Mass Problems and Randomness

Stephen G. Simpson
Department of Mathematics
Pennsylvania State University
simpson@math.psu.edu
http://www.math.psu.edu/simpson/

Draft: October 27, 2004

This research was partially supported by NSF grant DMS-0070718. I would like to thank Stephen Binns for discussing these topics with me.

AMS 2000 Subject Classifications: 03D30, 03D80, 03D25, 03F35, 03F15, 68Q30, 68Q15.

Bulletin of Symbolic Logic, **11**, 2005, pp. 1–27.

Abstract

A *mass problem* is a set of Turing oracles. If P and Q are mass problems, we say that P is *weakly reducible* to Q if every member of Q Turing computes a member of P. We say that P is *strongly reducible* to Q if every member of Q Turing computes a member of P via a fixed Turing functional. The *weak degrees* and *strong degrees* are the equivalence classes of mass problems under weak and strong reducibility, respectively. We focus on the countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ of weak and strong degrees of mass problems given by nonempty Π^0_1 subsets of 2^ω. Using an abstract Gödel/Rosser incompleteness property, we characterize the Π^0_1 subsets of 2^ω whose associated mass problems are of top degree in $\mathcal{P}_w$ and $\mathcal{P}_s$, respectively. Let R be the set of Turing oracles which are *random* in the sense of Martin-Löf, and let $\mathbf{r}$ be the weak degree of R. We show that $\mathbf{r}$ is a natural intermediate degree within $\mathcal{P}_w$. Namely, we characterize $\mathbf{r}$ as the unique largest weak degree of a Π^0_1 subset of 2^ω of positive measure. Within $\mathcal{P}_w$ we show that $\mathbf{r}$ is meet irreducible, does not join to $\mathbf{1}$, and is incomparable with all weak degrees of nonempty thin perfect Π^0_1 subsets of 2^ω. In addition, we present other natural examples of intermediate degrees in $\mathcal{P}_w$. We relate these examples to reverse mathematics, computational complexity, and Gentzen-style proof theory.

Contents

Abstract **1**

Contents **2**

1 Introduction **2**

2 Recursion-theoretic preliminaries **3**

3 Mass problems **4**

4 Recursively bounded Π^0_1 sets **6**

5 The lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ **9**

6 Weak and strong completeness **11**

7 Π^0_1 sets of positive measure **13**

8 Π^0_1 sets of random reals **15**

9 Thin Π^0_1 subsets of 2^ω **18**

10 Some additional natural examples in $\mathcal{P}_w$ **22**

References **25**

1 Introduction

Among the principal objects of study in recursion theory going back to the seminal work of Turing [59] and Post [44] have been the upper semilattice $\mathcal{D}_T$ of all *Turing degrees*, i.e., degrees of unsolvability, and its countable sub-semilattice $\mathcal{R}_T$ consisting of the *recursively enumerable Turing degrees*, i.e., the Turing degrees of recursively enumerable sets of positive integers. See for instance Sacks [46], Rogers [45], Lerman [36], Soare [56], Odifreddi [42, 43].

A major difficulty or obstacle in the study of $\mathcal{R}_T$ has been the lack of natural examples. Although it has long been known that $\mathcal{R}_T$ is infinite and structurally rich, to this day no specific, natural examples of recursively enumerable Turing degrees are known, beyond the two original examples noted by Turing: $\mathbf{0}'$ = the Turing degree of the Halting Problem, and $\mathbf{0}$ = the Turing degree of solvable problems. Furthermore, $\mathbf{0}'$ and $\mathbf{0}$ are respectively the top and bottom elements of $\mathcal{R}_T$. This lack of natural examples, although well known and a major source of frustration, has almost never been discussed in print, but see Rogers [45, Section 9.6]. In any case, the paucity of examples in $\mathcal{R}_T$ is striking, because it is well known that most other branches of mathematics are motivated and nurtured by a rich stock of natural examples. Clearly it ought to be of interest to somehow overcome this deficiency in the study of $\mathcal{R}_T$.

In recent years it has emerged that there are some natural, important, well-behaved degree structures, closely related to but different from $\mathcal{R}_T$, which do not suffer from the above mentioned deficiency. Simpson 1999 [50, 51, 54] called attention to the countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ of weak and strong degrees of mass problems given by nonempty Π^0_1 subsets of 2^ω, and noted the existence of specific, natural degrees which are intermediate between the top and bottom elements of $\mathcal{P}_w$ and $\mathcal{P}_s$. One of the natural intermediate degrees noted by Simpson was the weak degree $\mathbf{r}$ of the set of Turing oracles which are random in the sense of Martin-Löf [39]. The study of $\mathcal{P}_w$ and $\mathcal{P}_s$ has been continued by Simpson [53, 49], Cenzer/Hinman [9], Simpson/Slaman [55], Binns [3, 4, 5], Binns/Simpson [6], Terwijn [58].

The purpose of the present paper is to elucidate additional properties of previously noted natural degrees in $\mathcal{P}_w$ and $\mathcal{P}_s$, and to present some additional natural degrees in $\mathcal{P}_w$. Along the way we give a somewhat leisurely introduction to mass problems in general, and to $\mathcal{P}_w$ and $\mathcal{P}_s$ in particular, and we review other known results concerning $\mathcal{P}_w$ and $\mathcal{P}_s$.

In a later paper [48] we shall exhibit a natural embedding of the countable upper semilattice $\mathcal{R}_T$ into the countable distributive lattice $\mathcal{P}_w$. This embedding will be one-to-one and will preserve the top and bottom elements as well as the partial order relation and least upper bound operation from $\mathcal{R}_T$. In this way we shall see that $\mathcal{P}_w$ provides a satisfactory solution to several of the well known difficulties concerning $\mathcal{R}_T$.

2 Recursion-theoretic preliminaries

In this section we establish notation concerning recursive functionals and Turing degrees.

Throughout this paper we use standard recursion-theoretic notation and concepts from Rogers [45] and Soare [56]. We write $\omega = \{0, 1, 2, \ldots\}$ for the set of natural numbers. We write ω^ω for the space of total functions from ω into ω. We write 2^ω for the subspace of ω^ω consisting of the total functions from ω into $\{0, 1\}$. We sometimes identify a set $A \subseteq \omega$ with its characteristic function $\chi_A \in 2^\omega$ given by $\chi_A(n) = 1$ if $n \in A$, 0 if $n \notin A$. For $e, n, m \in \omega$ and $f \in \omega^\omega$ we write $\{e\}^f(n) = m$ to mean that the Turing machine with Gödel number e and oracle f and input n eventually halts with output m. Furthermore, $\{e\}^f(n)\downarrow$ means that $\{e\}^f(n)$ *is defined*, i.e., $\exists m\,(\{e\}^f(n) = m)$, and $\{e\}^f(n)\uparrow$ means that $\{e\}^f(n)$ *is undefined*, i.e., $\neg\exists m\,(\{e\}^f(n) = m)$. In the absence of an oracle f, we write simply $\{e\}(n) = m$, etc. For $P \subseteq \omega^\omega$ we consider *recursive functionals* $\Phi : P \to \omega^\omega$ given by $\Phi(f)(n) = \{e\}^f(n)$ for some $e \in \omega$ and all $f \in P$ and $n \in \omega$. In particular, a function $h : \omega \to \omega$ is said to be *recursive* or *computable* if there exists $e \in \omega$ such that $h(n) = \{e\}(n)$ for all $n \in \omega$. (The terms "recursive" and "computable" are synonymous.) A set $A \subseteq \omega$ is said to be *recursively enumerable* if it is the image of a recursive function, i.e., $A = \{m \mid \exists n\,(h(n) = m)\}$ for some recursive $h : \omega \to \omega$.

For $f, g \in \omega^\omega$ we write $f \leq_T g$ to mean that f is *Turing reducible* to g, i.e., $\exists e\,\forall n\,(f(n) = \{e\}^g(n))$. The *Turing degree* of f, denoted $\deg_T(f)$, is the set of all g such that $f \equiv_T g$, i.e., $f \leq_T g$ and $g \leq_T f$. The set $\mathcal{D}_T$ of all Turing degrees is partially ordered by putting $\deg_T(f) \leq \deg_T(g)$ if and only if $f \leq_T g$. Under this partial ordering, the bottom element of $\mathcal{D}_T$ is $\mathbf{0} = \{f \in \omega^\omega \mid f \text{ is recursive}\}$. It is known that $\mathcal{D}_T$ has no top element. Within $\mathcal{D}_T$, the least upper bound of $\deg_T(f)$ and $\deg_T(g)$ is given as $\deg_T(f \oplus g)$ where $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$ for all $n \in \omega$. The standard, natural example of a Turing degree $> \deg_T(f)$ is given by the *Turing jump operator*, $\deg_T(f) \mapsto \deg_T(f)' = \deg_T(f')$, where f' is (the characteristic function of) the *Halting Problem* relative to f, $H^f = \{e \mid \{e\}^f(0)\downarrow\}$. A Turing degree is said to be *recursively enumerable* if it is $\deg_T(f)$ where $f = \chi_A$ is the characteristic function of a recursively enumerable set $A \subseteq \omega$. The set of all recursively enumerable Turing degrees is denoted $\mathcal{R}_T$. Clearly $\mathcal{R}_T$ is countable, because there are only countably many recursively enumerable sets. It is known that $\mathcal{R}_T$ is closed under the least upper bound operation inherited from $\mathcal{D}_T$, and that $\mathbf{0}'$ and $\mathbf{0}$ are the top and bottom elements of $\mathcal{R}_T$. Thus $\mathcal{R}_T$ is a countable upper semilattice with a top and bottom element.

3 Mass problems

A *mass problem* is a subset of ω^ω. The underlying idea here is to view a set $P \subseteq \omega^\omega$ as a "problem" with a "solution" that does not necessarily exist and is not necessarily unique. The "solutions" of P are simply the members of P. In the special case when P is a singleton set, the "solution" exists and is unique, and the mass problem corresponds to a Turing degree.

In accordance with the conceptual scheme which was explained in the previous paragraph, one makes the following definitions.

Definition 3.1. Let P and Q be subsets of ω^ω. We say that P is *weakly reducible* to Q, written $P \leq_w Q$, if for all $g \in Q$ there exists $f \in P$ such that $f \leq_T g$. Conceptually this means that, given any "solution" of the mass problem Q, we can use it as an oracle to compute a "solution" of the mass problem P. The *weak degree* of P, written $\deg_w(P)$, is the set of all Q such that $P \equiv_w Q$, i.e., $P \leq_w Q$ and $Q \leq_w P$. The set $\mathcal{D}_w$ of all weak degrees is partially ordered by putting $\deg_w(P) \leq \deg_w(Q)$ if and only if $P \leq_w Q$.

Remark 3.2. The concept of weak reducibility goes back to Muchnik [41] and has sometimes been called *Muchnik reducibility*.

Definition 3.3. We say that P is *strongly reducible* to Q, written $P \leq_s Q$, if there exists $e \in \omega$ such that for all $g \in Q$ there exists $f \in P$ such that $f(n) = \{e\}^g(n)$ for all $n \in \omega$. In other words, $P \leq_s Q$ if and only if there exists a recursive functional $\Phi : Q \to P$. Note that strong reducibility is the uniform variant of weak reducibility. Just as for weak degrees, the *strong degree* of P, written $\deg_s(P)$, is the set of all Q such that $P \equiv_s Q$, i.e., $P \leq_s Q$ and $Q \leq_s P$. The set $\mathcal{D}_s$ of all strong degrees is partially ordered by putting $\deg_s(P) \leq \deg_s(Q)$ if and only if $P \leq_s Q$.

Remark 3.4. The concept of strong reducibility goes back to Medvedev [40] and has sometimes been called *Medvedev reducibility*.

Remark 3.5. Given $P, Q \subseteq \omega^\omega$, a *recursive homeomorphism* of P onto Q is a recursive functional $\Phi : P \to Q$ mapping P one-to-one onto Q such that the inverse functional $\Phi^{-1} : Q \to P$ is also recursive. In this case we say that P and Q are *recursively homeomorphic*. In addition, let us say that P is *Turing degree isomorphic* to Q if $\{\deg_T(f) \mid f \in P\} = \{\deg_T(g) \mid g \in Q\}$. Clearly recursive homeomorphism of P and Q implies strong equivalence and Turing degree isomorphism, either of which implies weak equivalence. No other implications hold.

Theorem 3.6. *$\mathcal{D}_w$ and $\mathcal{D}_s$ are distributive lattices. They have a bottom element, denoted* $\mathbf{0}$*, and a top element, denoted* ∞*.*

Proof. The least upper bound of $\deg_w(P)$ and $\deg_w(Q)$ in $\mathcal{D}_w$ or in $\mathcal{D}_s$ is given as $\deg_w(P \times Q)$ where

$$P \times Q = \{f \oplus g \mid f \in P \text{ and } g \in Q\}.$$

The greatest lower bound of $\deg_w(P)$ and $\deg_w(Q)$ in $\mathcal{D}_w$ is $\deg_w(P \cup Q)$, or $\deg_w(P + Q)$ where

$$P + Q = \{\langle 0\rangle^\frown f \mid f \in P\} \cup \{\langle 1\rangle^\frown g \mid g \in Q\}.$$

The greatest lower bound of $\deg_s(P)$ and $\deg_s(Q)$ in $\mathcal{D}_s$ is $\deg_s(P + Q)$. It is straightforward to check distributivity. The bottom element of $\mathcal{D}_w$ and $\mathcal{D}_s$ is

$$\mathbf{0} = \deg_w(\omega^\omega) = \deg_s(\omega^\omega) = \{P \subseteq \omega^\omega \mid \exists f\,(f \in P \text{ and } f \text{ is recursive})\}.$$

The top element of $\mathcal{D}_w$ and $\mathcal{D}_s$ is $\infty = \{\emptyset\}$, where $\emptyset$ denotes the empty set. □

Remark 3.7. There are obvious, natural embeddings of $\mathcal{D}_T$ into $\mathcal{D}_w$ and $\mathcal{D}_s$ given by $\deg_T(f) \mapsto \deg_w(\{f\})$ and $\deg_T(f) \mapsto \deg_s(\{f\})$ respectively. Here $\{f\}$ is the singleton set whose only member is $f \in \omega^\omega$. These embeddings are one-to-one and preserve $\mathbf{0}$ and the partial order relation and least upper bound operation from $\mathcal{D}_T$.

Remark 3.8. There is an obvious lattice homomorphism of $\mathcal{D}_s$ onto $\mathcal{D}_w$ given by $\deg_s(P) \mapsto \deg_w(P)$.

Remark 3.9. $\mathcal{D}_w$ is canonically isomorphic to the lattice of upward closed subsets of $\mathcal{D}_T$ under the set-theoretic operations of intersection and union. Namely, for each $P \subseteq \omega^\omega$, the weak degree $\deg_w(P) \in \mathcal{D}_w$ gets mapped to the upward closure of $\{\deg_T(f) \mid f \in P\}$ within $\mathcal{D}_T$. It follows that $\mathcal{D}_w$ is a complete distributive lattice. We do not know of an analogous set-theoretic representation of $\mathcal{D}_s$.

Remark 3.10. For a survey of general mass problems, see Sorbi [57].

4 Recursively bounded Π^0_1 sets

In this section we present some well known generalities concerning recursively bounded Π^0_1 sets and almost recursive functions. The reader is advised to skip most of this section now, and refer to it later as needed.

Definition 4.1. A predicate $R \subseteq \omega^\omega \times \omega$ is said to be *recursive* if

$$\exists e\, \forall f\, \forall n\, (\{e\}^f(n) = 1 \text{ if } R(f,n), \text{ and } \{e\}^f(n) = 0 \text{ if } \neg\, R(f,n)).$$

A set $P \subseteq \omega^\omega$ is said to be Π^0_1 if there exists a recursive predicate $R \subseteq \omega^\omega \times \omega$ such that $P = \{f \mid \forall n\, R(f,n)\}$.

Definition 4.2. A finite sequence of natural numbers $\sigma = \langle \sigma(0), \ldots, \sigma(k-1) \rangle$ is called a *string* of length k. We write $\mathrm{lh}(\sigma) = k$. The set of all strings is denoted $\omega^{<\omega}$. If σ, τ are strings of length k, l respectively, then the *concatenation*

$$\sigma^\frown\tau = \langle \sigma(0), \ldots, \sigma(k-1), \tau(0), \ldots, \tau(l-1) \rangle$$

is a string of length $k + l$. Note that $\sigma \subseteq \tau$ if and only if $\sigma^\frown\rho = \tau$ for some ρ, and this implies $\mathrm{lh}(\sigma) \leq \mathrm{lh}(\tau)$. If σ is a string of length k, then for all $f \in \omega^\omega$ we have $\sigma^\frown f \in \omega^\omega$ defined by $(\sigma^\frown f)(i) = \sigma(i)$ for $i < k$, $f(i-k)$ for $i \geq k$. Note that $\sigma \subset f$ if and only if $\sigma^\frown g = f$ for some $g \in \omega^\omega$. A *tree* is a set $T \subseteq \omega^{<\omega}$ such that, for all $\sigma \subseteq \tau \in T$, $\sigma \in T$. A *path* through T is an $f \in \omega^\omega$ such that $(\forall \sigma \subset f)\,(\sigma \in T)$. The set of all paths through T is denoted $[T]$. We sometimes identify a string σ with its Gödel number $\#(\sigma) \in \omega$. A tree T is said to be *recursive* if $\{\#(\sigma) \mid \sigma \in T\}$ is recursive, and Π^0_1 if $\{\#(\sigma) \mid \sigma \in T\}$ is Π^0_1.

Theorem 4.3. *$P \subseteq \omega^\omega$ is Π^0_1 if and only if $P = [T]$ for some recursive tree T.*

Proof. If T is a recursive tree, we have $[T] = \{f \mid \forall n\, R(f,n)\}$ where $R(f,n)$ is the recursive predicate asserting that $\langle f(0), \ldots, f(n-1) \rangle \in T$. Thus $[T]$ is Π^0_1. For the converse, assume that P is Π^0_1 with index e, i.e., via the Turing machine with Gödel number e. Thus for all $f \in \omega^\omega$ we have $f \in P$ if and only if $\forall n\, (\{e\}^f(n) = 1)$, if and only if $\forall n\, (\{e\}^f(n) \neq 0)$. Let us write $\{e\}^\sigma(n) = m$ to mean that $\{e\}^f(n) = m$ via a Turing machine computation using only oracle information from $\sigma \subset f$ and halting in $\leq \mathrm{lh}(\sigma)$ steps. Note that the 4-place relation $\{e\}^\sigma(n) = m$ and the 3-place relation $\{e\}^\sigma(n)\!\downarrow$ are primitive recursive. We have $f \in P$ if and only if $\forall n\, (\forall \sigma \subset f)\, (\{e\}^\sigma(n) \neq 0)$, if and only if $(\forall \sigma \subset f)\, (\forall n \leq \mathrm{lh}(\sigma))\, (\{e\}^\sigma(n) \neq 0)$. Thus $P = [T]$ where T is the primitive recursive tree consisting of all strings σ such that $(\forall n \leq \mathrm{lh}(\sigma))\, (\{e\}^\sigma(n) \neq 0)$. □

Theorem 4.4. *If $P, Q \subseteq \omega^\omega$ are Π^0_1 and $\Phi : P \to \omega^\omega$ is a recursive functional, then the preimage $\{f \in P \mid \Phi(f) \in Q\}$ is Π^0_1.*

Proof. By Theorem 4.3 let T be a recursive tree such that $Q = [T]$. Let e be an *index* of Φ, i.e., $\Phi(f)(n) = \{e\}^f(n)$ for all $f \in P$ and all n. Given $f \in P$, we have $\Phi(f) \in Q$ if and only if for all $\sigma \subset f$ and all $\tau \notin T$ there exists $n < \mathrm{lh}(\tau)$ such that $\{e\}^\sigma(n) \neq \tau(n)$. It follows that $\{f \in P \mid \Phi(f) \in Q\}$ is Π^0_1. □

Definition 4.5. A set $P \subseteq \omega^\omega$ is said to be *recursively bounded* if there exists a recursive function $h \in \omega^\omega$ such that $f(n) < h(n)$ for all $f \in P$ and $n \in \omega$.

Remark 4.6. Any subset of a recursively bounded set is recursively bounded. We shall be concerned with subsets of ω^ω which are recursively bounded and Π^0_1. In particular, 2^ω is recursively bounded and Π^0_1, and we are especially interested in Π^0_1 subsets of 2^ω. As in Theorem 4.3, P is a Π^0_1 subset of 2^ω if and only if $P = [T]$ for some recursive tree $T \subseteq 2^{<\omega}$. Here $2^{<\omega}$ denotes the set of all strings of 0's and 1's. See also Theorem 4.10 and Corollary 4.11 below.

Theorem 4.7. *Assume that $P \subseteq \omega^\omega$ is recursively bounded Π^0_1, and assume that $\Phi : P \to \omega^\omega$ is a recursive functional. Then the image $\{\Phi(f) \mid f \in P\}$ is recursively bounded Π^0_1. Moreover, there exists a total recursive functional $\Phi^* : \omega^\omega \to \omega^\omega$ such that Φ^** extends Φ*, i.e.,* $\Phi^*(f) = \Phi(f)$ *for all $f \in P$.*

Proof. The key to the proof is compactness. Let $h \in \omega^\omega$ be a recursive function such that $\forall i\, (\forall f \in P)\, (f(i) < h(i))$. Then P is a closed set in the product space

$$Q_h = \prod_{i \in \omega} \{0, 1, \ldots, h(i) - 1\} = \{f \in \omega^\omega \mid \forall i\, (f(i) < h(i))\}.$$

By general topology, Q_h is compact. Let T be a recursive tree such that $P = [T]$. Let e be an index of Φ, i.e., $\Phi(f)(n) = \{e\}^f(n)$ for all $f \in P$ and all n. Then for each n there is a covering of Q_h by clopen sets $\{f \mid \sigma \subset f\}$ where σ is a string such that either $\{e\}^\sigma(n)\downarrow$ or $\sigma \notin T$. By compactness of Q_h, there exists a finite subcovering. Since h and T are recursive, a particular finite subcovering $\sigma^0_n, \ldots, \sigma^{k_n}_n$ can be found effectively. Put

$$h^*(n) = \max\left\{\{e\}^{\sigma^i_n}(n) + 1 \,\middle|\, i \le k_n \text{ and } \sigma^i_n \in T\right\}.$$

Then $h^* : \omega \to \omega$ is a recursive function, and $\Phi(f)(n) < h^*(n)$ for all $f \in P$ and all n. Thus $\{\Phi(f) \mid f \in P\}$ is recursively bounded. For all $g \in \omega^\omega$ we have $g \in \{\Phi(f) \mid f \in P\}$ if and only if there is no finite covering of Q_h by strings σ such that either $\{e\}^\sigma(n)\downarrow$ and $\neq g(n)$ for some n, or else $\sigma \notin T$. Thus $\{\Phi(f) \mid f \in P\}$ is Π^0_1. We have now proved the first part of the lemma. To prove the second part, define a recursive functional $\Phi^* : \omega^\omega \to \omega^\omega$ by putting $\Phi^*(f)(n) = \{e\}^{\sigma^i_n}(n)$ where $i \le k_n$ is minimal such that $\sigma^i_n \subset f$ and $\sigma^i_n \in T$, or $\Phi^*(f)(n) = 0$ if no such i exists. Clearly Φ^* is recursive and extends Φ. ☐

Definition 4.8. In general, suppose that to each $n \in \omega$ we have effectively associated a finite sequence of ordered pairs $(\sigma^0_n, m^0_n), \ldots, (\sigma^{k_n}_n, m^{k_n}_n)$ where $\sigma^i_n \in \omega^{<\omega}$ and $m^i_n \in \omega$ for each $i \le k_n$. Define a recursive functional $\Phi : \omega^\omega \to \omega^\omega$ by putting $\Phi(f)(n) = m^i_n$ where $i \le k_n$ is minimal such that $\sigma^i_n \subset f$, or $\Phi(f)(n) = 0$ if no such i exists. Then Φ is called a *truth table functional.* For $f, g \in \omega^\omega$ we say that f is *truth table reducible* to g, written $f \le_{tt} g$, if there exists a truth table functional Φ such that $f = \Phi(g)$. Rogers [45, Chapter 8 and Section 9.6] provides general background on truth table reducibility.

Corollary 4.9. *Assume that $P \subseteq \omega^\omega$ is recursively bounded Π^0_1, and assume that $\Phi : P \to \omega^\omega$ is a recursive functional. Then Φ can be extended to a truth table functional. In particular, $\Phi(f) \leq_{tt} f$ for all $f \in P$.*

Proof. In the proof of Theorem 4.7, note that Φ^* is a truth table functional. □

Theorem 4.10. *Let P be a recursively bounded Π^0_1 set. Then P is recursively homeomorphic to a Π^0_1 subset of 2^ω.*

Proof. Given a recursively bounded Π^0_1 set $P \subseteq \omega^\omega$, put

$$P^* = \{G_f \mid f \in P\} \subseteq 2^\omega$$

where $G_f =$ (the characteristic function of) $\{2^m 3^n \mid f(m) = n\}$. Clearly $f \mapsto G_f$ is a recursive homeomorphism of P onto P^*. By Theorem 4.7, P^* is Π^0_1. □

Corollary 4.11. *The weak (strong) degrees of nonempty recursively bounded Π^0_1 sets are the same as the weak (strong) degrees of nonempty Π^0_1 subsets of 2^ω.*

Proof. This is immediate from Theorem 4.10. □

Definition 4.12. Given $P \subseteq \omega^\omega$, put

$$\mathrm{Ext}(P) = \{\sigma \mid (\exists f \in P)\,(\sigma \subset f)\} \subseteq \omega^{<\omega},$$

the set of *extendible nodes* of P. Note that $\mathrm{Ext}(P)$ is a tree, and $[\mathrm{Ext}(P)]$ is the topological closure of P in ω^ω. In particular, if P is Π^0_1, then $P = [\mathrm{Ext}(P)]$.

Lemma 4.13. *Let P be a recursively bounded Π^0_1 set. Then $\mathrm{Ext}(P)$ is Π^0_1.*

Proof. Let T be a recursive tree such that $P = [T]$. Let $h \in \omega^\omega$ be a recursive function such that $\forall n\,(\forall f \in P)\,(f(n) < h(n))$. As in the proof of Theorem 4.7, consider the compact space $Q_h = \{g \mid \forall n\,(g(n) < h(n))\}$. For each $\sigma \in \omega^{<\omega}$, we have $\sigma \notin \mathrm{Ext}(P)$ if and only if Q_h is covered by clopen sets $\{g \mid \tau \subset g\}$ such that either $\tau \notin T$ or τ is incompatible with σ. In this case, compactness of Q_h implies the existence of a finite subcovering. Moreover, since h and T are recursive, such a finite subcovering can be found effectively. Thus $\{\#(\sigma) \mid \sigma \notin \mathrm{Ext}(P)\}$ is recursively enumerable, i.e., Σ^0_1. It follows that $\mathrm{Ext}(P)$ is Π^0_1. □

Definition 4.14. For $P \subseteq \omega^\omega$, an *isolated point* of P is an $f \in P$ such that, for some string τ, f is the unique $g \in P$ such that $\tau \subset g$. We say that P is *perfect* if P has no isolated points.

Theorem 4.15. *Let P be a recursively bounded Π^0_1 set. If f is an isolated point of P, then f is recursive.*

Proof. By Theorem 4.10 we may assume that P is a Π^0_1 subset of 2^ω. Let $\tau \in 2^{<\omega}$ be such that f is the unique $g \in P$ such that $\tau \subset g$. Then, for all $\sigma \supseteq \tau$ in $2^{<\omega}$, we have $\sigma \subset f$ if and only if $\sigma \in \mathrm{Ext}(P)$. By Lemma 4.13, $\mathrm{Ext}(P)$ is Π^0_1, hence $A = 2^{<\omega} \setminus \mathrm{Ext}(P)$ is recursively enumerable. Now, given $\sigma \in 2^{<\omega}$ of length n, we have $\sigma \subset f$ if and only if $\rho \in A$ for all $\rho \in 2^{<\omega}$ of length n other than σ. Since $\{\rho \in 2^{<\omega} \mid \mathrm{lh}(\rho) = n\}$ is of cardinality 2^n, it follows that $\{\sigma \mid \sigma \subset f\}$ is recursively enumerable, so f is recursive. □

Corollary 4.16. *Let P be a recursively bounded Π^0_1 set. If the weak or strong degree of P is $> \mathbf{0}$, then P is perfect.*

Proof. For any $P \subseteq \omega^\omega$, we have $\deg_w(P) > \mathbf{0}$ or $\deg_s(P) > \mathbf{0}$ if and only if P has no recursive members. If P is recursively bounded Π^0_1 and has no recursive members, then by Theorem 4.15 P has no isolated points, i.e., P is perfect. □

Definition 4.17. We say that $g \in \omega^\omega$ is *almost recursive* if for all $f \leq_T g$ there exists $h \in \omega^\omega$ such that $\forall n\,(f(n) < h(n))$ and h is recursive. (Turing degrees which contain almost recursive functions have been known in the literature as *hyperimmune-free* Turing degrees.)

Theorem 4.18. *Suppose g is almost recursive. Then for all $f \leq_T g$ we have $f \leq_{tt} g$, i.e., f is truth table reducible to g. In particular, $f = \Phi(g)$ for some total recursive functional $\Phi : \omega^\omega \to \omega^\omega$.*

Proof. Let e be such that $f(n) = \{e\}^g(n)$ for all n. Define $f^* \in \omega^\omega$ by $f^*(n) =$ the least k such that $\{e\}^\tau(n) \downarrow$ where $\tau = \langle g(0), \ldots, g(k)\rangle$. Clearly $f^* \leq_T g$. Since g is almost recursive, there exists a recursive function h such that $\forall n\,(f^*(n) < h(n))$. Define $\Phi : \omega^\omega \to \omega^\omega$ by putting $\Phi(\overline{g})(n) = \{e\}^{\overline{\tau}}(n)$ where $\overline{\tau} = \langle \overline{g}(0), \ldots, \overline{g}(h(n))\rangle$, if $\{e\}^{\overline{\tau}}(n) \downarrow$, and $\Phi(\overline{g})(n) = 0$ otherwise, for all $\overline{g} \in \omega^\omega$ and $n \in \omega$. Then Φ is a truth table functional, and $f = \Phi(g)$. □

We end this section with the Almost Recursive Basis Theorem.

Theorem 4.19. *If $P \subseteq \omega^\omega$ is nonempty, recursively bounded, and Π^0_1, then there exists $g \in P$ such that g is almost recursive.*

Proof. This is the Hyperimmune-Free Basis Theorem of Jockusch/Soare [26, Theorem 2.4]. For completeness we present the proof here. Define inductively a sequence of nonempty Π^0_1 sets $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_e \supseteq P_{e+1} \supseteq \cdots$ as follows. Put $P_0 = P$. If $\exists n\,(\exists f \in P_e)\,(\{e\}^f(n)\uparrow)$, fix such an n and put $P_{e+1} = \{f \in P_e \mid \{e\}^f(n)\uparrow\}$. Otherwise, Theorem 4.7 gives us a recursive function $h = h_e$ such that $\forall n\,(\forall f \in P_e)\,(\{e\}^f(n) < h(n))$, and in this case we put $P_{e+1} = P_e$. By compactness, $\bigcap_{e=0}^\infty P_e$ is nonempty, so let $g \in \bigcap_{e=0}^\infty P_e$. By construction, g is almost recursive. □

5 The lattices $\mathcal{P}_w$ and $\mathcal{P}_s$

In this section we introduce the lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ which are the focus of this paper.

Remark 5.1. There is a large recursion-theoretic literature concerning Turing degrees of members of Π^0_1 subsets of ω^ω, and especially Turing degrees of members of recursively bounded Π^0_1 subsets of ω^ω. See for instance the classic paper of Jockusch and Soare [26] and the survey article by Cenzer and Remmel [10]. Mindful of this literature, we find it natural to view nonempty recursively bounded Π^0_1 sets as mass problems.

By Theorem 4.10 and Corollary 4.11, it suffices to consider Π^0_1 subsets of 2^ω.

Definition 5.2. $\mathcal{P}_w$ ($\mathcal{P}_s$) is the set of weak (strong) degrees of nonempty recursively bounded Π^0_1 sets. By Corollary 4.11, $\mathcal{P}_w$ ($\mathcal{P}_s$) is the same as the set of weak (strong) degrees of nonempty Π^0_1 subsets of 2^ω.

Theorem 5.3. *$\mathcal{P}_w$ and $\mathcal{P}_s$ are countable distributive lattices with a top and bottom element, denoted* **1** *and* **0** *respectively.*

Proof. If $P, Q \subseteq \omega^\omega$ are Π^0_1 and recursively bounded, then so are $P \cap Q$, $P \cup Q$, $P \times Q$, and $P + Q$. In particular, $\mathcal{P}_w$ and $\mathcal{P}_s$ are closed under the least upper bound and greatest lower bound operations inherited from the distributive lattices $\mathcal{D}_w$ and $\mathcal{D}_s$, respectively. It follows that $\mathcal{P}_w$ and $\mathcal{P}_s$ are distributive lattices. Clearly $\mathcal{P}_w$ and $\mathcal{P}_s$ are countable, because there are only countably many Π^0_1 subsets of 2^ω. Clearly $\mathbf{0} = \deg_w(2^\omega) = \deg_s(2^\omega)$ is the bottom element of $\mathcal{P}_w$ and of $\mathcal{P}_s$. It remains to show that $\mathcal{P}_w$ and $\mathcal{P}_s$ have a top element. Let PA be the set of completions of Peano Arithmetic. Identifying sentences with their Gödel numbers, we may view PA as a Π^0_1 subset of 2^ω. Since Peano Arithmetic is consistent, PA is nonempty. It is known that every nonempty Π^0_1 subset of 2^ω is $\leq_s$ PA, hence $\leq_w$ PA. We shall obtain this result and much more in Section 6 below, but see Jockusch/Soare [26]. Let us use **1** ambiguously to denote either $\deg_w(\mathrm{PA})$ or $\deg_s(\mathrm{PA})$. Thus **1** is the top element both of $\mathcal{P}_w$ and of $\mathcal{P}_s$. □

Remark 5.4. There is an obvious lattice homomorphism of $\mathcal{P}_s$ onto $\mathcal{P}_w$ given by $\deg_s(P) \mapsto \deg_w(P)$. Simpson and Slaman [55] have shown that every nonzero weak degree in $\mathcal{P}_w$ contains infinitely many strong degrees in $\mathcal{P}_s$.

Remark 5.5. In the context of recursively bounded Π^0_1 sets, there is reason to view weak reducibility as the mass problem analog of Turing reducibility, while strong reducibility is the mass problem analog of truth table reducibility. See Rogers [45, Sections 8.3 and 9.6] and Simpson [54, Remark 3.12]. Namely, if Q is recursively bounded Π^0_1 and $P \leq_s Q$, then by Corollary 4.9 the recursive functional $\Phi : Q \to P$ is given by truth tables, hence for each $g \in Q$ there exists $f = \Phi(g) \in P$ such that $f \leq_{tt} g$, i.e., f is truth table reducible to g. Thus we see that $\mathcal{P}_w$ is analogous to $\mathcal{R}_T$, the recursively enumerable Turing degrees, while $\mathcal{P}_s$ is more closely analogous to $\mathcal{R}_{tt}$, the recursively enumerable truth table degrees.

Remark 5.6. It is known that the countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ are structurally rich. Binns/Simpson [3, 6] have shown that every countable distributive lattice is lattice embeddable in every nontrivial initial segment of $\mathcal{P}_w$. A similar conjecture for $\mathcal{P}_s$ remains open, although partial results in this direction are known. Binns [3, 4] has obtained the $\mathcal{P}_w$ and $\mathcal{P}_s$ analogs of the Sacks Splitting Theorem. Namely, for all $\mathbf{b} > \mathbf{0}$ in $\mathcal{P}_w$ there exist $\mathbf{b}_1, \mathbf{b}_2 < \mathbf{b}$ in $\mathcal{P}_w$ such that $\sup(\mathbf{b}_1, \mathbf{b}_2) = \mathbf{b}$, and similarly for $\mathcal{P}_s$. Cenzer/Hinman [9] have obtained the $\mathcal{P}_s$ analog of the Sacks Density Theorem. Namely, for all $\mathbf{a} < \mathbf{b}$ in $\mathcal{P}_s$ there exists $\mathbf{c}$ in $\mathcal{P}_s$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$. A similar conjecture for $\mathcal{P}_w$ remains open. Binns [3, 4] has improved the result of Cenzer/Hinman [9] by showing that for all $\mathbf{a} < \mathbf{b}$ in $\mathcal{P}_s$ there exist $\mathbf{b}_1, \mathbf{b}_2 < \mathbf{b}$ in $\mathcal{P}_s$ such that $\mathbf{a} < \inf(\mathbf{b}_1, \mathbf{b}_2) < \sup(\mathbf{b}_1, \mathbf{b}_2) = \mathbf{b}$. These structural results for $\mathcal{P}_w$ and $\mathcal{P}_s$

are proved by means of priority arguments. They invite comparison with the older, known results for recursively enumerable Turing degrees, which were also proved by priority arguments.

6 Weak and strong completeness

In this section we obtain additional information concerning sets which are of weak or strong degree $\mathbf{1}$ in $\mathcal{P}_w$ or $\mathcal{P}_s$, respectively. We show that, if P is a nonempty recursively bounded Π^0_1 set, then $\deg_w(P) = \mathbf{1}$ if and only if P is Turing degree isomorphic to PA, and $\deg_s(P) = \mathbf{1}$ if and only if P is recursively homeomorphic to PA.

By Theorem 4.10 and Corollary 4.11, it suffices to consider Π^0_1 subsets of 2^ω.

Definition 6.1. Let P be a nonempty recursively bounded Π^0_1 set. P is *weakly complete* if $\deg_w(P) = \mathbf{1}$, i.e., $P \geq_w Q$ for all nonempty Π^0_1 sets $Q \subseteq 2^\omega$. P is *strongly complete* if $\deg_s(P) = \mathbf{1}$, i.e., $P \geq_s Q$ for all nonempty Π^0_1 sets $Q \subseteq 2^\omega$. These notions have sometimes been referred to as *Muchnik completeness* and *Medvedev completeness*, respectively.

Remark 6.2. We have seen in the proof of Theorem 5.3 that PA is strongly complete, hence weakly complete. In addition, there are natural examples of recursively bounded Π^0_1 sets which are weakly complete but not strongly complete. See Definition 7.9 and Remark 7.10 below.

Theorem 6.3. *Let P and Q be nonempty Π^0_1 subsets of 2^ω.*

1. *If P and Q are strongly complete, then P is recursively homeomorphic to Q.*
2. *If P is strongly complete, then we can find a recursive functional $\Phi : P \to Q$ which is* onto *Q, i.e., $Q = \{\Phi(f) \mid f \in P\}$.*
3. *P is strongly complete if and only if P is* productive*, i.e., given an index of a nonempty Π^0_1 set $P' \subseteq P$ we can effectively find a canonically indexed clopen set $U \subseteq 2^\omega$ such that both $P' \cap U$ and $P' \setminus U$ are nonempty.*

Proof. See Simpson [54, Section 3]. □

Corollary 6.4. *Let P be a nonempty recursively bounded Π^0_1 set. If P is strongly complete, then the set of Turing degrees of members of P is upward closed.*

Proof. For any P, the set of Turing degrees of members of $P \times 2^\omega$ is obviously upward closed. Now assume that P is strongly complete. Then clearly $P \times 2^\omega$ is strongly complete. Hence, by Theorem 4.10 and part 1 of Theorem 6.3, P and $P \times 2^\omega$ are recursively homeomorphic to each other. Since the set of Turing degrees of members of $P \times 2^\omega$ is upward closed, it follows that the set of Turing degrees of members of P is upward closed. □

Corollary 6.5. *Let P be a nonempty recursively bounded Π^0_1 set. Then P is strongly complete if and only if P is recursively homeomorphic to* PA.

Proof. Recall that PA is the set of completions of Peano Arithmetic. By the Gödel/Rosser Theorem for Peano Arithmetic, PA is productive. Hence, by part 3 of Theorem 6.3, PA is strongly complete. Our corollary now follows by Theorem 4.10 and part 1 of Theorem 6.3. □

The next corollary is originally due to Robert M. Solovay.

Corollary 6.6. *The set of Turing degrees of members of* PA *is upward closed.*

Proof. This is immediate from Corollaries 6.4 and 6.5. □

Remark 6.7. Instead of Peano Arithmetic, we could have used any consistent recursively axiomatizable theory T which is *effectively essentially incomplete*, i.e., has the property given by the Gödel/Rosser Theorem. The required property of T is as follows. Given a consistent recursively axiomatizable theory T' extending T, we can effectively find a sentence φ in the language of T which is *independent of T'*, i.e., $T' \nvdash \varphi$ and $T' \nvdash \neg\varphi$. Compare this with our notion of productivity from part 3 of Theorem 6.3, which may be viewed as an abstract Gödel/Rosser property for Π^0_1 subsets of 2^ω.

Theorem 6.8. *Let P be a nonempty recursively bounded Π^0_1 set. Then P is weakly complete if and only if P is Turing degree isomorphic to* PA.

Proof. By Corollary 6.5, PA is strongly complete. (This is the only property of PA which we shall need.) Hence PA is weakly complete, so any P which is Turing degree isomorphic to PA is weakly complete. For the converse, let P be a nonempty Π^0_1 subset of 2^ω which is weakly complete. In particular PA $\leq_w P$. It follows by Corollary 6.6 that the Turing degrees of members of P are included in the Turing degrees of members of PA.

In order to finish the proof of Theorem 6.8, we need the following lemma, which exposes an interesting relationship between weak reducibility and strong reducibility.

Lemma 6.9. *Let P and Q be nonempty recursively bounded Π^0_1 sets. If $P \leq_w Q$, then we can find a nonempty Π^0_1 set $\overline{Q} \subseteq Q$ such that $P \leq_s \overline{Q}$.*

Proof. By the Almost Recursive Basis Theorem 4.19, let $g \in Q$ be almost recursive. Since $P \leq_w Q$, let $f \in P$ be such that $f \leq_T g$. By Theorem 4.18 we can find a total recursive functional $\Phi : \omega^\omega \to \omega^\omega$ such that $f = \Phi(g)$. Put $\overline{Q} = \{\overline{g} \in Q \mid \Phi(\overline{g}) \in P\}$. By Theorem 4.4 we have that $\overline{Q}$ is a Π^0_1 subset of Q. Since $f = \Phi(g) \in P$, we have $g \in \overline{Q}$, hence $\overline{Q}$ is nonempty. Putting $\overline{\Phi}$ = the restriction of Φ to $\overline{Q}$, we have $\overline{\Phi} : \overline{Q} \to P$, so $P \leq_s \overline{Q}$. □

Now, since our P is $\geq_w$ PA, apply Lemma 6.9 to get a nonempty Π^0_1 set $\overline{P} \subseteq P$ such that $\overline{P} \geq_s$ PA. Since PA is strongly complete, $\overline{P}$ is strongly complete. Hence, by Theorem 4.10 and part 1 of Theorem 6.3, $\overline{P}$ is recursively homeomorphic to PA. It follows that $\overline{P}$ is Turing degree isomorphic to PA. We now have $\{\deg_T(f) \mid f \in P\} \subseteq \{\deg_T(f) \mid f \in \mathrm{PA}\} = \{\deg_T(f) \mid f \in \overline{P}\} \subseteq \{\deg_T(f) \mid f \in P\}$, so P is Turing degree isomorphic to PA. This completes the proof of Theorem 6.8. □

Corollary 6.10. *Let P and Q be nonempty recursively bounded Π^0_1 sets. If P and Q are weakly complete, then P is Turing degree isomorphic to Q.*

Proof. By Theorem 6.8, P and Q are Turing degree isomorphic to PA, hence to each other. □

We can now strengthen Corollary 6.4 as follows.

Corollary 6.11. *Let P be a nonempty recursively bounded Π^0_1 set. If P is weakly complete, then the set of Turing degrees of members of P is upward closed.*

Proof. This is immediate from Corollary 6.6 and Theorem 6.8. □

7 Π^0_1 sets of positive measure

Definition 7.1. The *fair coin probability measure* on 2^ω is defined by

$$\mu(\{f \in 2^\omega \mid f(n) = m\}) = \frac{1}{2}$$

for all $m \in \{0, 1\}$ and $n \in \omega$. A set $P \subseteq 2^\omega$ is said to be *of positive measure* if $\mu(P) > 0$.

In this section we prove a "non-helping" theorem for weak and strong degrees of subsets of 2^ω which are of positive measure.

Lemma 7.2. *Let F_n, $n \in \omega$ be a sequence of finite subsets of ω of bounded cardinality. Put*

$$S = \textstyle\prod_{n\in\omega} F_n = \{f \in \omega^\omega \mid \forall n\,(f(n) \in F_n)\} \subseteq \omega^\omega.$$

Let $P \subseteq 2^\omega$ be of positive measure. Let $Q \subseteq \omega^\omega$ be arbitrary. If $S \leq_s P \times Q$, then $S \leq_s Q$.

Proof. We generalize an argument of Jockusch/Soare [26, Theorem 5.3]. Let $k \geq 2$ be such that, for all n, F_n is of cardinality $< k$. Our hypothesis concerning P is that $\mu(P) > 0$. By measure theory, let $V \supseteq P$ be an open set in 2^ω such that $\mu(V \setminus P) < \mu(P)/4k$. Let $U \subseteq V$ be a clopen set such that $\mu(V \setminus U) < \mu(P)/4k$. It follows that $\mu(U \setminus P) < \mu(U)/k$. Note that $\mu(U)$ is a positive rational number. Since $S \leq_s P \times Q$, let Φ be a recursive functional such that $\Phi(f \oplus g) \in S$ for all $f \in P$ and $g \in Q$. Given $g \in Q$ and $n \in \omega$, we can effectively find $m = \Psi(g)(n) \in \omega$ such that $\mu(\{f \in U \mid \Phi(f \oplus g)(n) = m\}) > \mu(U)/k$. It follows that $m \in F_n$. Thus Ψ is a recursive functional, and $\Psi(g) \in S$ for all $g \in Q$. Hence $S \leq_s Q$. □

Lemma 7.3. *Same as Lemma 7.2 with strong reducibility, $\leq_s$, replaced by weak reducibility, $\leq_w$.*

Proof. Assume $S \leq_w P \times Q$. Fix $g \in Q$. We have $S \leq_w P \times \{g\}$. By countable additivity of μ, since there are only countably many recursive functionals, there exists $P_g \subseteq P$ such that $\mu(P_g) > 0$ and $S \leq_s P_g \times \{g\}$. By Lemma 7.2 it follows that $S \leq_s \{g\}$. This implies $S \leq_w Q$, since $g \in Q$ is arbitrary. □

Definition 7.4. For $A, B \subseteq \omega$ we say that $f \in 2^\omega$ *separates* A, B if $f(n) = 1$ for all $n \in A$, and $f(n) = 0$ for all $n \in B$. A nonempty Π^0_1 set $S \subseteq 2^\omega$ is said to be *separating* if there exist recursively enumerable sets $A, B \subseteq \omega$ such that $S = \{f \in 2^\omega \mid f \text{ separates } A, B\}$. In this case we say that the weak degree $\deg_w(S)$ and the strong degree $\deg_s(S)$ are *separating.*

Theorem 7.5. *Let S, P, Q be Π^0_1 subsets of 2^ω. Assume that S is separating and that P is of positive measure. Let $\mathbf{s}, \mathbf{p}, \mathbf{q} \in \mathcal{P}_w$ be the weak degrees of S, P, Q respectively. If $\mathbf{s} \leq \sup(\mathbf{p}, \mathbf{q})$, then $\mathbf{s} \leq \mathbf{q}$. The same holds for strong degrees.*

Proof. It suffices to note that S is of the form required by Lemmas 7.2 and 7.3. Namely, $S = \prod_{n\in\omega} F_n$ where $F_n = \{1\}$ if $n \in A$, $\{0\}$ if $n \in B$, $\{0, 1\}$ otherwise. □

Corollary 7.6. *Let P and Q be nonempty Π^0_1 subsets of 2^ω. Assume that P is of positive measure. Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_w$ be the weak degrees of P, Q respectively. If $\mathbf{q} < \mathbf{1}$, then $\sup(\mathbf{p}, \mathbf{q}) < \mathbf{1}$. The same holds for strong degrees.*

Proof. By Theorem 7.5, it suffices to note that $\mathbf{1}$ is separating. Namely, $\mathbf{1}$ is the weak or strong degree of the Π^0_1 set $S = \{f \in 2^\omega \mid f \text{ separates } A, B\}$ where $A = \{n \mid \{n\}(n) = 0\}$ and $B = \{n \mid \{n\}(n) = 1\}$. Or, we could take A and B to be the set of Gödel numbers of provable and refutable sentences of Peano Arithmetic. See also Jockusch/Soare [26] and Simpson [54, Section 3]. □

Corollary 7.7. *Let P be a Π^0_1 subset of 2^ω of positive measure. Let $\mathbf{p} \in \mathcal{P}_w$ be the weak degree of P. Then $\mathbf{p} < \mathbf{1}$. The same holds for strong degrees.*

Proof. This follows from Corollary 7.6 by setting $\mathbf{q} = \mathbf{0}$. □

Remark 7.8. In [48] we shall give an example of a Π^0_1 set $Q \subseteq 2^\omega$ whose Turing upward closure $\widehat{Q} = \{f \in 2^\omega \mid (\exists g \leq_T f)\,(g \in Q)\}$ is of positive measure yet does not contain any Π^0_1 set of positive measure.

Definition 7.9. Following Jockusch [25], for $k \geq 2$ we define

$$\mathrm{DNR}_k = \{f \in \omega^\omega \mid \forall n\,(f(n) < k \text{ and } f(n) \neq \{n\}(n))\}.$$

Thus DNR_k is the set of k-bounded, diagonally nonrecursive functions. Note that DNR_k is recursively bounded and Π^0_1.

Remark 7.10. By Jockusch [25, Theorem 5] each DNR_k is weakly complete, i.e., of weak degree $\mathbf{1}$. Let $\mathbf{d}^*_k \in \mathcal{P}_s$ be the strong degree of DNR_k. It is well known (see also the proof of Corollary 7.6 above) that DNR_2 is strongly complete, i.e., $\mathbf{d}^*_2 = \mathbf{1}$ in $\mathcal{P}_s$. By Jockusch [25, Theorem 6] we have

$$\mathbf{1} = \mathbf{d}_2^* > \mathbf{d}_3^* > \cdots > \mathbf{d}_k^* > \mathbf{d}_{k+1}^* > \cdots$$

in $\mathcal{P}_s$. See also Simpson [54, Remark 3.21].

We have the following new result.

Corollary 7.11. *Let P and Q be Π^0_1 subsets of 2^ω. Assume that P is of positive measure. Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_s$ be the strong degrees of P, Q respectively. For each $k \geq 2$, if $\mathbf{d}_k^* \leq \sup(\mathbf{p}, \mathbf{q})$, then $\mathbf{d}_k^* \leq \mathbf{q}$. In particular we have*

$$\mathbf{1} = \sup(\mathbf{p}, \mathbf{d}_2^*) > \sup(\mathbf{p}, \mathbf{d}_3^*) > \cdots > \sup(\mathbf{p}, \mathbf{d}_k^*) > \sup(\mathbf{p}, \mathbf{d}_{k+1}^*) > \cdots.$$

Proof. It suffices to note that DNR_k is of the form required by Lemma 7.2. Namely, $\mathrm{DNR}_k = \prod_{n\in\omega} F_n$ where $F_n = \{m < k \mid \{n\}(n) \neq m\}$. □

Remark 7.12. In Corollary 7.11, we do not know whether it is necessarily the case that $\mathbf{d}_3^* \geq \mathbf{p}$, or $\mathbf{d}_k^* \geq \mathbf{p}$ for all $k \geq 3$.

8 Π^0_1 sets of random reals

In this section we exhibit a particular degree $\mathbf{r} \in \mathcal{P}_w$ and note some of its degree-theoretic properties.

As in Section 7, let μ denote the fair coin probability measure on 2^ω.

Definition 8.1. An *effective null G_δ* is a set $S \subseteq 2^\omega$ of the form $S = \bigcap_{n\in\omega} U_n$ where $\{U_n\}_{n\in\omega}$ is a recursive sequence of Σ^0_1 subsets of 2^ω with $\mu(U_n) \leq 1/2^n$ for all n. A point $f \in 2^\omega$ is said to be *random* if $f \notin S$ for all effective null G_δ sets $S \subseteq 2^\omega$.

Remark 8.2. The notion of randomness in Definition 8.1 is due to Martin-Löf [39] and has been studied extensively. It appears to be the most general and natural notion of algorithmic randomness for infinite sequences of 0's and 1's. It has also been called *Martin-Löf randomness* (Li/Vitányi [37, Section 2.5]), *1-randomness* (Kurtz [28], Kautz [27]), and *the NAP property* (Kučera [29, 30, 31, 32, 33, 34, 35]). It is closely related to Kolmogorov complexity (see Li/Vitányi [37]).

The following theorem is well known. It says that the union of all effective null G_δ sets is an effective null G_δ set.

Theorem 8.3. *$\{f \in 2^\omega \mid f$ is not random$\}$ is an effective null G_δ set.*

Proof. This result is essentially due to Martin-Löf [39]. See also Kučera [29, Theorems 1 and 2]. For the sake of completeness, we present the proof here. For each $e \in \omega$ define a Σ^0_1 set $V_e \subseteq 2^\omega$ as follows. Compute $\{e\}(e)$. If $\{e\}(e)$ is undefined, $V_e =$ the empty set. If $\{e\}(e) = m$, let $V_e =$ the Σ^0_1 subset of 2^ω with Σ^0_1 index m enumerated so long as its measure is $\leq 1/2^e$. Put $R = \bigcup_{k=0}^\infty R_k$ where $R_k = 2^\omega \setminus \bigcup_{e=k+1}^\infty V_e$. We have

$$\mu(2^\omega \setminus R_k) \leq \sum_{e=k+1}^\infty \mu(V_e) \leq \sum_{e=k+1}^\infty \frac{1}{2^e} = \frac{1}{2^k}$$

and R_k is uniformly Π^0_1. Thus $2^\omega \setminus R = \bigcap_{k=0}^\infty (2^\omega \setminus R_k)$ is an effective null G_δ set. Hence every random $f \in 2^\omega$ belongs to R. We claim that, conversely, every $f \in R$ is random. To see this, consider an effective null G_δ set $S = \bigcap_{n=0}^\infty U_n$, where $\mu(U_n) \leq 1/2^n$. It suffices to show that $R \cap S = \emptyset$. Given $k \in \omega$, let $e \geq k+1$ be such that, for all n, $\{e\}(n)$ is a Σ^0_1 index of U_n. In particular, $\{e\}(e)$ is a Σ^0_1 index of U_e. Since $\mu(U_e) \leq 1/2^e$, it follows that $V_e = U_e$. Since R_k is disjoint from V_e, it follows that R_k is disjoint from S. But k is arbitrary, so R is disjoint from S. This completes the proof that $R = \{f \in 2^\omega \mid f$ is random$\}$. □

Corollary 8.4. *There exists a nonempty Π^0_1 set*

$$P \subseteq R = \{f \in 2^\omega \mid f \textit{ is random}\}.$$

Proof. Trivially any effective null G_δ set is Π^0_2. In particular, by Theorem 8.3, R is Σ^0_2. Hence R is a union of Π^0_1 sets. Let P be any one of these Π^0_1 sets. Alternatively, we could let P be any one of the sets R_k as in the proof of Theorem 8.3. Each of these sets is Π^0_1. □

Notation 8.5. We use the following notation for shifts: $f^{(k)}(n) = f(k+n)$. Note that $f \mapsto f^{(k)}$ is a mapping of 2^ω into 2^ω.

Lemma 8.6. *For all $f \in 2^\omega$ and $k \in \omega$, f is random if and only if $f^{(k)}$ is random.*

Proof. The proof is straightforward. □

The next lemma is an effective version of the Zero-One Law of probability theory.

Lemma 8.7. *Let f be random. Let $P \subseteq 2^\omega$ be Π^0_1 with $\mu(P) > 0$. Then $\exists k\, (f^{(k)} \in P)$.*

Proof. This is due to Kučera [29]. For completeness we present the proof here. Let P be the set of paths through T, where $T \subseteq 2^{<\omega}$ is a recursive tree. Put

$$\widetilde{T} = \{\sigma^\frown\langle i\rangle \mid \sigma \in T, i \in \{0,1\}, \sigma^\frown\langle i\rangle \notin T\}.$$

For $n \geq 1$, put

$$T^n = \{\tau_1{}^\frown \cdots {}^\frown\tau_m{}^\frown\sigma \mid m < n, \tau_1, \ldots, \tau_m \in \widetilde{T}, \sigma \in T\},$$

and let $P^n = [T^n]$, the set of paths through T^n. We have

$$\mu(P^n) = 1 - (1-\mu(P))^n,$$

hence $2^\omega \setminus \bigcup_{n=1}^\infty P^n$ is an effective null G_δ set. Hence $f \in P^n$ for some n. Hence for some $m < n$ and $\tau_1, \ldots, \tau_m \in \widetilde{T}$ we have $f = \tau_1{}^\frown \cdots {}^\frown\tau_m{}^\frown g$ where $g \in P$. Putting $k =$ length of $\tau_1{}^\frown \cdots {}^\frown\tau_m$, we have $g = f^{(k)}$. □

Lemma 8.8. *Let $f \in 2^\omega$ be random. Then for all Π^0_1 sets $P \subseteq 2^\omega$, if $f \in P$ then $\mu(P) > 0$.*

Proof. Since P is Π^0_1, let $P = \bigcap_s P_s$ where P_s, $s \in \omega$, is a recursive sequence of canonically indexed clopen sets in 2^ω with

$$P_0 \supseteq P_1 \supseteq \cdots \supseteq P_s \supseteq P_{s+1} \supseteq \cdots .$$

Assume $\mu(P) = 0$. By countable additivity, $\lim_s \mu(P_s) = 0$. Define a recursive function $h : \omega \to \omega$ by $h(n) =$ least s such that $\mu(P_s) \leq 1/2^n$. Putting $U_n = P_{h(n)}$, we see that $P = \bigcap_n U_n$ is an effective null G_δ set. Hence $f \notin P$, a contradiction. $\square$

Lemma 8.9. *Let P and R be as in Corollary 8.4. Then $\mu(P) > 0$, and $P \equiv_w R$.*

Proof. By Lemma 8.8 $\mu(P) > 0$, and by Lemma 8.7 $(\forall f \in R)\, \exists k\, (f^{(k)} \in P)$. Thus $P \leq_w R$. On the other hand, since $P \subseteq R$, $P \geq_w R$, so $P \equiv_w R$. $\square$

Theorem 8.10. *Let $\mathbf{r} = \deg_w(R)$ where $R = \{f \in 2^\omega \mid f$ is random$\}$. Then $\mathbf{r}$ can be characterized as the unique largest weak degree of a Π^0_1 set $P \subseteq 2^\omega$ such that $\mu(P) > 0$.*

Proof. By Lemma 8.7 we have that, for any Π^0_1 set $P \subseteq 2^\omega$ with $\mu(P) > 0$, $P \leq_w R$. By Corollary 8.4 let P' be a nonempty Π^0_1 subset of R. By Lemma 8.9 we have $\mu(P') > 0$ and $P' \equiv_w R$. This completes the proof. $\square$

Remark 8.11. Theorem 8.10 tells us that, among all weak degrees of Π^0_1 sets of positive measure, there exists a unique largest degree. Simpson/Slaman [55] and independently Terwijn [58] have shown that, among all strong degrees of Π^0_1 sets of positive measure, there is no largest or even maximal degree.

We end this section by noting some additional properties of the particular weak degree $\mathbf{r}$ which was defined in Theorem 8.10.

Theorem 8.12. *Let $\mathbf{r}$ be the weak degree of $R = \{f \in 2^\omega \mid f$ is random$\}$. Then:*

1. *$\mathbf{r} \in \mathcal{P}_w$, and $\mathbf{0} < \mathbf{r} < \mathbf{1}$.*
2. *For all $\mathbf{q} \in \mathcal{P}_w$, if $\mathbf{q} < \mathbf{1}$ then $\sup(\mathbf{q}, \mathbf{r}) < \mathbf{1}$.*
3. *For all $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{P}_w$, if $\mathbf{r} \geq \inf(\mathbf{q}_1, \mathbf{q}_2)$ then either $\mathbf{r} \geq \mathbf{q}_1$ or $\mathbf{r} \geq \mathbf{q}_2$.*
4. *There is no separating $\mathbf{s} \in \mathcal{P}_w$ such that $\mathbf{0} < \mathbf{s} \leq \mathbf{r}$.*

Proof. Since R has no recursive members, $\mathbf{r} > \mathbf{0}$. Theorem 8.10 implies that $\mathbf{r} \in \mathcal{P}_w$ and contains a Π^0_1 subset of 2^ω of positive measure. By Corollary 7.7 it follows that $\mathbf{r} < \mathbf{1}$, completing the proof of part 1 of the theorem. Corollary 7.6 gives part 2. To prove part 3, let Q_1, Q_2 be Π^0_1 subsets of 2^ω of weak degree $\mathbf{q}_1, \mathbf{q}_2$ respectively. We are assuming that $R \geq_w Q_1 \cup Q_2$. By Corollary 8.4 let P be a nonempty Π^0_1 subset of R. Then $P \geq_w Q_1 \cup Q_2$. By Lemma 6.9 we can find a nonempty Π^0_1 set $\overline{P} \subseteq P$ such that $\overline{P} \geq_s Q_1 \cup Q_2$. Thus there is a recursive functional $\Phi : \overline{P} \to Q_1 \cup Q_2$. Put $\overline{P}_1 = \overline{P} \cap \Phi^{-1}(Q_1)$ and $\overline{P}_2 = \overline{P} \cap \Phi^{-1}(Q_2)$. We have $\overline{P} = \overline{P}_1 \cup \overline{P}_2$, hence at least one of $\overline{P}_1$ and $\overline{P}_2$ is nonempty, say $\overline{P}_1$. Then $\overline{P}_1 \geq_s Q_1$. Note also that $\overline{P}_1$ is a Π^0_1 subset of R, hence by Lemma 8.9 we have $\overline{P}_1 \equiv_w R$. It follows that $R \geq_w Q_1$, and this proves part 3. Part 4 is a consequence of Theorem 7.5. □

Corollary 8.13. *The weak degree* $\mathbf{r} \in \mathcal{P}_w$ *is meet irreducible and does not join to* $\mathbf{1}$ *in* $\mathcal{P}_w$.

Proof. This follows from parts 1, 2 and 3 of Theorem 8.12. □

9 Thin Π^0_1 subsets of 2^ω

In this section we discuss an interesting class of degrees in $\mathcal{P}_w$, each of which is incomparable with the particular degree $\mathbf{r} \in \mathcal{P}_w$ of Section 8.

We begin with some generalities concerning thin Π^0_1 sets.

Definition 9.1. A Π^0_1 set $Q \subseteq \omega^\omega$ is said to be *thin* if, for all Π^0_1 sets $Q' \subseteq Q$, the set-theoretic difference $Q \setminus Q'$ is Π^0_1.

Lemma 9.2. *Let* Q *be a recursively bounded* Π^0_1 *set. Then* Q *is thin if and only if all* Π^0_1 *subsets of* Q *are* trivial*, i.e., they are of the form*

$$Q' = \{g \in Q \mid \sigma_1 \subset g \text{ or } \cdots \text{ or } \sigma_k \subset g\}$$

for some finite set of strings $\sigma_1, \ldots, \sigma_k$.

Proof. If Q is any Π^0_1 set, and if $Q' \subseteq Q$ is trivial, then clearly Q' and $Q \setminus Q'$ are Π^0_1. It remains to prove that if Q is a recursively bounded Π^0_1 set, and if $Q' \subseteq Q$ is such that Q' and $Q \setminus Q'$ are Π^0_1, then Q' is trivial. To see this, let h be a recursive function such that $\forall n\,(\forall g \in Q)\,(g(n) < h(n))$. As in the proof of Theorem 4.7, note that Q is a closed set in the compact space $Q_h = \{g \mid \forall n\,(g(n) < h(n))\}$. By Theorem 4.3, since Q' and $Q \setminus Q'$ are Π^0_1, let T' and T'' be recursive trees such that $Q' = [T']$ and $Q \setminus Q' = [T'']$. Then Q_h is covered by clopen sets of the form $\{g \mid \tau \subset g\}$ where either $\tau \notin T'$ or $\tau \notin T''$. Let $\tau_1, \ldots, \tau_l$ be a finite subcovering. Let $\sigma_1, \ldots, \sigma_k$ consist of those τ_i, $1 \leq i \leq l$, such that $\tau_i \notin T''$. Then $Q' = \{g \in Q \mid \sigma_1 \subset g \text{ or } \cdots \text{ or } \sigma_k \subset g\}$, so Q' is trivial. □

Theorem 9.3. *Let* Q *be a nonempty thin recursively bounded* Π^0_1 *set. Then* $f \in Q$ *is isolated if and only if* f *is recursive. In particular,* Q *is perfect if and only if and only if* Q *has no recursive members, i.e.,* $\deg_w(Q) > \mathbf{0}$.

Proof. If $f \in Q$ is isolated, then f is recursive by Theorem 4.15. Conversely, suppose $f \in Q$ is recursive. Then the singleton set $\{f\}$ is a Π^0_1 subset of Q. Since Q is thin and recursively bounded Π^0_1, by Lemma 9.2 there is a finite set of strings $\sigma_1, \ldots, \sigma_k$ such that f is the unique $g \in Q$ such that $\sigma_1 \subset g$ or $\cdots$ or $\sigma_k \subset g$. It follows that f is isolated. This proves the first part of the theorem. The second part follows immediately. □

Lemma 9.4. *Let Q be a thin recursively bounded Π^0_1 set. Then:*

1. *Every Π^0_1 subset of Q is thin and recursively bounded.*
2. *Let $P = \{\Phi(g) \mid g \in Q\}$ be the image of Q under a recursive functional $\Phi : Q \to \omega^\omega$. Then P is a thin recursively bounded Π^0_1 set.*

Proof. Part 1 is straightforward. For part 2, note first that, since Q is recursively bounded and Π^0_1, so is P, by Theorem 4.7. It remains to show that P is thin. Given a Π^0_1 set $P' \subseteq P$, let $Q' = \{g \in Q \mid \Phi(g) \in P'\}$ be the preimage of P'. By Theorem 4.4, Q' is Π^0_1. Since Q is thin, $Q \setminus Q'$ is also Π^0_1. It follows by Theorem 4.7 that $P \setminus P' = \{\Phi(g) \mid g \in Q \setminus Q'\}$ is Π^0_1. Since P' is an arbitrary Π^0_1 subset of P, we see that P is thin. □

Theorem 9.5. *If Q is a thin recursively bounded Π^0_1 set, then Q is recursively homeomorphic to a thin Π^0_1 set $Q^* \subseteq 2^\omega$. Moreover, Q is perfect if and only if Q^* is perfect.*

Proof. This follows from Theorems 4.10 and 9.3 and part 2 of Lemma 9.4. □

Remark 9.6. There is a large literature on thin perfect Π^0_1 subsets of 2^ω going back to Martin/Pour-El [38]. See Downey/Jockusch/Stob [15, 16] and Cholak et al [11]. Typically, thin perfect Π^0_1 subsets of 2^ω are constructed by means of priority arguments. In this sense, thin perfect Π^0_1 subsets of 2^ω and their weak and strong degrees are artificial or unnatural. In particular, thin perfect Π^0_1 subsets of 2^ω have been used by Binns/Simpson [6] to embed countable distributive lattices into $\mathcal{P}_w$ and $\mathcal{P}_s$.

Remark 9.7. Let $\mathbf{q} = \deg_w(Q)$ where Q is any nonempty thin perfect recursively bounded Π^0_1 set. Obviously $\mathbf{q} \in \mathcal{P}_w$. Let $\mathbf{r} = \deg_w(R)$ where $R = \{f \in 2^\omega \mid f \text{ is random}\}$. We have seen in Theorem 8.10 that $\mathbf{r} \in \mathcal{P}_w$. Our goal in this section is to prove Theorem 9.15, which says that $\mathbf{q}$ and $\mathbf{r}$ are incomparable, i.e., $\mathbf{q} \not\leq \mathbf{r}$ and $\mathbf{r} \not\leq \mathbf{q}$. By Theorem 9.5, it suffices to prove this in the special case when Q is a nonempty thin perfect Π^0_1 subset of 2^ω.

Lemma 9.8. *Let Q be a thin Π^0_1 subset of 2^ω. Then $\mu(Q) = 0$.*

Proof. For $f, g \in 2^\omega$ we write $f <_{\text{lex}} g$ to mean that there exists j such that $(\forall i < j)\,(f(i) = g(i))$ and $f(j) < g(j)$. Note that $<_{\text{lex}}$ is a linear ordering of 2^ω, the *lexicographical ordering.* For $\sigma, \tau \in 2^{<\omega}$ we write $\sigma <_{\text{lex}} \tau$ to mean that there exists $j < \min(\text{lh}(\sigma), \text{lh}(\tau))$ such that $(\forall i < j)\,(\sigma(i) = \tau(i))$ and $\sigma(j) < \tau(j)$. Note that, for each $n \in \omega$, the restriction of $<_{\text{lex}}$ to strings of length n is a linear ordering, the *lexicographical ordering.*

Let Q be a thin Π^0_1 subset of 2^ω. Assume for a contradiction that $\mu(Q) > 0$. Fix $p \in \omega$ such that $\mu(Q) > 1/2^p$. Put $T = \text{Ext}(Q) = \{\tau \in 2^{<\omega} \mid (\exists g \in Q)\,(\tau \subset g)\}$, the set of extendible nodes of Q. By Lemma 4.13 T is a Π^0_1 tree, and $Q = [T]$. Define inductively a sequence of strings $\tau_n \in T$, $n \in \omega$, as follows: $\tau_n =$ the lexicographically least $\tau \in T$ of length $p + n + 1$ such that $\tau >_{\text{lex}} \tau_m$

for all $m < n$. The existence of τ_n is assured by the fact that $\{g \in Q \mid \tau_0 \subset g$ or $\cdots$ or $\tau_{n-1} \subset g\}$ is a lexicographically initial segment of Q of measure

$$\sum_{m=0}^{n-1} \mu(\{g \in Q \mid \tau_m \subset g\}) \leq \sum_{m=0}^{n-1} \frac{1}{2^{p+m+1}} < \frac{1}{2^p} < \mu(Q).$$

Put $Q' = \{g \in Q \mid \neg \exists n\, (\tau_n \subset g)\}$. Thus Q' is a lexicographically final segment of Q. Moreover $Q \setminus Q' = \{g \in Q \mid \exists n\, (\tau_n \subset g)\}$ is clearly not compact, hence not of the form $\{g \in Q \mid \sigma_1 \subset g$ or $\cdots$ or $\sigma_k \subset g\}$ where $\sigma_1, \ldots, \sigma_k$ is a finite set of strings. It follows that Q' is also not of this form. In the next paragraph we shall show that Q' is a Π^0_1 subset of Q. Hence Q is not thin, so our lemma will be proved.

It remains to show that Q' is Π^0_1. Since T is Π^0_1, let T^s, $s \in \omega$ be a recursive sequence of recursive trees such that

$$T^0 \supseteq T^1 \supseteq \cdots \supseteq T^s \supseteq T^{s+1} \supseteq \cdots$$

and $\bigcap_{s=0}^{\infty} T^s = T$. For each s define inductively $\tau^s_n =$ the lexicographically least $\tau \in T^s$ of length $p + n + 1$ such that $\tau >_{\text{lex}} \tau^s_m$ for all $m < n$. The double sequence τ^s_n, $n, s \in \omega$ is recursive, and for each n we have

$$\tau^0_n \leq_{\text{lex}} \tau^1_n \leq_{\text{lex}} \cdots \leq_{\text{lex}} \tau^s_n \leq_{\text{lex}} \tau^{s+1}_n \leq_{\text{lex}} \cdots$$

and $\tau_n = \lim_s \tau^s_n$. Since Q' is a lexicographically final segment of $Q = [T]$, it follows that

$$Q' = \{g \in Q \mid \neg \exists n\, (\tau_n \subset g)\} = \{g \in Q \mid \neg \exists s\, \exists n\, (\tau^s_n \subset g)\}.$$

Thus Q' is Π^0_1, and our lemma is proved. □

Remark 9.9. We do not know the answer to the following question. If Q is a thin perfect Π^0_1 subset of 2^ω, does it follow that the Turing upward closure $\widehat{Q} = \{f \in 2^\omega \mid (\exists g \leq_T f)\, (g \in Q)\}$ is of measure 0?

Lemma 9.10. *Let Q be a nonempty thin Π^0_1 subset of 2^ω. If f is random, and if $g \in Q$ is almost recursive, then $f \not\leq_T g$. In particular, $R \not\leq_w Q$.*

Proof. Assume for a contradiction that $f \in R$ and $f \leq_T g$. Since g is almost recursive, by Theorem 4.18 we have $f \leq_{tt} g$, i.e., $f = \Phi(g)$ where $\Phi : 2^\omega \to 2^\omega$ is a total recursive functional. Since R is Σ^0_2, let P be a Π^0_1 subset of R such that $f \in P$. Put $\overline{Q} = \{\overline{g} \in Q \mid \Phi(\overline{g}) \in P\}$. Then $\overline{Q}$ is a Π^0_1 subset of Q. By part 1 of Lemma 9.4, $\overline{Q}$ is thin. Note also that $g \in \overline{Q}$, since $\Phi(g) = f \in P$. Put $\overline{P} = \{\Phi(\overline{g}) \mid \overline{g} \in \overline{Q}\}$. Clearly $\overline{P} \subseteq P \subseteq R$, and by Theorem 4.7 we have that $\overline{P}$ is Π^0_1. Moreover $f = \Phi(g) \in \overline{P}$, so $\overline{P}$ is nonempty, so by Lemma 8.8 we have $\mu(\overline{P}) > 0$. On the other hand, by part 2 of Lemma 9.4, $\overline{P}$ is thin, hence by Lemma 9.8 we have $\mu(\overline{P}) = 0$. This contradiction proves the first part of our lemma. For the second part, since Q is nonempty, the Almost Recursive Basis Theorem 4.19 gives $g \in Q$ such that g is almost recursive. By the first part of the lemma, $f \not\leq_T g$ for all $f \in R$. It follows that $R \not\leq_w Q$. □

Lemma 9.11. *Let $f \in 2^\omega$ be random. If $g \leq_{tt} f$ is nonrecursive, then there exists $\overline{f} \in 2^\omega$ such that $\overline{f} \equiv_T g$ and $\overline{f}$ is random.*

Proof. This lemma has been stated by Demuth [14, Lemma 30]. The proof is in Kautz's thesis [27, Theorem IV.3.16]. □

Lemma 9.12. *Let $f \in 2^\omega$ be random and almost recursive. If $g \leq_T f$ is nonrecursive, then there exists $\overline{f} \in 2^\omega$ such that $\overline{f} \equiv_T g$ and $\overline{f}$ is random.*

Proof. This follows from Lemma 9.11 because, by Theorem 4.18, if f is almost recursive then $g \leq_T f$ implies $g \leq_{tt} f$. □

Lemma 9.13. *Let Q be a nonempty thin perfect Π^0_1 subset of 2^ω. If $g \in Q$, and if $f \in 2^\omega$ is random and almost recursive, then $g \not\leq_T f$. In particular $Q \not\leq_w R$, and $Q \not\leq_w P$ for all Π^0_1 sets $P \subseteq 2^\omega$ of positive measure.*

Proof. Assume for a contradiction that $g \in Q$ and $g \leq_T f$. Since f is almost recursive, g is almost recursive. Since f is random and almost recursive, Lemma 9.12 gives us $\overline{f} \equiv_T g$ such that $\overline{f}$ is random. This contradicts Lemma 9.10. The first part of our lemma is now proved. For the second part, let $P \subseteq 2^\omega$ be Π^0_1 of positive measure. Since R is Σ^0_2 of measure 1, we can find a Π^0_1 set $P' \subseteq P \cap R$ which is of positive measure. By the Almost Recursive Basis Theorem 4.19, let $f \in P'$ be almost recursive. By the first part of our lemma, we have $g \not\leq_T f$ for all $g \in Q$. Thus $Q \not\leq_w P'$. It follows that $Q \not\leq_w P$ and $Q \not\leq_w R$. □

Summarizing, we have:

Lemma 9.14. *Let $f, g \in 2^\omega$ be almost recursive. Assume that f is random, and assume that $g \in Q$ where Q is a thin perfect Π^0_1 subset of 2^ω. Then $f \not\leq_T g$ and $g \not\leq_T f$, i.e., the Turing degrees of f and g are incomparable.*

Proof. This is immediate from Lemmas 9.10 and 9.13. □

Theorem 9.15. *Let $\mathbf{q} = \deg_w(Q)$ where Q is a nonempty thin perfect Π^0_1 subset of 2^ω. Let $\mathbf{r} = \deg_w(R)$ where $R = \{f \in 2^\omega \mid f \text{ is random}\}$. Then $\mathbf{q}$ and $\mathbf{r}$ are incomparable weak degrees in $\mathcal{P}_w$.*

Proof. Obviously $\mathbf{q} \in \mathcal{P}_w$. Theorem 8.10 implies that $\mathbf{r} \in \mathcal{P}_w$. By Lemma 9.10 we have $\mathbf{r} \not\leq \mathbf{q}$. By Lemma 9.13 we have $\mathbf{q} \not\leq \mathbf{r}$. □

Corollary 9.16. *There exist $\mathbf{0} < \mathbf{q} < \mathbf{q}^*$ in $\mathcal{P}_w$ such that $\mathbf{q}$ is separating and $\mathbf{q}^*$ is not separating. Indeed, every separating $\mathbf{s} \in \mathcal{P}_w$ which is $\leq \mathbf{q}^*$ is $\leq \mathbf{q}$.*

Proof. By Martin/Pour-El [38] let $Q \subseteq 2^\omega$ be a thin perfect Π^0_1 set which is separating. Put $\mathbf{q} = \deg_w(Q)$ and $\mathbf{q}^* = \sup(\mathbf{q}, \mathbf{r})$. By Theorem 9.15 we have $\mathbf{q} < \mathbf{q}^*$. If $\mathbf{s}$ were separating and $\leq \mathbf{q}^*$ but not $\leq \mathbf{q}$, then this would contradict Theorem 7.5. □

10 Some additional natural examples in $\mathcal{P}_w$

In this section we present some additional natural examples in $\mathcal{P}_w$, including a hierarchy of weak degrees in $\mathcal{P}_w$ corresponding to the transfinite Ackermann hierarchy from proof theory.

Definition 10.1. Put $\mathrm{DNR} = \{f \in \omega^\omega \mid \forall n\,(f(n) \neq \{n\}(n))\}$, the set of *diagonally nonrecursive* functions. The set of Turing degrees of members of DNR has been studied by Jockusch [25]. Note that DNR is nonempty and Π^0_1. If $h : \omega \to \omega$ is a recursive function such that $h(n) \geq 2$ for all n, put $\mathrm{DNR}_h = \{f \in \mathrm{DNR} \mid \forall n\,(f(n) < h(n))\}$, the set of h-bounded DNR functions. In addition, put $\mathrm{DNR}_{\mathrm{REC}} = \bigcup\{\mathrm{DNR}_h \mid h \text{ is recursive}\}$, the set of recursively bounded DNR functions.

Remark 10.2. Trivially

$$\mathrm{DNR} \supset \mathrm{DNR}_{\mathrm{REC}} \supset \mathrm{DNR}_h,$$

hence $\mathrm{DNR} \leq_s \mathrm{DNR}_{\mathrm{REC}} \leq_s \mathrm{DNR}_h$, hence $\mathrm{DNR} \leq_w \mathrm{DNR}_{\mathrm{REC}} \leq_w \mathrm{DNR}_h$. According to Ambos-Spies et al [2, Theorems 1.8 and 1.9], we have strict inequalities

$$\mathrm{DNR} <_w \mathrm{DNR}_{\mathrm{REC}} <_w \mathrm{DNR}_h.$$

As in Section 8, let R be the set of random reals. An argument of Kurtz (see Jockusch [25, Proposition 3]) shows that $\mathrm{DNR}_h \leq_w R$ provided h is such that $\sum_{n=0}^{\infty} 1/h(n) < \infty$, for example $h(n) = \max(n^2, 2)$.

Remark 10.3. Since DNR_h is nonempty, recursively bounded, and Π^0_1, we have $\deg_s(\mathrm{DNR}_h) \in \mathcal{P}_s$ and $\deg_w(\mathrm{DNR}_h) \in \mathcal{P}_w$. Although DNR and $\mathrm{DNR}_{\mathrm{REC}}$ are not recursively bounded, it will be shown in Simpson [48] that $\deg_w(\mathrm{DNR}) \in \mathcal{P}_w$ and $\deg_w(\mathrm{DNR}_{\mathrm{REC}}) \in \mathcal{P}_w$. We do not know whether $\deg_s(\mathrm{DNR}) \in \mathcal{P}_s$, or whether $\deg_s(\mathrm{DNR}_{\mathrm{REC}}) \in \mathcal{P}_s$. Put $\mathbf{d} = \deg_w(\mathrm{DNR})$, $\mathbf{d}_{\mathrm{REC}} = \deg_w(\mathrm{DNR}_{\mathrm{REC}})$, $\mathbf{d}_h = \deg_w(\mathrm{DNR}_h)$, $\mathbf{r} = \deg_w(R)$. Summarizing, we have the following result.

Theorem 10.4. *In $\mathcal{P}_w$ we have*

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathrm{REC}} < \mathbf{d}_h < \mathbf{r} < \mathbf{1}$$

for all sufficiently fast-growing recursive functions $h : \omega \to \omega$.

Proof. This follows from part 1 of Theorem 8.12 plus the results of Ambos-Spies et al [2] and Simpson [48] which were mentioned in Remarks 10.2 and 10.3 above. □

Remark 10.5. Some of our natural weak degrees are closely related to certain formal systems which arise naturally in the foundations of mathematics. Namely, the weak degrees $\mathbf{1}$, $\mathbf{r}$, $\mathbf{d}$ correspond to the systems WKL_0, WWKL_0, $\mathsf{RCA}_0 + \mathrm{DNR}$ respectively. Each of these subsystems of second order arithmetic is of interest in connection with the well known foundational program of reverse mathematics. See Simpson [52, Chapter IV and Section X.1], Yu/Simpson [61], Brown/Giusto/Simpson [7], and Giusto/Simpson [23]. The standard reference for reverse mathematics is Simpson [52].

Remark 10.6. From the recursion-theoretic viewpoint, there are some subtle issues concerning naturalness of the mass problems DNR, $\mathrm{DNR}_{\mathrm{REC}}$, DNR_h and of their weak degrees $\mathbf{d}$, $\mathbf{d}_{\mathrm{REC}}$, $\mathbf{d}_h$. First, DNR, $\mathrm{DNR}_{\mathrm{REC}}$, DNR_h are not invariant under recursive permutations of ω, and on this basis it is possible to question their recursion-theoretic naturalness. (See also the discussion of the recursion-theoretic Erlanger Programm in Rogers [45, Chapter 4].) On the other hand, this objection clearly does not apply to the weak degrees $\mathbf{d}$, $\mathbf{d}_{\mathrm{REC}}$, $\mathbf{d}_h$, because all weak and strong degrees are invariant under recursive permutations of ω. Second, one may note that our definitions of DNR, $\mathrm{DNR}_{\mathrm{REC}}$, DNR_h and their weak degrees $\mathbf{d}$, $\mathbf{d}_{\mathrm{REC}}$, $\mathbf{d}_h$ depend upon a particular choice of Gödel numbering of Turing machines, because the function $n \mapsto \{n\}(n)$ is defined in terms of such a Gödel numbering. (See also the discussion of acceptable Gödel numberings in Rogers [45].) We shall now present a method of overcoming this objection. Our idea is to replace the particular partial recursive function $n \mapsto \{n\}(n)$ by an arbitrary partial recursive function $n \mapsto \psi(n)$. This will answer the objection, because the extensional concept "partial recursive function" is independent of the choice of Gödel numbering.

Definition 10.7. Let D be the set of $g \in \omega^\omega$ such that for all partial recursive functions ψ there exists $f \leq_T g$ such that $\forall n\,(f(n) \neq \psi(n))$. Let $\mathrm{D}_{\mathrm{REC}}$ be the set of $g \in \omega^\omega$ such that for all partial recursive functions ψ there exists $f \leq_T g$ such that $\forall n\,(f(n) < h(n)$ and $f(n) \neq \psi(n))$ for some recursive function $h : \omega \to \omega$.

Remark 10.8. Using the S-m-n Theorem, it is easy to see that $\mathrm{DNR} \equiv_w \mathrm{D}$ and $\mathrm{DNR}_{\mathrm{REC}} \equiv_w \mathrm{D}_{\mathrm{REC}}$. (See also the proof of Theorem 10.10 below.) Thus the weak degrees $\mathbf{d}$ and $\mathbf{d}_{\mathrm{REC}}$ are natural in the sense that they can be defined in a way that does not depend on the choice of Gödel numbering. What about $\mathbf{d}_h$ where h is a fixed recursive function? Let D_h be the set of $g \in \omega^\omega$ such that for all partial recursive functions ψ there exists $f \leq_T g$ such that $\forall n\,(f(n) < h(n)$ and $f(n) \neq \psi(n))$. It is not clear that $\mathrm{DNR}_h \equiv_w \mathrm{D}_h$ for a fixed recursive function h, but we have the following definition and theorem for classes of recursive functions.

Definition 10.9. If C is a class of recursive functions, put $\mathrm{DNR}_C = \bigcup_{h \in C} \mathrm{DNR}_h$. Let D_C be the set of $g \in \omega^\omega$ such that for all partial recursive functions ψ there exists $f \leq_T g$ such that $\forall n\,(f(n) < h(n)$ and $f(n) \neq \psi(n))$ for some $h \in C$.

Theorem 10.10. *If C is closed under composition with primitive recursive functions, then $\mathrm{DNR}_C \equiv_w \mathrm{D}_C$. If there exists a uniform recursive enumeration of C, then $\deg_w(\mathrm{DNR}_C) \in \mathcal{P}_w$.*

Proof. Given $g \in \mathrm{D}_C$, let $f \leq_T g$ be as in the definition of D_C for the particular partial recursive function $\psi(n) \simeq \{n\}(n)$. Clearly $f \in \mathrm{DNR}_C$, and this shows that $\mathrm{DNR}_C \leq_w \mathrm{D}_C$. Conversely, to show that $\mathrm{DNR}_C \geq_w \mathrm{D}_C$, let $f \in \mathrm{DNR}_C$ be given, say $f \in \mathrm{DNR}_h$ where $h \in C$. Given a partial recursive function ψ, apply the S-m-n Theorem to get a primitive recursive function $p : \omega \to \omega$ such that $\{p(n)\}(p(n)) \simeq \psi(n)$ for all n. Then we have $\forall n\,(f(p(n)) < h(p(n))$ and

$f(p(n)) \neq \psi(n))$. Moreover, the function $n \mapsto f(p(n))$ is Turing reducible to f, and the function $n \mapsto h(p(n))$ belongs to C. Thus $f \in \mathrm{D}_C$. We have now shown that $\mathrm{DNR}_C \subseteq \mathrm{D}_C$. It follows that $\mathrm{DNR}_C \geq_w \mathrm{D}_C$, and we have proved the first part of the theorem. To prove the second part, let h_n, $n \in \omega$ be a uniform recursive enumeration of C. Putting $P_n = \mathrm{DNR}_{h_n}$, we note that P_n is uniformly recursively bounded and Π^0_1. As in the proof of Theorem 4.10, put $P^*_n = \{G_f \mid f \in P_n\}$. Thus P^*_n is a uniformly Π^0_1 subset of 2^ω which is uniformly recursively homeomorphic to P_n. Put $S^* = \bigcup_n P^*_n$. Then S^* is a Σ^0_2 subset of 2^ω, and by construction S^* is Turing degree isomorphic to $\bigcup_n P_n = \mathrm{DNR}_C$. Now apply Theorem 10.11 to find a Π^0_1 set $P^* \subseteq 2^\omega$ such that P^* is Turing degree isomorphic to S^*. Then $\deg_w(\mathrm{DNR}_C) = \deg_w(S^*) = \deg_w(P^*) \in \mathcal{P}_w$. $\square$

Theorem 10.11. *Let S be a Σ^0_2 subset of 2^ω. Then we can find a Π^0_1 set $P \subseteq 2^\omega$ such that P is Turing degree isomorphic to S.*

Proof. We may safely assume that S is nonempty. By hypothesis $S = \bigcup_n P_n$ where P_n, $n \in \omega$ is a recursive sequence of nonempty Π^0_1 subsets of 2^ω. We use a construction from Binns/Simpson [6, Definition 4.2]. Let T_n, $n \in \omega$ be a recursive sequence of infinite recursive subtrees of $2^{<\omega}$ such that $P_n = [T_n]$, the set of paths through T_n. Put

$$\widetilde{T_0} = \{\sigma^\frown\langle i\rangle \mid \sigma \in T_0, i \in \{0,1\}, \sigma^\frown\langle i\rangle \notin T_0\}.$$

We may safely assume that $\widetilde{T_0}$ is infinite. Note that the strings in $\widetilde{T_0}$ are pairwise incompatible. Let τ_n, $n \in \omega$ be a one-to-one recursive enumeration of $\widetilde{T_0}$. Put $T = T_0 \cup \bigcup_n \{\tau_n{}^\frown\sigma \mid \sigma \in T_n\}$. Thus T is an infinite recursive subtree of $2^{<\omega}$. Let $P = [T]$, the set of paths through T. Thus P is a nonempty Π^0_1 subset of 2^ω. By construction we have $P = P_0 \cup \bigcup_n \{\tau_n{}^\frown f \mid f \in P_n\}$, hence P is Turing degree isomorphic to $\bigcup_n P_n = S$. $\square$

Remark 10.12. In the proof of Theorem 10.11, note that $\mathbf{p} = \inf_n \mathbf{p}_n$, where $\mathbf{p} = \deg_w(P)$ and $\mathbf{p}_n = \deg_w(P_n)$. Thus the proof shows that $\mathcal{P}_w$ is closed under effective infima.

Remark 10.13. If C is a class of recursive functions satisfying the hypotheses of Theorem 10.10, put $\mathbf{d}_C = \deg_w(\mathrm{DNR}_C)$. We have seen that $\mathbf{d}_C \in \mathcal{P}_w$ and that $\mathbf{d}_C$ is natural in the sense that it can be defined in a way which does not depend on the choice of Gödel numbering. Moreover, if $C^* \supset C$ is another such class, then $\mathbf{d}_{C^*} \leq \mathbf{d}_C$, and according to Ambos-Spies et al [2, Theorem 1.9] we have strict inequality $\mathbf{d}_{C^*} < \mathbf{d}_C$ provided C^* contains a function which "grows much faster than" all functions in C. There are many examples and problems here.

Example 10.14. For each constructive ordinal α, let C_α be the class of recursive functions obtained at levels $< \omega \cdot (1 + \alpha)$ of the transfinite Ackermann hierarchy. (See for instance Wainer [60].) Thus C_0 is the class of primitive recursive functions, C_1 is the class of functions which are primitive recursive relative to the Ackermann function, etc. Putting $\mathbf{d}_\alpha = \mathbf{d}_{C_\alpha}$ we have a transfinite descending sequence

$$\mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \mathbf{d}_{\alpha+1} > \cdots$$

in $\mathcal{P}_w$. Moreover, if α is a limit ordinal, then $\mathbf{d}_\alpha = \inf_{\beta<\alpha} \mathbf{d}_\beta$. Thus we see a rich set of natural degrees in $\mathcal{P}_w$ which are related to subrecursive hierarchies of the kind that arise in Gentzen-style proof theory.

Remark 10.15. Let us assume that we are using one of the standard Gödel numberings of Turing machines which appear in the literature. Then the function $p(n)$ in the proof of Theorem 10.10 can be chosen to be bounded by a linear function. Therefore, instead of assuming that C is closed under composition with primitive recursive functions, we could assume merely that for all $h \in C$ and $c \geq 1$ there exists $h^*_c \in C$ such that $h^*_c(n) \geq h(m)$ for all $m \leq c \cdot (n+1)$. In particular, we can take C to be various well known computational complexity classes such as PTIME, EXPTIME, etc. For each such class, Theorem 10.10 shows that the weak degree $\mathbf{d}_C \in \mathcal{P}_w$ is natural in that its definition does not depend on the choice of a standard Gödel numbering.

Example 10.16. In $\mathcal{P}_w$ we have

$$\mathbf{d}_{\mathrm{PTIME}} > \mathbf{d}_{\mathrm{EXPTIME}} > \cdots$$

etc. Thus we see a rich set of natural degrees in $\mathcal{P}_w$ which are related to computational complexity.

References

[1] K. Ambos-Spies, G. H. Müller, and G. E. Sacks, editors. *Recursion Theory Week*. Number 1432 in Lecture Notes in Mathematics. Springer-Verlag, 1990. IX + 393 pages.

[2] Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A. Slaman. Comparing DNR and WWKL. *Journal of Symbolic Logic*, 69:1089–1104, 2004.

[3] Stephen Binns. *The Medvedev and Muchnik Lattices of* Π^0_1 *Classes*. PhD thesis, Pennsylvania State University, August 2003. VII + 80 pages.

[4] Stephen Binns. A splitting theorem for the Medvedev and Muchnik lattices. *Mathematical Logic Quarterly*, 49:327–335, 2003.

[5] Stephen Binns. Small Π^0_1 classes. *Archive for Mathematical Logic*, 45:393–410, 2006.

[6] Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Archive for Mathematical Logic*, 43:399–414, 2004.

[7] Douglas K. Brown, Mariagnese Giusto, and Stephen G. Simpson. Vitali's theorem and WWKL. *Archive for Mathematical Logic*, 41:191–206, 2002.

[8] P. Bruscoli and A. Guglielmi, editors. *Summer School and Workshop on Proof Theory, Computation and Complexity, Dresden, 2003.* Electronic Notes in Theoretical Computer Science. Elsevier, 2004. To appear.

[9] Douglas Cenzer and Peter G. Hinman. Density of the Medvedev lattice of Π^0_1 classes. *Archive for Mathematical Logic*, 42:583–600, 2003.

[10] Douglas Cenzer and Jeffrey B. Remmel. Π^0_1 classes in mathematics. In [20], pages 623–821, 1998.

[11] Peter Cholak, Richard Coles, Rod Downey, and Eberhard Herrmann. Automorphisms of the lattice of Π^0_1 classes; perfect thin classes and ANC degrees. *Transactions of the American Mathematical Society*, 353:4899–4924, 2001.

[12] COMP-THY e-mail list. `http://listserv.nd.edu/archives/comp-thy.html`, September 1995 to the present.

[13] S. B. Cooper, T. A. Slaman, and S. S. Wainer, editors. *Computability, Enumerability, Unsolvability: Directions in Recursion Theory.* Number 224 in London Mathematical Society Lecture Notes. Cambridge University Press, 1996. VII + 347 pages.

[14] Oswald Demuth. A notion of semigenericity. *Commentationes Mathematicae Universitatis Carolinae*, 28:71–84, 1987.

[15] Rodney G. Downey, Carl G. Jockusch, Jr., and Michael Stob. Array nonrecursive sets and multiple permitting arguments. In [1], pages 141–174, 1990.

[16] Rodney G. Downey, Carl G. Jockusch, Jr., and Michael Stob. Array nonrecursive degrees and genericity. In [13], pages 93–105, 1996.

[17] F. R. Drake and J. K. Truss, editors. *Logic Colloquium '86.* Studies in Logic and the Foundations of Mathematics. North-Holland, 1988. X + 342 pages.

[18] H.-D. Ebbinghaus, J. Fernandez-Prida, M. Garrido, D. Lascar, and M. Rodriguez Artalejo, editors. *Logic Colloquium '87.* Studies in Logic and the Foundations of Mathematics. North-Holland, 1989. X + 375 pages.

[19] H.-D. Ebbinghaus, G. H. Müller, and G. E. Sacks, editors. *Recursion Theory Week.* Number 1141 in Lecture Notes in Mathematics. Springer-Verlag, 1985. IX + 418 pages.

[20] Y. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel, editors. *Handbook of Recursive Mathematics*. Studies in Logic and the Foundations of Mathematics. North-Holland, 1998. Volumes 1 and 2, XLVI + 1372 pages.

[21] J.-E. Fenstad, I. T. Frolov, and R. Hilpinen, editors. *Logic, Methodology and Philosophy of Science VIII*. Studies in Logic and the Foundations of Mathematics. North-Holland, 1989. XVII + 702 pages.

[22] FOM e-mail list. `http://www.cs.nyu.edu/mailman/listinfo/fom/`, September 1997 to the present.

[23] Mariagnese Giusto and Stephen G. Simpson. Located sets and reverse mathematics. *Journal of Symbolic Logic*, 65:1451–1480, 2000.

[24] J. Gruska, B. Rovan, and J. Wiedermann, editors. *Mathematical Foundations of Computer Science*. Number 233 in Lecture Notes in Computer Science. Springer-Verlag, 1986. IX + 650 pages.

[25] Carl G. Jockusch, Jr. Degrees of functions with no fixed points. In [21], pages 191–201, 1989.

[26] Carl G. Jockusch, Jr. and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:35–56, 1972.

[27] Steven M. Kautz. *Degrees of Random Sets*. PhD thesis, Cornell University, 1991. X + 89 pages.

[28] Stuart A. Kurtz. *Randomness and Genericity in the Degrees of Unsolvability*. PhD thesis, University of Illinois at Urbana-Champaign, 1981. VII + 131 pages.

[29] Antonín Kučera. Measure, Π^0_1 classes and complete extensions of PA. In [19], pages 245–259, 1985.

[30] Antonín Kučera. An alternative, priority-free, solution to Post's problem. In [24], pages 493–500, 1986.

[31] Antonín Kučera. On the role of $\mathbf{0}'$ in recursion theory. In [17], pages 133–141, 1988.

[32] Antonín Kučera. On the use of diagonally nonrecursive functions. In [18], pages 219–239, 1989.

[33] Antonín Kučera. Randomness and generalizations of fixed point free functions. In [1], pages 245–254, 1990.

[34] Antonín Kučera. On relative randomness. *Annals of Pure and Applied Logic*, 63:61–67, 1993.

[35] Antonín Kučera and Sebastiaan A. Terwijn. Lowness for the class of random sets. *Journal of Symbolic Logic*, 64:1396–1402, 1999.

[36] Manuel Lerman. *Degrees of Unsolvability.* Perspectives in Mathematical Logic. Springer-Verlag, 1983. XIII + 307 pages.

[37] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and its Applications.* Graduate Texts in Computer Science. Springer-Verlag, 2nd edition, 1997. XX + 637 pages.

[38] Donald A. Martin and Marian B. Pour-El. Axiomatizable theories with few axiomatizable extensions. *Journal of Symbolic Logic*, 35:205–209, 1970.

[39] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[40] Yuri T. Medvedev. Degrees of difficulty of mass problems. *Doklady Akademii Nauk SSSR, n.s.*, 104:501–504, 1955. In Russian.

[41] A. A. Muchnik. On strong and weak reducibilities of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. In Russian.

[42] Piergiorgio Odifreddi. *Classical Recursion Theory.* Number 125 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1989. XVII + 668 pages.

[43] Piergiorgio Odifreddi. *Classical Recursion Theory, Volume 2.* Number 143 in Studies in Logic and the Foundations of Mathematics. North-Holland, 1999. XVI + 949 pages.

[44] Emil L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, 50:284–316, 1944.

[45] Hartley Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, 1967. XIX + 482 pages.

[46] Gerald E. Sacks. *Degrees of Unsolvability.* Number 55 in Annals of Mathematics Studies. Princeton University Press, 1963. IX + 174 pages.

[47] S. G. Simpson, editor. *Reverse Mathematics 2001*, volume 21 of *Lecture Notes in Logic.* Association for Symbolic Logic, 2005. X + 401 pages.

[48] Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society.* Preprint, August 2004, 15 pages, accepted for publication.

[49] Stephen G. Simpson. Mass problems. Preprint, May 24, 2004, 24 pages, submitted for publication.

[50] Stephen G. Simpson. FOM: natural r.e. degrees; Pi01 classes. FOM e-mail list [22], August 13, 1999.

[51] Stephen G. Simpson. FOM: priority arguments; Kleene-r.e. degrees; Pi01 classes. FOM e-mail list [22], August 16, 1999.

[52] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, 1999. XIV + 445 pages.

[53] Stephen G. Simpson. Medvedev degrees of nonempty Pi^0_1 subsets of 2^omega. COMP-THY e-mail list [12], June 9, 2000.

[54] Stephen G. Simpson. Π^0_1 sets and models of WKL_0. In [47], pages 352–378. 2005.

[55] Stephen G. Simpson and Theodore A. Slaman. Medvedev degrees of Π^0_1 subsets of 2^ω. Preprint, July 2001, 4 pages, in preparation.

[56] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic. Springer-Verlag, 1987. XVIII + 437 pages.

[57] Andrea Sorbi. The Medvedev lattice of degrees of difficulty. In [13], pages 289–312, 1996.

[58] Sebastiaan A. Terwijn. The Medvedev lattice of computably closed sets. *Archive for Mathematical Logic*. Preprint, June 2004, 22 pages, accepted for publication.

[59] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.

[60] Stanley S. Wainer. A classification of the ordinal recursive functions. *Archiv für Mathematische Logik und Grundlagenforschung*, 13:136–153, 1970.

[61] Xiaokang Yu and Stephen G. Simpson. Measure theory and weak König's lemma. *Archive for Mathematical Logic*, 30:171–180, 1990.