv_T g$, if $f \leq_T g$ and $g \leq_T f$. Clearly $\equiv_T$ is an equivalence relation on $\omega^\omega$.

Definition 1.8. The Turing degrees, or degrees of unsolvability, are the equivalence classes of members of $\omega^\omega$ under the equivalence relation $\equiv_T$. The Turing degree of $f \in \omega^\omega$ is denoted $\text{deg}_T(f)$. We use $D_T$ to denote the set of Turing degrees. Thus we have
\[
\text{deg}_T(f) = \{g \in \omega^\omega \mid f \equiv_T g\}
\]
and
\[
D_T = \omega^\omega / \equiv_T = \{\text{deg}_T(f) \mid f \in \omega^\omega\}.
\]

Definition 1.9. In Definition 1.3, if there is no oracle $g$, we write simply
\[
\{e\}(m) = n.
\]
Thus $f \in \omega^\omega$ is said to be Turing computable, or recursive, if there exists $e \in \omega$ such that $\{e\}(m) = f(m)$ for all $m \in \omega$.

Remark 1.10. We partially order $D_T$ by putting $\text{deg}_T(f) \leq_T \text{deg}_T(g)$ if and only if $f \leq_T g$. Under this partial ordering, $D_T$ is an upper semilattice, with least upper bound operation given by $f \oplus g \in \omega^\omega$, where
\[
(f \oplus g)(2n) = f(n), \quad (f \oplus g)(2n + 1) = g(n),
\]
for all $f, g \in \omega^\omega$. Moreover $D_T$ has a bottom element.
\[ 0 = \deg_T(\lambda n.0) = \{ f \in \omega^\omega \mid f \text{ is recursive} \}. \]

It can be shown that \( \mathcal{D}_T \) has no top element and is not a lattice.

We now compare the Baire space, \( \omega^\omega \), to the Cantor space, \( 2^\omega \).

**Theorem 1.11.** For each \( f \in \omega^\omega \) there exists \( X \in 2^\omega \) such that \( f \equiv_T X \).

**Proof.** Let \( X \) be the characteristic function of \( \{ 2^m3^n \mid f(m) = n \} \). \( \square \)

**Remark 1.12.** In view of Theorem 1.11, the Baire space \( \omega^\omega \) and the Cantor space \( 2^\omega \) are identical as to Turing degrees. Thus we may write

\[ \mathcal{D}_T = 2^\omega/\equiv_T = \{ \deg_T(X) \mid X \in 2^\omega \}. \]

**Remark 1.13.** The idea behind Turing reducibility is that each non-recursive (i.e., non-computable) \( X \in 2^\omega \) is regarded as an “unsolvable problem”, viz., the problem of computing \( X \). Then \( Y \leq_T X \) means that “the problem” \( Y \) is “reducible” to “the problem” \( X \) in the sense that, if there were an oracle for solving \( X \), then, with the help of this oracle, we could solve \( Y \). Thus \( \deg_T(X) \), the Turing degree of \( X \), is a measure of the “unsolvability” of “the problem” \( X \). In particular, the Turing degree \( 0 \) corresponds to “solvable problems”, i.e., recursive members of \( 2^\omega \).

**Example 1.14.** Turing’s original example of an unsolvable problem is the Halting Problem, i.e., the (characteristic function of the) set

\[ H = \{ e \in \omega \mid |e|0 \downarrow \} \]

\[ = \{ \text{Gödel numbers of Turing machines which eventually halt} \}. \]

The Turing degree of the Halting Problem is denoted \( 0' \). Turing’s famous theorem on unsolvability of the Halting Problem amounts to saying that \( H \) is nonrecursive, i.e., \( 0' > 0 \).

In addition, it is known that there are infinitely many Turing degrees \( a \) in the interval \( 0 \leq a \leq 0' \). Moreover, the Turing degrees in this interval do not form a lattice.

**Definition 1.15.** A set \( A \subseteq \omega \) is said to be recursively enumerable, abbreviated r. e., if \( A = \{ f(m) \mid m \in \omega \} \) for some recursive function \( f : \omega \to \omega \). In this case, an index of \( A \) is just an index of \( f \). A Turing degree is said to be recursively enumerable if it is the Turing degree of (the characteristic function of) an r. e. subset of \( \omega \). The set of r. e. Turing degrees is denoted \( \mathcal{R}_T \).

**Definition 1.16.** An r. e. set \( C \subseteq \omega \) is said to be Turing complete if for every r. e. set \( A \subseteq \omega \) we have \( A \leq_T C \). An r. e. Turing degree is said to be Turing complete if it is the Turing degree of a Turing complete r. e. set.

**Remark 1.17.** It is known that the halting set \( H \) and its Turing degree \( 0' \) are recursively enumerable and Turing complete. Moreover, if \( a \) and \( b \) are r. e. Turing degrees, then so is their least upper bound, \( \sup(a, b) \). Thus \( \mathcal{R}_T \) is a countable upper semilattice with a top element, \( 0' \), and a bottom element, \( 0 \). It is known that \( \mathcal{R}_T \) is infinite and properly included in the interval \( 0 \leq a \leq 0' \) in \( \mathcal{D}_T \), and is not a lattice.
Remark 1.18. More generally, for any Turing degree $a = \text{deg}_T(X)$, define $a' = \text{deg}_T(H^X)$ where $H^X$ is (the characteristic function of)
\[ H^X = \{ e \mid \{ e \}^X(0) \downarrow \}, \]
the halting set relative to $X$. It is known that the Turing jump operator $a \rightarrow a'$ is well defined, because $X \leq_T Y$ implies $H^X \leq_T H^Y$. It is known that $a' > a$, and that $a'$ is r. e. relative to $a$ and Turing complete relative to $a$.

Remark 1.19. The semilattice $\mathcal{D}_T$ of all Turing degrees, and its subsemilattice $\mathcal{R}_T$ consisting of the r. e. Turing degrees, have been studied intensively for many years. See Sacks [31], Rogers [30], Lerman [23], Simpson [36] [37], Soare [42].

2 Weak and Strong Degrees

Definition 2.1. A mass problem is a subset of $\omega^\omega$. We use, $P, Q, R, \ldots$ to denote subsets of $\omega^\omega$.

Definition 2.2. For $P, Q \subseteq \omega^\omega$, we say that $P$ is weakly reducible to $Q$, abbreviated $P \leq_w Q$, if for each $g \in Q$ there exists $f \in P$ such that $f \leq_T g$. (Recall that $\leq_T$ is Turing reducibility.) The notion of weak reducibility was introduced by Muchnik [28] and has sometimes been called Muchnik reducibility. Clearly $\leq_w$ is a reflexive, transitive relation on the powerset of $\omega^\omega$.

Definition 2.3. We say that $P, Q \subseteq \omega^\omega$ are weakly equivalent, or Muchnik equivalent, abbreviated $P \equiv_w Q$, if $P \leq_w Q$ and $Q \leq_w P$. Clearly $\equiv_w$ is an equivalence relation on the powerset of $\omega^\omega$. The equivalence classes are called weak degrees, or Muchnik degrees. The weak degree of $P$ is denoted $\text{deg}_w(P)$. We use $\mathcal{D}_w$ to denote the set of weak degrees. Thus we have
\[ \text{deg}_w(P) = \{ Q \subseteq \omega^\omega \mid P \equiv_w Q \} \]
and
\[ \mathcal{D}_w = \{ \text{deg}_w(P) \mid P \subseteq \omega^\omega \}. \]

We partially order $\mathcal{D}_w$ by putting $\text{deg}_w(P) \leq \text{deg}_w(Q)$ if and only if $P \leq_w Q$.

Remark 2.4. The idea behind weak reducibility is that an arbitrary subset of $\omega^\omega$ may be regarded as a “mass problem”, i.e., a problem whose solution is not necessarily unique. If $P \subseteq \omega^\omega$ is viewed a mass problem, the “solutions” of $P$ are just the members of $P$. A mass problem is regarded as “unsolvable” if it has no recursive solution. Compare Remark 1.13. Viewing $P$ and $Q$ as mass problems, $P \leq_w Q$ means that $P$ is “reducible” to $Q$ in the sense that, given any oracle for a solution of $Q$, we could use this oracle to compute a solution of $P$. Thus the weak degree of $P$ is a measure of the “difficulty” of $P$ qua mass problem.
Example 2.5. Let $T$ be a consistent, recursively axiomatizable theory. For instance, we may take $T = \text{PA} =$ first order Peano Arithmetic, or $T = \text{ZF} =$ Zermelo/Fraenkel Set Theory. By a completion of $T$ we mean a maximal consistent theory extending $T$ with the same vocabulary as $T$. Note that $T$ is incomplete if and only if $T$ has more than one completion. The Gödel Incompleteness Theorem implies that this is the case for $T = \text{PA}$ or $T = \text{ZF}$.

We identify sentences with their Gödel numbers in $\omega$. We identify theories as sets of (Gödel numbers of) sentences. Define

$$P_T = \{X \in 2^\omega \mid X \text{ is (the characteristic function of) a completion of } T\}.$$ 

We may then view $P_T$ as a mass problem, viz., the “problem” of finding a completion of $T$. The mass problem $P_T$ is regarded as “unsolvable” if and only if the theory $T$ has no decidable completion. This is equivalent to saying that $T$ is essentially undecidable, i.e., there is no consistent, decidable theory extending $T$. By a result of Tarski, this is the case for $\text{PA}$, $\text{ZF}$, and many other theories which arise in the foundations of mathematics.

We now introduce strong reducibility, a variant of weak reducibility.

Definition 2.6. For $P, Q \subseteq \omega^\omega$, we say that $P$ is strongly reducible to $Q$, abbreviated $P \leq_s Q$, if there exists a partial recursive functional $\Phi : Q \to P$, i.e., a partial recursive functional $\Phi$ from $\omega^\omega$ to $\omega^\omega$ such that the domain of definition of $\Phi$ includes $Q$ and for all $g \in Q$ we have $\Phi(g) \in P$. This notion was introduced by Medvedev [27] and has sometimes been called Medvedev reducibility. See also Rogers [30, Section 13.7]. Clearly $\leq_s$ is a reflexive, transitive relation on the powerset of $\omega^\omega$.

Remark 2.7. Thus strong reducibility is a uniform variant of weak reducibility. Note that $P \leq_s Q$ implies $P \leq_w Q$, but the converse often fails. Later we shall see an analogy

weak reducibility / strong reducibility =

Turing reducibility / truth table reducibility.

See Remark 3.9 below.

Definition 2.8. We define strong degrees in terms of $\leq_s$, just as weak degrees were defined in terms of $\leq_w$ in Definition 2.3.

Explicitly, we say that $P, Q \subseteq \omega^\omega$ are strongly equivalent, or Medvedev equivalent, abbreviated $P \equiv_s Q$, if $P \leq_s Q$ and $Q \leq_s P$. Clearly $\equiv_s$ is an equivalence relation on the powerset of $\omega^\omega$. The equivalence classes are called strong degrees, or Medvedev degrees. The strong degree of $P$ is denoted $\text{deg}_s(P)$. We use $\mathcal{D}_s$ to denote the set of strong degrees. Thus we have

$$\text{deg}_s(P) = \{Q \subseteq \omega^\omega \mid P \equiv_s Q\}$$

and

$$\mathcal{D}_s = \{\text{deg}_s(P) \mid P \subseteq \omega^\omega\}.$$ 

We partially order $\mathcal{D}_s$ by putting $\text{deg}_s(P) \leq \text{deg}_s(Q)$ if and only if $P \leq_s Q$. 


Remark 2.9. We are mainly interested in weak reducibility and weak degrees. We discuss strong reducibility and strong degrees for technical reasons only.

Theorem 2.10. The lattices of weak degrees and strong degrees, $D_w$ and $D_s$, are distributive lattices with a bottom element.

Proof. The least upper bound operation for weak or strong reducibility is given by
\[ P \times Q = \{ f \oplus g \mid f \in P, g \in Q \}. \]
The greatest lower bound operation for weak reducibility is given by $P \cup Q$. The greatest lower bound operation for weak or strong reducibility is given by
\[ P + Q = \{ (0) \wedge f \mid f \in P \} \cup \{ (1) \wedge g \mid g \in Q \}. \]
It is straightforward but instructive to check that the weak degrees form a distributive lattice under these operations. Similarly for the strong degrees.

The bottom element of $D_w$ is $0 = \deg_w(\omega\omega)$, and similarly for $D_s$. Actually, the $0$ of $D_w$ and the $0$ of $D_s$ are identical, namely
\[ 0 = \{ P \subseteq \omega\omega \mid P \text{ contains a recursive member} \}. \]

Theorem 2.11. In both $D_w$ and $D_s$, the bottom element $0$ is meet irreducible, i.e., it is not the greatest lower bound of two nonzero degrees.

Proof. This is obvious, because $P + Q$ (or $P \cup Q$) contains a recursive member if and only if at least one of $P$ and $Q$ contains a recursive member.

Theorem 2.12. $D_T$ is canonically embeddable into $D_w$ and into $D_s$. The embeddings preserve order and least upper bound, and carry $0$ to $0$.

Proof. The embedding of $D_T$ into $D_w$ is given by $\deg_T(X) \mapsto \deg_w(\{X\})$, and similarly for $D_s$. Here $\{X\}$ denotes the singleton set whose unique element is $X$. We have $X \leq_T Y$ if and only if $\{X\} \leq_w \{Y\}$, if and only if $\{X\} \leq_s \{Y\}$. Moreover, $\{X\} \times \{Y\} = \{X \oplus Y\}$.

Theorem 2.13. $D_w$ is a complete distributive lattice.

Proof. It is straightforward to check that $D_w$ under $\leq$ is canonically isomorphic to the partial ordering of upward closed subsets of $D_T$ under reverse inclusion. This is clearly a complete lattice ordering. The isomorphism is given by $\deg_w(P) \mapsto \{ \deg_T(g) \mid P \leq_w \{g\} \}$.

We now compare the Baire space, $\omega\omega$, to the Cantor space, $2\omega$.

Definition 2.14. $P, Q \subseteq \omega\omega$ are recursively homeomorphic if there exist one-to-one, onto, partial recursive functionals $\Phi : P \to Q$ and $\Phi^{-1} : Q \to P$.

Theorem 2.15. For each $P \subseteq \omega\omega$ there exists $P^* \subseteq 2\omega$ such that $P$ is recursively homeomorphic to $P^*$. It follows that $P \equiv_s P^*$, hence $P \equiv_w P^*$.
Proof. Identifying subsets of \( \omega \) with their characteristic functions in \( 2^\omega \), let 
\[
P^* = \{ \{ 2^m3^n \mid f(m) = n \} \mid f \in P \}.
\]
Compare Theorem 1.11.

Remark 2.16. In view of Theorem 2.15, subsets of \( 2^\omega \) and subsets of \( \omega^\omega \) are identical as to their weak and strong degrees. Compare Remark 1.12. Thus we may write
\[
D_w = \{ \deg_w(P) \mid P \subseteq 2^\omega \}
\]
and similarly for \( D_s \).

Remark 2.17. On the other hand, the spaces \( 2^\omega \) and \( \omega^\omega \) are different in some ways. For instance, \( 2^\omega \) is compact while \( \omega^\omega \) is not compact. Furthermore, the differences will be important to us. In particular, we shall see that the lattice of weak degrees of nonempty \( \Pi^0_1 \) subsets of \( 2^\omega \) has a top element, but this is not the case for \( \omega^\omega \). See Section 4 below.

Remark 2.18. Our main object of study will be the lattice \( \mathcal{P}_w \) of weak degrees of nonempty \( \Pi^0_1 \) subsets of the Cantor space, \( 2^\omega \). See Definition 3.7 below. We mention \( \Pi^0_1 \) subsets of \( \omega^\omega \) for technical reasons only.

Remark 2.19. Sorbi [43] gives a general survey of the lattices \( D_w \) and \( D_s \) of all weak and strong degrees. However, Sorbi does not discuss the sublattices \( \mathcal{P}_w \) and \( \mathcal{P}_s \) of weak and strong degrees of nonempty \( \Pi^0_1 \) subsets of \( 2^\omega \). The explicit study of \( \mathcal{P}_w \) and \( \mathcal{P}_s \) began only recently, in 1999, with Simpson [38].

3 Trees and \( \Pi^0_1 \) Sets

Definition 3.1. A predicate \( R \subseteq \omega^\omega \times \omega \) is said to be recursive if its characteristic function \( \chi_R : \omega^\omega \times \omega \rightarrow \{0, 1\} \), given by
\[
\chi_R(f, n) = \begin{cases} 
1 & \text{if } R(f, n), \\
0 & \text{if } \neg R(f, n),
\end{cases}
\]
is a recursive functional.

Definition 3.2. A set \( P \subseteq \omega^\omega \) is said to be \( \Pi^0_1 \) if there is a recursive predicate \( R \subseteq \omega^\omega \times \omega \) such that
\[
P = \{ f \in \omega^\omega \mid \forall n R(f, n) \}.
\]

Example 3.3. Let \( T \) be a consistent, recursively axiomatizable theory. As in Example 2.5, let \( P_T \) be the set of (characteristic functions of) completions of \( T \). It is easy to see that \( P_T \) is a nonempty \( \Pi^0_1 \) subset of \( 2^\omega \). Thus weak and strong reducibility can be used to compare such theories.

In more detail, let \( B(T) \) be the Lindenbaum algebra of \( T \), i.e., the Boolean algebra of sentences in the vocabulary of \( T \) modulo provable equivalence over \( T \). Thus \( B(T) \) is a recursively presented Boolean algebra, i.e., the quotient of a free recursive Boolean algebra modulo a recursively enumerable ideal. It can
be shown that $P_T$ is (recursively homeomorphic to) the Stone space of $B(T)$. Thus we have an effective version of Stone Duality, with nonempty $\Pi^0_1$ subsets of $2^\omega$ as the Stone spaces. If $T_1$ and $T_2$ are two such theories, then recursive homomorphisms

$$h : B(T_1) \to B(T_2)$$

are in canonical one-to-one correspondence with recursive functionals

$$\Phi : P_{T_2} \to P_{T_1}.$$ 

In particular, $B(T_1)$ is recursively isomorphic to $B(T_2)$ if and only if $P_{T_1}$ is recursively homeomorphic to $P_{T_2}$.

**Remark 3.4.** Conversely, one can show that every nonempty $\Pi^0_1$ subset of $2^\omega$ is recursively homeomorphic to $P_T$ for some consistent, recursively axiomatizable theory $T$. See Theorem 3.18 and Remark 3.19 below.

**Remark 3.5.** Many other interesting examples of $\Pi^0_1$ subsets of $2^\omega$ arise from logic, algebra, analysis, geometry, combinatorics, computability theory, etc. There is a large literature on this subject. See for instance Cenzer/Remmel [8] and Simpson [39, Chapter IV and Section VIII.2].

**Remark 3.6.** If $P$ and $Q$ are $\Pi^0_1$ subsets of $2^\omega$, then $P \times Q, P \cup Q, P + Q$ are $\Pi^0_1$ subsets of $2^\omega$. Thus the weak degrees of nonempty $\Pi^0_1$ subsets of $2^\omega$ form a sublattice of the lattice of all weak degrees. Similarly for strong degrees, and similarly for subsets of $\omega^\omega$.

**Definition 3.7.** We use $\mathcal{P}_w$ to denote the set of weak degrees of nonempty $\Pi^0_1$ subsets of $2^\omega$. Thus $\mathcal{P}_w$ is a countable sublattice of $D_w$. Similarly $\mathcal{P}_s$, the set of strong degrees of nonempty $\Pi^0_1$ subsets of $2^\omega$, is a countable sublattice of $D_s$.

**Remark 3.8.** The countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ are known to have a rich structure.

Binns/Simpson [4, 6] have shown that every countable distributive lattice is lattice embeddable in $\mathcal{P}_w$. A similar conjecture for $\mathcal{P}_s$ remains open, although partial results in this direction are known. A special case is Corollary 9.4 below. Binns [4, 5] has shown that for every $b > 0$ in $\mathcal{P}_w$ there exist $b_1, b_2 < b$ in $\mathcal{P}_w$ such that $b = \text{sup}(b_1, b_2)$, and similarly for $\mathcal{P}_s$. Cenzer/Hinman [7] have shown that, for all $a, b \in \mathcal{P}_s$ such that $a < b$, there exists $c \in \mathcal{P}_s$ such that $a < c < b$. A similar conjecture for $\mathcal{P}_w$ remains open. Binns [4, 5] has improved the result of Cenzer/Hinman [7] by showing that, for all $a, b \in \mathcal{P}_s$ such that $a < b$, there exist $b_1, b_2 < b$ in $\mathcal{P}_s$ such that $a < b_1, b_2$ and $b = \text{sup}(b_1, b_2)$.

These recent results for $\mathcal{P}_w$ and $\mathcal{P}_s$ are proved by means of priority arguments. They invite comparison with the older, known results for r.e. Turing degrees in Sacks [31] and Soare [42].

**Remark 3.9.** It is known that $\mathcal{P}_s$ behaves somewhat differently from $\mathcal{P}_w$. To bring out one of the differences, let $P, Q$ be $\Pi^0_1$ subsets of $2^\omega$ with $P \leq_s Q$. Thus we have a recursive functional $\Phi : Q \to P$. Using compactness of $2^\omega$, we
can find a total recursive functional \( \hat{\Phi} : 2^\omega \rightarrow 2^\omega \) which extends \( \Phi \), i.e., \( \Phi = \text{the restriction of } \hat{\Phi} \text{ to } Q \). Thus, for every \( Y \in Q \), \( \Phi(Y) \) is not only \( \leq_T Y \) but also \( \leq_T Y \), where \( \leq_T \) denotes truth table reducibility. A general discussion of truth table reducibility is in Rogers [30]. See also Simpson [35, Section 3] and the proof of Lemma 6.5 below. Thus we have an analogy

\[
\text{weak reducibility / strong reducibility = Turing reducibility / truth table reducibility.}
\]

In addition, it is known that every nonzero weak degree in \( P_w \) includes infinitely many distinct strong degrees in \( P_s \). See Simpson/Slaman [41].

We now present a useful characterization of \( \Pi^0_1 \) sets, in terms of trees.

**Definition 3.10.** We use \( \text{Seq} \) to denote the set of finite sequences of natural numbers. We use \( \rho, \sigma, \tau, \ldots \) to denote members of \( \text{Seq} \). The length of \( \sigma \in \text{Seq} \) is denoted \( \text{lh}(\sigma) \). The concatenation \( \sigma \) followed by \( \tau \) is denoted \( \sigma \sqsupset \tau \). Thus \( \text{lh}(\sigma \sqsupset \tau) = \text{lh}(\sigma) + \text{lh}(\tau) \). Given \( f \in \omega^\omega \) and \( n \in \omega \), we put

\[
f[n] = \langle f(0), f(1), \ldots, f(n-1) \rangle.
\]

Thus \( f[n] \in \text{Seq} \) and \( \text{lh}(f[n]) = n \). We write \( \sigma \subset f \) to mean that \( \sigma = f[n] \) for some \( n \). Given \( \tau \in \text{Seq} \) and \( n \leq \text{lh}(\tau) \), we put

\[
\tau[n] = \langle \tau(0), \tau(1), \ldots, \tau(n-1) \rangle.
\]

We write \( \sigma \subseteq \tau \) to mean that \( \sigma = \tau[n] \) for some \( n \leq \text{lh}(\tau) \).

**Definition 3.11.** For \( e, m, n, t \in \omega \) and \( \sigma \in \text{Seq} \), we write

\[
\{e\}^\sigma_t(m) = n
\]

to mean that \( \{e\}^\sigma_t(m) = n \) for some (or any) \( f \in \omega^\omega \) such that \( f \supset \sigma \), via a Turing computation which halts in at most \( t \) steps and uses only oracle information from \( \sigma \). Compare Definition 1.3.

**Lemma 3.12.** We have:

1. If \( \{e\}^s_{t'}(m) = n \) and \( t' \geq t \) and \( s' \geq s \), then \( \{e\}^{s'}_{t'}(m) = n \).
2. \( \{e\}^{s}_{t}(m) = n \) if and only if \( \{e\}^{s}_{t}(m) = n \) for some \( s, t \in \omega \).
3. \( \{e\}^{s}_{t}(m) = n \) if and only if \( \{e\}^{s}_{t}(m) = n \) for some \( s \in \omega \).
4. The relations \( \{e\}^{\tau}_{t}(m) = n \) and \( \{e\}^{\tau}_{t}(m) \downarrow \) are recursive.

**Proof.** Straightforward.

**Definition 3.13.** A tree is a set \( T \subseteq \text{Seq} \) such that, for all \( \tau \in T \) and all \( n < \text{lh}(\tau) \), \( \tau[n] \in T \). A path through \( T \) is a function \( f \in \omega^\omega \) such that, for all \( n \in \omega \), \( f[n] \in T \).
Theorem 3.14. A set \( P \subseteq \omega^\omega \) is \( \Pi^0_1 \) if and only if there exists a recursive tree \( T \subseteq \text{Seq} \) such that

\[
P = \{ f \in \omega^\omega \mid f \text{ is a path through } T \}.
\]

Proof. Clearly the set of paths through a recursive tree is \( \Pi^0_1 \). Conversely, given a \( \Pi^0_1 \) set \( P = \{ f \mid \forall n R(f, n) \} \), let \( e \) be an index of the partial recursive functional \( \Phi(f, m) \simeq \text{least } n \) such that \( \neg R(f, n) \). Then \( P = \{ f \mid \{ e \} f(0) \uparrow \} \). Putting \( T = \{ \sigma \mid \{ e \} \text{lh}(\sigma)(0) \uparrow \} \) we see that \( T \) is a recursive tree and \( P = \{ \text{paths through } T \} \).

Definition 3.15. We use \( \text{Seq}_2 \) to denote the set of finite sequences of 0’s and 1’s. Since \( \text{Seq}_2 \subseteq \text{Seq} \), the notations introduced in Definitions 3.10 and 3.11 apply.

Definition 3.16. A tree \( T \) is said to be bounded if for each \( n \in \omega \) there are only finitely many \( \sigma \in T \) such that \( \text{lh}(\sigma) = n \). Note that \( \text{Seq}_2 \) is a bounded tree, while \( \text{Seq} \) is an unbounded tree.

Corollary 3.17. A set \( P \subseteq 2^\omega \) is \( \Pi^0_1 \) if and only if there exists a recursive tree \( T \subseteq \text{Seq}_2 \) such that

\[
P = \{ X \in 2^\omega \mid X \text{ is a path through } T \}.
\]

Moreover, \( P \) is nonempty if and only if \( T \) is infinite.

Proof. The first assertion is a special case of Theorem 3.14. The second assertion follows from compactness of \( 2^\omega \), in the form of König’s Lemma. Namely, a bounded tree has a path if and only if it is infinite.

We now use Corollary 3.17 to obtain the converse of Example 3.3.

Theorem 3.18.

1. If \( T \) is a consistent, recursively axiomatizable theory, then \( P_T \), the set of completions of \( T \), is a nonempty \( \Pi^0_1 \) subset of \( 2^\omega \).

2. Conversely, if \( P \) is a nonempty \( \Pi^0_1 \) subset of \( 2^\omega \), we can find a consistent, recursively axiomatizable theory, \( T \), such that \( P \) is recursively homeomorphic to \( P_T \).

Proof. Part 1 has already been noted in Example 3.3. For part 2, let \( S \) be an infinite recursive tree such that \( P = \{ \text{paths through } S \} \). We use \( S \) to construct a theory \( T \) in the propositional calculus with atoms \( A_n, n \in \omega \). Writing \( A^1 = A \) and \( A^0 = \neg A \), the axioms of \( T \) are all sentences of the form \( \neg (A_0^n \land A_1^i \land \cdots \land A_k^j) \) where \( \langle i_0, i_1, \ldots, i_k \rangle \in \text{Seq}_2 \setminus S \). Then \( P \) is recursively homeomorphic to \( P_T \), via \( X \mapsto \text{the completion of } T \) with axioms \( A_n^{X(n)}, n \in \omega \).

Remark 3.19. Surprisingly, it is known that every \( \Pi^0_1 \) subset of \( 2^\omega \) is recursively homeomorphic to \( P_T \) for some finitely axiomatizable theory \( T \) in the predicate calculus. See Peretyatkin [29].
We now discuss $\Pi^0_1$ subsets of $\omega^\omega$ and compare them to $\Pi^0_1$ subsets of $2^\omega$.

**Definition 3.20.** A set $P \subseteq \omega^\omega$ is said to be *recursively bounded* if there exists a recursive function $g \in \omega^\omega$ such that for all $f \in P$, $f(n) < g(n)$ for all $n$.

**Remark 3.21.** Clearly any subset of $2^\omega$ is recursively bounded, viz., by the constant function $\lambda n$. The next theorem implies that, up to recursive homeomorphism, the study of recursively bounded $\Pi^0_1$ subsets of $\omega^\omega$ is equivalent to the study of $\Pi^0_1$ subsets of $2^\omega$.

**Theorem 3.22.** For each recursively bounded $\Pi^0_1$ set $P \subseteq \omega^\omega$, we can find a $\Pi^0_1$ set $P^* \subseteq 2^\omega$ such that $P$ is recursively homeomorphic to $P^*$. It follows that $P \equiv_s P^*$, hence $P \equiv_r P^*$.

**Proof.** Define $P^*$ as in the proof of Theorem 2.15. It is straightforward to show that, if $P$ is $\Pi^0_1$ and recursively bounded, then $P^*$ is $\Pi^0_1$.

**Remark 3.23.** Conversely, if $P \subseteq 2^\omega$ is $\Pi^0_1$, then for any recursive functional $\Phi : P \rightarrow \omega^\omega$, the range $\{\Phi(f) \mid f \in P\}$ is $\Pi^0_1$ and recursively bounded. This is a consequence of compactness of $2^\omega$.

**Remark 3.24.** By Theorem 3.22, the weak degrees of recursively bounded $\Pi^0_1$ subsets of $\omega^\omega$ belong to $\mathcal{P}_w$, and similarly for strong degrees. On the other hand, there are plenty of nonempty $\Pi^0_1$ subsets of $\omega^\omega$ whose weak degrees do not belong to $\mathcal{P}_w$.

**Example 3.25.** It is known from hyperarithmetical theory (see Sacks [32, Part A] or Simpson [39, Section VIII.3]) that for any hyperarithmetical $X \in 2^\omega$ there exists a hyperarithmetical $g \in \omega^\omega$ such that $X \leq_T g$ and the singleton set $\{g\} \subseteq \omega^\omega$ is $\Pi^0_1$. If $g$ is not recursive, the GKT Basis Theorem (see Simpson [39, Section VIII.2]) implies that $\deg_w(\{g\}) \nleq_w P$ for any nonempty $\Pi^0_1$ set $P \subseteq 2^\omega$.

**Example 3.26.** Another interesting $\Pi^0_1$ subset of $\omega^\omega$ is

$$\text{DNR} = \{f \in \omega^\omega \mid \forall n f(n) \neq \{n\}(n)\},$$

i.e., the set of $f : \omega \rightarrow \omega$ which are *diagonally non-recursive*. We shall comment more on this later. See Corollary 7.3 and Remark 7.5 below.

## 4 Weak and Strong Completeness

**Definition 4.1.** A nonempty $\Pi^0_1$ set $P \subseteq 2^\omega$ is said to be *weakly complete*, or *Muchnik complete*, if every nonempty $\Pi^0_1$ subset of $2^\omega$ is weakly reducible to $P$.

**Definition 4.2.** A nonempty $\Pi^0_1$ set $P \subseteq 2^\omega$ is said to be *strongly complete*, or *Medvedev complete*, if every nonempty $\Pi^0_1$ subset of $2^\omega$ is strongly reducible to $P$.

**Remark 4.3.** We use $1$ to denote the weak degree of any nonempty $\Pi^0_1$ subset of $2^\omega$ which is weakly complete. Thus $1$ is the top element of $\mathcal{P}_w$. Similarly for strong degrees and $\mathcal{P}_s$. 
Example 4.4. The following $\Pi^0_1$ subsets of $2^\omega$ are known to be strongly complete, hence weakly complete.

1. $P = \{\text{completions of PA}\}$. Instead of PA we could use any effectively axiomatizable, effectively essentially undecidable theory. This is related to the Gödel/Rosser Theorem. See also Scott/Tennenbaum [33].

2. $P = \{f \in 2^\omega \mid f \text{ separates } A \text{ and } B\}$, where $A = \{n \mid \{n\}(n) \simeq 0\}$ and $B = \{n \mid \{n\}(n) \simeq 1\}$. See Jockusch/Soare [20].

3. We can also give an explicit, recursion-theoretic construction of a $\Pi^0_1$ set $P$ with the desired property. Namely, $P = \prod_{e=0}^{\infty} P_e^+$ where $P_e^+$ is the nonempty $\Pi^0_1$ subset of $2^\omega$ indexed by $e$. See Simpson [35, Lemma 3.3].

Theorem 4.5 (Simpson 2000). Any two strongly complete $\Pi^0_1$ subsets of $2^\omega$ are recursively homeomorphic.

Proof. The proof is by an effective back-and-forth argument, using the Recursion Theorem. See Simpson [35, Section 3]. It is analogous to the proof of Myhill’s result that any two creative, recursively enumerable subsets of $\omega$ are recursively isomorphic. Myhill’s result is expounded in Rogers [30].

Corollary 4.6. A nonempty $\Pi^0_1$ subset of $2^\omega$ is strongly complete if and only if it is recursively homeomorphic to the set of completions of PA.

The proof of Theorem 4.5 also gives the following.

Corollary 4.7. Let $P$ and $Q$ be nonempty $\Pi^0_1$ subsets of $2^\omega$. If $P$ is strongly complete, then there is a recursive functional $\Phi: P \to Q$ which maps $P$ onto $Q$, i.e., $Q = \{\Phi(f) \mid f \in P\}$.

Proof. See Simpson [35, Section 3].

The following example shows that strong completeness is not the same as weak completeness.

Example 4.8 (Jockusch 1989). For $k \geq 2$ let DNR$_k$ be the set of functions $f: \omega \to \{1, \ldots, k\}$ which are DNR. It is easy to see that the sets DNR$_k$, $k = 2, 3, \ldots$, are $\Pi^0_1$ and recursively bounded, and that DNR$_2$ is strongly complete. Jockusch [19] has shown that the the sets DNR$_k$, $k = 2, 3, \ldots$ are weakly complete but of different strong degrees. Thus we have DNR$_2 \equiv_w \text{DNR}_3 \equiv_w \ldots$ yet DNR$_2 \succ_s \text{DNR}_3 \succ_s \ldots$.

An interesting relationship between weak and strong reducibility is given by the following theorem.

Theorem 4.9 (Simpson 2001). Let $P, Q \subseteq 2^\omega$ be nonempty $\Pi^0_1$ sets. If $P \leq_w Q$, then there exists a nonempty $\Pi^0_1$ set $Q' \subseteq Q$ such that $P \leq_s Q'$.

Proof. We shall prove this later, as a consequence of the Almost Recursive Basis Theorem. See Theorem 6.6.
Corollary 4.10. If \( Q \subseteq \mathcal{P}_\omega^0 \) and weakly complete, then there is a \( \Pi^0_1 \) set \( Q' \subseteq Q \) such that \( Q' \) is strongly complete.

Definition 4.11. \( P, Q \subseteq \mathcal{P}_\omega \) are said to be Turing degree isomorphic if there exists a Turing-degree-preserving one-to-one correspondence between \( P \) and \( Q \). Clearly recursive homeomorphism implies Turing degree isomorphism.

Theorem 4.12 (Simpson 2001). Any two weakly complete \( \Pi^0_1 \) subsets of \( \mathcal{P}_\omega \) are Turing degree isomorphic.

Proof. This follows easily from Theorem 4.5 and Corollary 4.10.

Corollary 4.13. A nonempty \( \Pi^0_1 \) subset of \( \mathcal{P}_\omega \) is weakly complete if and only if it is Turing degree isomorphic to the set of completions of \( \text{PA} \).

Corollary 4.14. Any two nonempty \( \Pi^0_1 \) subsets of \( \bigcup_{k=0}^{\infty} \text{DNR}_k \) are Turing degree isomorphic.

Corollary 4.15. If \( P \) is weakly complete, then the set of Turing degrees of members of \( P \) is upward closed.

Proof. Let \( P \) be weakly complete. Put \( Q = P \times \mathcal{P}_\omega \). Clearly \( Q \) is weakly complete, and the set of Turing degrees of members of \( Q \) is upward closed. By Theorem 4.12, \( P \) and \( Q \) are Turing degree isomorphic.

Corollary 4.16 (Solovay). The set of Turing degrees of completions of \( \text{PA} \) is upward closed.

5 1-Randomness

In this section we present an explicit, natural example of a weak degree in \( \mathcal{P}_\omega \) which is strictly between 0 and 1. Our example is based on Martin-Löf’s theory of randomness.

We use the “fair coin” probability measure on \( \mathcal{P}_\omega \). Thus for all \( n \in \omega \) we have

\[
\mu(\{ X \in \mathcal{P}_\omega \mid X(n) = 0 \}) = \mu(\{ X \in \mathcal{P}_\omega \mid X(n) = 1 \}) = 1/2.
\]

Definition 5.1. An effective null \( G_\delta \) is a set \( S \subseteq \mathcal{P}_\omega \) of the form \( S = \bigcap_{n=0}^{\infty} U_n \) where \( U_n, n \in \omega \), is a recursively indexed sequence of \( \Sigma^0_1 \) sets such that \( \mu(U_n) < 1/2^n \) for all \( n \).

Definition 5.2. \( X \in \mathcal{P}_\omega \) is 1-random if \( X \notin S \) for all effective null \( G_\delta \) sets \( S \). The set of 1-random \( X \in \mathcal{P}_\omega \) is denoted \( R_1 \). Clearly \( \mu(R_1) = 1 \).

Theorem 5.3 (Martin-Löf 1966). The union of all effective null \( G_\delta \) sets is an effective null \( G_\delta \) set.

Proof. This result is due to Martin-Löf [26]. The proof is by a diagonal argument. See also Kučera [22].
Corollary 5.4. $2^\omega \setminus R_1$ is an effective null $G_\delta$ set. Hence $R_1$ is $\Sigma^0_0$.  

Corollary 5.5. $R_1 = \bigcup_{n=0}^\infty P_n$ where $P_n$, $n \in \omega$, is a sequence of $\Pi^0_1$ sets.  

Theorem 5.6. Let $Q \subseteq 2^\omega$ be $\Pi^0_1$ of measure 0. Then $Q$ is an effective null $G_\delta$ set.  

Proof. Straightforward. □  

Corollary 5.7. Let $Q \subseteq 2^\omega$ be $\Pi^0_1$. We have $\mu(Q) > 0$ if and only if $Q \cap R_1 \neq \emptyset$. In this case we actually have $Q \cap R_1 \supseteq P \neq \emptyset$, where $P$ is $\Pi^0_1$ and $\mu(P) > 0$.  

Theorem 5.8 (Kučera 1985). Let $Q \subseteq 2^\omega$ be $\Pi^0_1$ with $\mu(Q) > 0$. Then for all 1-random $X \in 2^\omega$ we have that $X^{(k)} \in Q$ for some $k \in \omega$. Here $X^{(k)}(n) = X(k+n)$ for all $n \in \omega$.  

Proof. See Kučera [22]. Let $T$ be a recursive tree such that $Q$ is the set of paths through $T$. Let $\tilde{T}$ be the set of all $\tau \upharpoonright \langle i \rangle \in \text{Seq}_2$ such that $\tau \in T$ and $\tau \upharpoonright \langle i \rangle \notin \tilde{T}$.

Let $Q^2$ be the set of paths through the tree $T^2 = T \cup \{\sigma \upharpoonright \tau \mid \sigma \in \tilde{T}, \tau \in T\}$. Note that $Q^2$ is $\Pi^0_1$ and $\mu(Q^2) = 1 - \mu(Q)^2$. Define $Q^n$ similarly for all $n \geq 1$. Since $Q^n$ is $\Pi^0_1$ and $\mu(Q^n) = 1 - (1 - \mu(Q))^n$, we have that $2^\omega \setminus \bigcup_{n=1}^\infty Q^n$ is an effective null $G_\delta$ set. Hence $X \in Q^n$ for some $n$. It follows that $X^{(k)} \in Q$ for some $k$. □  

Corollary 5.9. Let $Q \subseteq 2^\omega$ be $\Pi^0_1$ with $\mu(Q) > 0$. Then $Q \leq_w R_1$.  

Corollary 5.10. Let $Q$ be a nonempty $\Pi^0_1$ subset of $R_1$. Then $Q \equiv_w R_1$.  

Corollary 5.11. Among all weak degrees of $\Pi^0_1$ sets $Q \subseteq 2^\omega$ with $\mu(Q) > 0$, there is a largest one, and it is the same as the weak degree of $R_1$. Call this weak degree $r_1$.  

Theorem 5.12. Let $A, B \subseteq \omega$ be recursively inseparable. Then

$$
\mu(\{X \in 2^\omega \mid \exists Y \leq_T X (Y \text{ separates } A, B)\}) = 0.
$$

Proof. Not difficult. See Jockusch/Soare [20]. □  

Corollary 5.13. The weak degree $r_1 = \deg_w(R_1) \in \mathcal{P}_w$ of Corollary 5.11 is not weakly complete. We have $0 < r_1 < 1$.  

Remark 5.14. More generally, for all weak degrees $a \in \mathcal{D}_w$, if $\sup(a, r_1) \geq 1$ then $a \geq 1$. This result is due to Simpson [40].  

Remark 5.15. The weak degree $r_1$ is the first explicit, natural example of a weak degree in $\mathcal{P}_w$ strictly between $0$ and $1$. This is especially interesting because no explicit, natural examples of $r$. e. Turing degrees strictly between $0$ and $0'$ are known. See Simpson [38].
6 The Almost Recursive Basis Theorem

**Definition 6.1.** $X$ is almost recursive (a.k.a., hyperimmune-free) if, for each function $f : \omega \to \omega$ recursive in $X$, there exists a recursive function $g : \omega \to \omega$ such that $f(m) < g(m)$ for all $m \in \omega$.

The following theorem is from Jockusch/Soare [20]. We call it the Almost Recursive Basis Theorem.

**Theorem 6.2.** Let $P$ be a nonempty $\Pi_0^1$ subset of $2^\omega$. Then there exists $X \in P$ such that $X$ is almost recursive.

**Proof.** Define a sequence of nonempty $\Pi_0^1$ sets $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n \supseteq \cdots$ as follows. Put $P_0 = P$. If $\exists m (\exists X \in P_n) \{n\}^X(m) \uparrow$, fix such an $m$ and put $P_{n+1} = \{X \in P_n \mid \{n\}^X(m) \uparrow\}$. Otherwise, put $P_{n+1} = P_n$. Clearly there is a unique $X \in \bigcap_{n=0}^\infty P_n$. By Remark 3.23, $X$ is almost recursive. \qed

**Corollary 6.3.** There exists a completion of PA which is almost recursive.

**Corollary 6.4.** There exists a $1$-random $X \in 2^\omega$ which is almost recursive.

**Lemma 6.5.** Suppose $X$ is almost recursive and $X \geq_T Y$. Then $Y$ is truth table reducible to $X$. In particular, there exists a total recursive functional $\Phi : 2^\omega \to 2^\omega$ such that $\Phi(Y) = X$.

**Proof.** Let $e$ be such that $Y = \{e\}^X$. Define $f : \omega \to \omega$ by $f(m) = \text{the least } s \text{ such that } \{e\}^X_s(m) \downarrow$. Clearly $f \leq_T X$. Let $g : \omega \to \omega$ be recursive such that $f(m) \leq g(m)$ for all $n$. Define a truth table functional $\Phi : 2^\omega \to 2^\omega$ by putting $\Phi(Z)(m) = \{e\}^Z_{g(m)}(m)$ if this is defined, and $\Phi(Z)(m) = 0$ otherwise. Clearly $\Phi(X) = Y$. \qed

The following theorem from Simpson [40] provides an interesting relationship between $\leq_w$ and $\leq_s$.

**Theorem 6.6.** Let $P, Q \subseteq 2^\omega$ be nonempty $\Pi_0^1$ sets. If $P \leq_w Q$, then there is a nonempty $\Pi_0^1$ set $Q' \subseteq Q$ such that $P \leq_s Q'$.

**Proof.** Assume $P \leq_w Q$. By Theorem 6.2 let $Y \in Q$ be almost recursive. Let $X \in P$ be such that $X \leq_T Y$. By Lemma 6.5 let $\Phi : 2^\omega \to 2^\omega$ be a truth table functional such that $\Phi(Y) = X$. Put $Q' = Q \cap \Phi^{-1}(P)$. Then $Q'$ is a nonempty $\Pi_0^1$ subset of $Q$, and $P \leq_s Q'$ via $\Phi$. \qed

**Corollary 6.7.** Let $X$ be $1$-random and almost recursive. Then there is no completion of PA which is $\leq_T X$.

**Proof.** Otherwise, by the proof of Theorem 6.6, there would be a strongly complete $\Pi_0^1$ set $Q \subseteq 2^\omega$ with $\mu(Q) > 0$. This would contradict Theorem 5.12. \qed
7 The $\Sigma^0_3 \rightarrow \Pi^0_1$ Embedding Theorem

The next theorem, due to Simpson [40], tells us that the weak degrees of many naturally occurring mass problems belong to $\mathcal{P}_w$, even when they do not naturally occur as recursively bounded $\Pi^0_1$ sets.

**Definition 7.1.** A set $S \subseteq \omega^\omega$ is said to be $\Sigma^0_3$ if there exists a recursive predicate $R \subseteq \omega^\omega \times \omega^3$ such that

$$ S = \{ f \in \omega^\omega \mid \exists n_1 \forall n_2 \exists n_3 \ R(f, n_1, n_2, n_3) \}. $$

A set $P \subseteq \omega^\omega$ is said to be $\Pi^0_1$ if its complement $\omega^\omega \setminus P$ is $\Sigma^0_3$. One defines $\Sigma^0_k$ and $\Pi^0_k$ similarly for all $k \geq 1$. See Rogers [30, Chapter 15].

**Theorem 7.2.** If $S \subseteq \omega^\omega$ is $\Sigma^0_3$, then for all nonempty $\Pi^0_1$ sets $P \subseteq 2^\omega$ we can find a $\Pi^0_1$ set $Q \subseteq 2^\omega$ such that $Q \equiv_w P \cup S$.

**Proof.** First use a Skolem function technique to reduce to the case where $S$ is a $\Pi^0_1$ subset of $\omega^\omega$. Namely, replace $S$ by the set of all $\langle k \rangle \in \omega^\omega$ such that $\forall m (g(m) = \text{the least } n \text{ such that } R(f, k, m, n))$. Clearly this set is $\equiv_w S$ and $\Pi^0_1$. After that, let $T_S$ be a recursive subtree of Seq such that $S$ is the set of paths through $T_S$. Let $T_P$ be a recursive subtree of Seq such that $P$ is the set of paths through $T_P$. We may assume that, for all $\tau \in T_S$ and $n < \ lh(\tau)$, $\tau(n) \geq 2$. Define $T_Q$ to be the set of sequences $\rho \in \text{Seq}$ of the form

$$ \sigma_0 \langle n_0 \rangle \ldots \langle n_k-1 \rangle \sigma_k $$

where $\langle n_0, n_1, \ldots, n_{k-1} \rangle \in T_S$, $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$, and $\rho(m) \leq m + 2$ for all $m < \ lh(\rho)$. Thus $T_Q$ is a recursive subtree of Seq. Let $Q \subseteq \omega^\omega$ be the set of paths through $T_Q$. It is not hard to see that $Q \equiv_w P \cup S$. Note that $Q$ is $\Pi^0_1$ and recursively bounded. Hence by Theorem 3.22 there is a $\Pi^0_1$ set $Q^* \subseteq 2^\omega$ which is recursively homeomorphic to $Q$. \[\Box\]

**Corollary 7.3.** There is a $\Pi^0_1$ set $D \subseteq 2^\omega$ such that $D \equiv_w \text{DNR}$. \[\Box\]

**Proof.** Apply Theorem 7.2 with $P = \text{DNR}_2$ and $S = \text{DNR}$. \[\Box\]

**Remark 7.4.** Put $d = \deg_w(D) = \deg_w(\text{DNR})$. By Kumabe [21] (see also Ambos-Spies/Kjos-Hanssen/Lempp/Slaman [2]) we have

$$ 0 < d < r_1 < 1. $$

The weak degrees $1$, $r_1$, and $d$ correspond to the system $\text{WKL}_0$ and two of its subsystems which have arisen in the foundations of mathematics. See respectively Simpson [39], Yu/Simpson [45], and Giusto/Simpson [18].

**Remark 7.5.** Jockusch [19] has shown that the following mass problems are pairwise Turing degree isomorphic, hence weakly equivalent.

1. DNR = $\{ f \in \omega^\omega \mid f$ is diagonally non-recursive, i.e., $\forall e f(e) \neq \{e\}(e) \}$. 


2. \( \text{FPF} = \{ f \in \omega^\omega \mid f \text{ is fixed point free, i.e., } \forall e \exists m \{ f(e)\}(m) \neq \{e\}(m) \} \).

3. \( \text{EI} = \{ A \subseteq \omega \mid A \text{ is effectively immune} \} \).
   This means that \( A \) is infinite and, given an index of an r. e. set \( C \subseteq A \),
we can effectively find a finite upper bound for the cardinality of \( C \).

4. \( \text{EBI} = \{ A \subseteq \omega \mid A \text{ is effectively bi-immune} \} \).
   This means that both \( A \) and \( \omega \setminus A \) are effectively immune.

**Definition 7.6.** A member of \( 2^\omega \) is said to be 2-\textit{random} if it is 1-random relative
to \( 0' \), the Turing degree of the Halting Problem. The set of 2-random \( X \in 2^\omega \)
is denoted \( R^2_2 \). We write \( r^2_2 = \deg_w(R^2_2) \).

**Corollary 7.7.** There is a \( \Pi^0_1 \) set \( R^2_2 \subseteq 2^\omega \) such that \( R^2_2 \equiv_w R^2_2 \cup P \), where
\( P = \{ \text{completions of } PA \} \). Put \( r^2_1 = \inf(r^2_2, 1) = \deg_w(R^2_2) \).

**Proof.** Relativizing Corollary 5.4 we see that \( R^2_2 \) is a \( \Sigma^0_3 \) subset of \( 2^\omega \). Our result
then follows by Theorem 7.2.

**Theorem 7.8.** If \( X \) is 2-random, then \( X \) is not almost recursive.

**Proof.** Martin [24] has shown that \( \mu(\{ X \in 2^\omega \mid X \text{ is almost recursive} \}) = 0 \).
Our theorem follows from an analysis of Martin’s proof. See also the exposition
of Martin’s result in Dobrinen/Simpson [11].

**Theorem 7.9.** We have \( 0 < d < r_1 < r^*_2 < 1 \).

**Proof.** From Remark 7.4 we have \( 0 < d < r_1 \), and obviously \( r_1 \leq r^*_2 \leq 1 \).
Theorem 5.12 implies that \( r^*_2 < 1 \). The fact that \( r_1 < r^*_2 \) follows from Corollaries
6.4 and 6.7 and Theorem 7.8.

**Remark 7.10.** Additional examples of naturally occurring mass problems whose
weak degrees belong to \( P_w \) are in Simpson [40].

### 8 Embedding the R. E. Turing Degrees

Recall that \( \mathcal{R}_T \) is the upper semilattice of Turing degrees of recursively enumerable
subsets of \( \omega \), and \( \mathcal{P}_w \) (\( \mathcal{P}_s \)) is the lattice of weak (strong) degrees of
nonempty \( \Pi^0_1 \) subsets of \( 2^\omega \). See Remark 1.17 and Definition 3.7.

In this section we use the \( \Sigma^0_3 \rightarrow \Pi^0_1 \) Embedding Theorem 7.2 to embed \( \mathcal{R}_T \)
into \( \mathcal{P}_w \). We do not know whether there exists an embedding of \( \mathcal{R}_T \) into \( \mathcal{P}_s \).

**Theorem 8.1.** Let \( A \in 2^\omega \) be \( \Delta^0_2 \), i.e., \( \deg_T(A) \leq_T 0' \). Then there is a \( \Pi^0_1 \) set
\( P_A \subseteq 2^\omega \) such that \( P_A \equiv_w P \cup \{ A \} \), where \( P = \{ \text{completions of } PA \} \). We have
\( P_{A \oplus B} \equiv_w P_A \times P_B \).

**Proof.** The first statement follows from Theorem 7.2 since the singleton set \( \{ A \} \)
is \( \Pi^0_2 \). The second statement is straightforward.
Theorem 8.2 (Arslanov Completeness Criterion). Let \( A \subseteq \omega \) be recursively enumerable. If \( f \in \text{DNR} \) and \( f \leq_T A \), then \( A \) is Turing complete, i.e., \( \deg_T(A) = 0' \).

Proof. See Soare’s book [42, Section V.5]. Note that we are identifying \( A \subseteq \omega \) with its characteristic function \( \chi_A \in 2^{\omega} \).

Theorem 8.3. Let \( A, B \subseteq \omega \) be recursively enumerable. Then \( A \leq_T B \) if and only if \( \text{PA} \leq_w \text{PB} \).

Proof. Obviously \( A \leq_T B \) implies \( \text{PA} \leq_w \text{PB} \). For the converse, recall that \( P \) is strongly complete, hence recursively homeomorphic to \( \text{DNR}_2 \). In particular, for all \( X \in P \) there is a DNR function \( f \leq_T X \). Assume now that \( \text{PA} \leq_w \text{PB} \). In particular we can find \( X \in P \cup \{A\} \) such that \( X \leq_T B \). If \( X \in P \), then by the Arslanov Completeness Criterion, \( B \) is Turing complete, hence \( A \leq_T B \). If \( X \notin P \), then \( X = A \), hence again \( A \leq_T B \).

Remark 8.4. Thus our embedding of the r. e. Turing degrees into the weak lattice \( \text{PW} \) is given by \( \deg_T(A) \mapsto \deg_w(\text{PA} \cup \{A\}) \), where \( \text{P} = \{\text{completions of PA}\} \). The embedding is one-to-one, order preserving, least upper bound preserving, and carries \( 0 \) to \( 0 \) and \( 0' \) to \( 1 \).

Remark 8.5. Instead of \( P = \{\text{completions of PA}\} \), we could use any nonempty \( \Pi_0^1 \) set \( P \subseteq 2^\omega \) such that \( \text{DNR} \leq_w P \). Compare Corollary 7.3. Thus, for any \( c \in \text{Pw} \) such that \( c \geq d = \deg_w(\text{DNR}) \), we obtain an embedding of the r. e. Turing degrees into \( \{a \in \text{Pw} \mid 0 \leq a \leq c\} \). The embedding is one-to-one, order preserving, least upper bound preserving, and carries \( 0 \) to \( 0 \) and \( 0' \) to \( c \).

9 A Priority Argument

In this section we sketch the construction of a \( \Pi_1^0 \) set \( P \subseteq 2^\omega \) with several interesting properties. The construction uses a priority argument.

Definition 9.1. A \( \Pi_1^0 \) set \( P \subseteq 2^\omega \) is said to be thin if, for all \( \Pi_1^0 \) sets \( Q \subseteq P \), there is a finite set \( \sigma_1, \ldots, \sigma_n \in \text{Seq}_2 \) such that

\[ Q = \{X \in P \mid \sigma_1 \subset X \lor \cdots \lor \sigma_n \subset X\}. \]

This is equivalent to saying that, for all \( \Pi_1^0 \) sets \( Q \subseteq P \), \( P \setminus Q \) is \( \Pi_1^0 \). See also references [25, 12, 13, 9].

Definition 9.2. A family of Turing degrees \( \{a_i \mid i \in I\} \) is said to be independent if for all finite \( \{i_0, i_1, \ldots, i_n\} \subseteq I \), \( a_{i_0} \leq \text{sup}(a_{i_1}, \ldots, a_{i_n}) \) implies \( i_0 \in \{i_1, \ldots, i_n\} \).

Theorem 9.3. We can construct a \( \Pi_1^0 \) set \( P \subseteq 2^\omega \) with the following properties:

1. \( P \) is thin and of cardinality \( 2^{\aleph_0} \).

2. \( P \) has no recursive members.
3. The Turing degrees $\text{deg}_T(X), X \in P$, are independent.

4. For all $X \in P$, putting $a = \text{deg}_T(X)$, we have $a' = \sup(a, 0')$.

Here $a'$ denotes the Turing jump of $a$.

Furthermore, given a $\Pi^0_1$ set $Q \subseteq 2^\omega$ with no recursive members, we can arrange that no member of $P$ Turing computes a member of $Q$.

**Sketch of proof.** We follow Binns/Simpson [6] building on the techniques of Martin/Pour-El [25] and Jockusch/Soare [20, Theorem 4.7].

By a treemap we mean a function $h : \text{Seq}_2 \to \text{Seq}_2$ such that $h(\sigma) \cup \{i\} \subseteq h(\sigma \cup \{i\})$ for all $\sigma \in \text{Seq}_2$, $i \in \{0, 1\}$.

Starting with $h_0 =$ the identity map, we construct a recursive sequence of recursive treemaps $h_s$, $s \in \omega$, which are nested in the sense that for all $s$ and all $\sigma \in \text{Seq}_2$ there exists $\tau \in \text{Seq}_2$ such that $h_{s+1}(\sigma) = h_s(\tau)$. After presenting the recursive construction, we argue that, for all $\sigma \in \text{Seq}_2$, the limit $h(\sigma) = \lim_s h_s(\sigma)$ exists and is finite. If follows that $h = \lim_s h_s$ is a treemap, and we define

$$P = \{X \in 2^\omega \mid \forall n (\exists \sigma \text{ of length } n) h(\sigma) \subset X\}.$$ 

Clearly $P$ will be $\Pi^0_1$ and of cardinality $2^{\aleph_0}$.

In order to insure that $P$ is thin, we arrange that for all $e \in \omega$ and all $\sigma \in \text{Seq}_2$ of length $e$, $\{e\}^{h(\sigma)}(0)$ ↓ “if possible”. Then for all $X \in P$ we have

$$H^X = \{e \mid (\exists \sigma \text{ of length } e) (\{e\}^{h(\sigma)}(0) \downarrow \text{ and } h(\sigma) \subset X)\},$$

so $H^X \leq_T H \oplus X$ and this gives property 4. Now, given a $\Pi^0_1$ set $Q \subseteq 2^\omega$, let $e$ be such that $Q = \{X \mid \{e\}^X(0) \uparrow\}$. (See the proof of Theorem 3.14.) Then

$$Q = \{X \in P \mid (\exists \sigma \text{ of length } e) (\{e\}^{h(\sigma)}(0) \uparrow \text{ and } h(\sigma) \subset X)\},$$

and this gives thinness.

The strategy for property 3 is similar. For example, to insure $X \not\leq_T Y$ for all $X, Y \in P$ with $X \neq Y$, we arrange that for all $e$ and all $\sigma, \tau \in \text{Seq}_2$ of length $e$ with $\sigma \neq \tau$, $\exists m < \text{lh}(h(\sigma)) (\{e\}^{h(\tau)}(m) \downarrow \neq h(\sigma)(m))$ “if possible”.

The final property is obtained by means of a Sacks preservation strategy. See Binns/Simpson [6].

**Corollary 9.4.** Every finite distributive lattice is lattice embeddable in $\mathcal{P}_w$ and in $\mathcal{P}_s$.

**Proof.** First note that any finite distributive lattice is lattice embeddable in the free distributive lattice on $n$ generators, for sufficiently large $n$. Now let $P$ be as in Theorem 9.3, and let $P_1, \ldots, P_n$ be nonempty, pairwise disjoint, $\Pi^0_1$ subsets of $P$. In view of property 3, the weak or strong degrees of $P_1, \ldots, P_n$ are independent and hence freely generate a free distributive lattice. Details are in Binns/Simpson [6]. □
Corollary 9.5. For any $b > 0$ in $\mathcal{P}_w$ or $\mathcal{P}_s$, every finite distributive lattice is lattice embeddable in the interval $0 \leq a \leq b$.

Proof. Let $Q \subseteq 2^\omega$ be $\Pi_1^0$ with $b = \deg_w(Q)$ or $\deg_s(Q)$ as the case may be. Let $P$ be as in Theorem 9.3 such that no member of $P$ Turing computes a member of $Q$. Proceed as in the proof of Corollary 9.4, replacing $P_1, \ldots, P_n$ by $P_1 + Q, \ldots, P_n + Q$.

Remark 9.6. By Theorem 9.3, let $P$ be a nonempty thin $\Pi_1^0$ subset of $2^\omega$ with no recursive members. Then $P$ is of measure 0, and in fact, $\deg_w(P)$ is incomparable with $r_1$. These results are due to Simpson [40].

Remark 9.7. By Theorem 9.3 and Remark 3.19, let $T$ be a consistent, finitely axiomatizable, essentially undecidable theory such that $P_T$ is thin. Then any recursively axiomatizable theory extending $T$ with the same vocabulary as $T$ is finitely axiomatizable. Compare Martin/Pour-El [25].

References


(22) Antonín Kučera. Measure, $\Pi_1^0$ classes and complete extensions of PA. In [14], pages 245–259, 1985.


[38] Stephen G. Simpson. FOM: natural r.e. degrees; Pi01 classes. FOM e-mail list [17], August 13, 1999.


