Reverse mathematics and Hilbert's program

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Potential infinity versus actual infinity.

An important philosophical distinction:

Definition. A <u>potential infinity</u> is a quantity which is finite but indefinitely large. For instance, when we enumerate the natural numbers as $0, 1, 2, \ldots, n, n+1, \ldots$, the enumeration is finite at any point in time, but it grows indefinitely and without bound. Another example is the enumeration of all finite sequences of 0's and 1's.

Definition. An <u>actual infinity</u> is a completed infinite totality.
Examples: $\mathbb{N}$, $\mathbb{R}$, $C[0,1]$, $L_2[0,1]$, etc.
Other examples: gods, devils, etc.

Reference:

Aristotle, Metaphysics, Books M and N.

Four philosophical positions.

Ultrafinitism: Infinities, both potential and actual, do not exist and are not acceptable in mathematics.

Finitism: Potential infinities exist and are acceptable in mathematics. Actual infinities do not exist and we must limit or eliminate their role in mathematics.

Predicativism: We may accept the natural numbers but not the real numbers as a completed infinite totality. Quantification over $\mathbb{N}$ is acceptable, but quantification over $\mathbb{R}$ or the powerset of $\mathbb{N}$ is unacceptable.

Infinitism: Actual infinities of all kinds are welcome in mathematics, so long as they are consistent and intuitively natural.

Of these four positions, the finitist one seems to be the most objective.

Definition. By the real world we mean the real world around us and out there, as grasped by our minds. Objectivity is a relationship between our minds and the real world, wherein we grasp reality.

The real world contains many potential infinities. Examples are: counting, the rotation of the earth about the sun, the human reproduction cycle, division of a piece of metal into smaller pieces, the accumulation of wealth, etc.

However, the real world does not appear to contain any actual infinities. For this reason, actual infinities are suspect.

In order to maintain objectivity in mathematics, it seems necessary to limit the use of actual infinities. We introduce them only as "convenient fictions," and use them only in a way which leads to conclusions that are objectively justifiable.

Theories corresponding to these positions:

Ultrafinitism: ???

Finitism:

PRA = Primitive Recursive Arithmetic. It includes the scheme of primitive recursion: $f(-,0) = g(-)$, $f(-,n+1) = h(-,n,f(-,n))$. For instance, each of the functions $A_k(n)$ for $k = 0, 1, 2, \ldots$ defined by $A_0(n) = 2n$, $A_{k+1}(n) = \underbrace{A_k A_k \cdots A_k}_{n \text{ times}}(1)$, is primitive recursive. For instance, $A_1(n) = 2^n$, and $A_2(n) = 2_n =$ a stack of 2's of height n. However, the Ackermann function $A(n) = A_n(n)$ is not primitive recursive.

Predicativism: Feferman's system IR.

Solomon Feferman, Systems of predicative analysis I, II, Journal of Symbolic Logic, 29, 1964, 1–30, and 33, 1968, 193–220.

Infinitism: ZFC+ large cardinals ?

Hilbert's program: finitistic reductionism.

Let S be a subsystem of Z_2.

Definition.
S is finitistically reducible if all Π^0_2 sentences which are provable in S are provable in PRA.

Finitistic reducibility means: if we use S to prove a finitistically meaningful sentence, then that same sentence is provable finitistically. Thus, the non-finitistic part of S can be "eliminated" from the proof.
In other words, S is a "convenient fiction."

Also, if S proves a Π^0_2 sentence $\forall m\,\exists n\,\Phi(m,n)$, then PRA proves $\forall m\,\Phi(m,f(m))$ for some primitive recursive function f.

Theorem. The following systems are finitistically reducible.

1. RCA_0.
2. WKL_0.
3. $\mathsf{WKL}_0 + \Sigma^0_2$-bounding.
4. WKL_0^+, which includes a useful version of the Baire category theorem.

References.

David Hilbert, Über das Unendliche, Mathematische Annalen, 95, 1926, 161–190.

William W. Tait, Finitism, Journal of Philosophy, 78, 1981, 524–546.

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Douglas K. Brown and Stephen G. Simpson, The Baire category theorem in weak subsystems of second order arithmetic, Journal of Symbolic Logic, 58, 1993, 557–578.

References (continued).

Stephen G. Simpson, Subsystems of Second Order Arithmetic, Springer-Verlag, 1999, XIV + 445 pages; 2nd edition, Association for Symbolic Logic, 2009, XVI + 444 pages.
See especially Chapter IX.

Richard Zach, Hilbert's Program,
Stanford Encyclopedia of Philosophy, 2003,
http://plato.stanford.edu/entries/hilbert-program/.

Stephen G. Simpson, Toward objectivity in mathematics, in: Infinity and Truth, edited by C.-T. Chong, Q. Feng, T. A. Slaman, and W. H. Woodin, World Scientific, 2014, 157–169.

Stephen G. Simpson, An objective justification for actual infinity?, same volume, 2014, 225–228.

Some systems which are <u>*not*</u> finitistically reducible:

1. $I\Sigma_2$
2. $\mathsf{RCA}_0 + \Sigma^0_2$ induction.
3. $\mathsf{RCA}_0 + \mathsf{WO}(\omega^\omega)$.
4. ACA_0 and stronger systems.

Each of these systems proves that the Ackermann function is total, i.e., $\forall n\, \exists j\, (A(n) = j)$. They also prove the consistency of PRA, which by Gödel is not provable in PRA. These are finitistically meaningful sentences whose proofs are not finitistically reducible.

Remark. Let $\mathsf{RT}(2,2)$ = Ramsey's Theorem for exponent 2. It is <u>*unknown*</u> whether $\mathsf{RCA}_0 + \mathsf{RT}(2,2)$ is finitistically reducible.

Predicative reductionism.

Definition.
S is predicatively reducible if all Π^1_1 sentences which are provable in S are provable in IR.

Predicative reducibility means: if we use S to prove a predicatively meaningful sentence, then that same sentence is provable predicatively. Thus, the non-predicative part of S can be "eliminated" from the proof. In other words, S is a "convenient fiction."

Theorem. ATR_0 is predicatively reducible.

However, Π^1_1-CA_0 and stronger systems are *not* predicatively reducible. For instance, they prove the consistency of IR, which is certainly not provable in IR.

References:

Harvey Friedman, Kenneth McAloon, and Stephen G. Simpson, A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis, in: Patras Logic Symposion, edited by G. Metakides, North-Holland, Amsterdam, 1982, pp. 197–220.

Stephen G. Simpson, Predicativity: the outer limits, in Reflections on the Foundations of Mathematics: Essays in Honor of Solomon Feferman, edited by W. Sieg, R. Sommer, and C. Talcott, Lecture Notes in Logic, Volume 15, Association for Symbolic Logic, 2001, pp. 134–140.

Proof-theoretic ordinals.

Definition. The proof-theoretic ordinal of S is $|S|$ = the supremum of all ordinals α such that S proves $\mathsf{WO}(\alpha)$. Here $\mathsf{WO}(\alpha)$ means that α is well ordered.

The proof-theoretic ordinals of the Big Five:

$|\mathsf{RCA}_0| = |\mathsf{WKL}_0| = \omega^\omega$.

$|\mathsf{ACA}_0| = \varepsilon_0$.

$|\mathsf{ATR}_0| = \Gamma_0$.

$|\Pi^1_1\text{-}\mathsf{CA}_0| = \Psi_0(\Omega_\omega)$.

In every case we have $S \vdash \mathsf{WO}(\alpha)$ for all $\alpha < |S|$, and $S \nvdash \mathsf{WO}(|S|)$. For instance, $\mathsf{ACA}_0 \nvdash \mathsf{WO}(\varepsilon_0)$ and $\mathsf{RCA}_0 \nvdash \mathsf{WO}(\omega^\omega)$.

Reference:
Subsystems of Second Order Arithmetic, Chapter IX, Section 5.