RECURSIVE ASPECTS
OF
DESCRIPTIVE SET THEORY

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with a chapter by Stephen Simpson

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Chapter 9

Bqo Theory and Fraïssé’s Conjecture
by Stephen G. Simpson

Let \( L \) and \( M \) be linearly ordered sets. We write \( L \leq M \) to mean that \( L \) is embeddable into \( M \), i.e. there exists a subset of \( M \) isomorphic to \( L \). We write \( L \equiv M \) to mean that \( L \leq M \) and \( M \leq L \). We write \( L < M \) to mean that \( L \leq M \) and \( M \not\leq L \). We write \( L \mid M \) to mean that \( L \) and \( M \) are incomparable under embeddability, i.e. \( L \not\leq M \) and \( M \not\leq L \).

Fraïssé’s conjecture [2] is the statement that, among countable linearly ordered sets, there are no infinite descending sequences

\[ L_0 > L_1 > \ldots > L_n > \ldots (n \in \omega) \]

and no infinite antichains

\[ L_i \mid L_j \quad (i, j \in \omega, i \neq j). \]

The purpose of this chapter is to explain Laver’s proof [7] of Fraïssé’s conjecture. The proof depends heavily on Nash-Williams’ theory [12] of better quasiorderings. The latter theory will be presented here as an application of a theorem of Galvin and Prikry [3] on Borel partitions.

In order to motivate Nash-Williams’ concept of better quasiordering (bqo), we first discuss the closely related but simpler concept of well quasiordering (wqo).

A quasiordered (i.e., qo) set is a set \( Q \) endowed with a binary relation \( \leq \) which is transitive \((x \leq y, y \leq z \text{ imply } x \leq z)\) and reflexive \((x \leq x \text{ for }\)

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For all $x \in Q$. For $x,y \in Q$ we write $x \equiv y$ to mean that $x \leq y$ and $y \leq x$; $x < y$ to mean that $x \leq y$ and $y \not\leq x$; and $x | y$ to mean that $z \not\leq y$ and $y \not\leq x$. A quasiordered set $Q$ is said to be well quasiordered (wqo) if it contains no infinite descending sequence $x_0 > x_1 > \ldots > x_n > \ldots$ $(n \in \omega)$ and no infinite antichain $z_i \mid z_j (i,j \in \omega, i \neq j)$.

Thus Fraïssé's conjecture may be rephrased as follows: the set of all countable linearly ordered sets is wqo under embeddability. An obvious strategy would be to construct an elaborate theory stating that certain large classes of qo sets are wqo. One could then hope for Fraïssé's conjecture to fall out as a corollary.

Such an elaborate wqo theory does in fact exist. One of the major theorems of wqo theory reads as follows. Let $Q$ be a wqo. Let $Q^{<\omega}$ be the set of finite sequences of elements of $Q$ quasiordered by $(a_1 \ldots a_m) \leq (b_1 \ldots b_n)$ if and only if there exist $k_1 < \ldots < k_m \leq n$ such that $a_i \leq b_{k_i}$. Then $Q^{<\omega}$ is wqo. (This theorem is due to Graham Higman. For the proof, plus an excellent survey of wqo theory, see Laver [9, § 1].)

Unfortunately, it turns out that wqo theory alone is not sufficiently far-reaching to provide a proof of Fraïssé's conjecture. The difficulty is that the class of wqo sets fails to be closed under certain infinitary closure operations. For instance, let $Q^\omega$ be the set of $\omega$-sequences of elements of $Q$ quasiordered by $(a_n)_{n \in \omega} \leq (b_n)_{n \in \omega}$ if and only if there exist $n_0 < n_1 < \ldots < n_i < \ldots (i \in \omega)$ such that $a_i \leq b_{n_i}$. It is not hard to devise a wqo $Q_1$ such that $(Q_1)^\omega$ is not wqo. Namely, let $Q_1 = \{(i,j) : i < j < \omega\}$ quasiordered by $(i,j) \leq (k,l)$ if and only if either $i = k$ and $j \leq l$, or $j < k$. This counterexample is due to Richard Rado (see Laver [9, § 1]).

In order to prove Fraïssé's conjecture we need a concept stronger than well quasiordering, namely better quasiordering (bqo). It will turn out that every bqo is wqo and that every non-pathological wqo is bqo. Furthermore the class of bqos enjoys strong infinitary closure properties; for instance, if $Q$ is bqo then $Q^\omega$ is bqo (indeed $Q^\alpha$ is bqo for all ordinals $\alpha$). Fraïssé's conjecture will be proved by showing that the class of countable linearly ordered sets (indeed the wider class of scattered linearly ordered sets) is bqo under embeddability.
The following exercises are provided for the convenience of the reader and are not essential for the rest of the chapter.

9.1. Exercise. Verify that \( Q_1 \) is wqo but \( (Q_1)^\omega \) is not wqo.

9.2. Exercise. An \( \omega \)-sequence \( (a_n) \in Q^\omega \) is called bad if \( a_m \preceq a_n \) for all \( n \) and \( m < n \). Show that the following assertions are pairwise equivalent.

(i) \( Q \) is wqo.

(ii) There is no bad \( \omega \)-sequence \( (a_n) \in Q^\omega \).

(iii) For all \( (a_n) \in Q^\omega \) there exist \( n_0 < n_1 \ldots < n_i < \ldots (i \in \omega) \) such that \( a_{n_0} \preceq a_{n_1} \preceq \ldots \preceq a_{n_i} \preceq \ldots (i \in \omega) \).

(Hint: use Ramsey's Theorem.)

9.3. Exercise.

(i) Show that any well-ordered set is wqo.

(ii) Show that if \( Q \) is the union of two subsets each wqo in the induced quasiordering, then \( Q \) is wqo.

(iii) Show that if \( Q_1 \) and \( Q_2 \) are wqo then \( Q_1 \times Q_2 \) with the product quasiordering is wqo.

(iv) Show that a wqo sum of wqos is wqo.

9.4. Exercise. A bad \( \omega \)-sequence \( (a_n) \in Q^\omega \) is called minimal bad if there is no bad \( \omega \)-sequence \( (b_n) \in Q^\omega \) such that \( \forall m \exists n b_m \preceq a_n \) and \( \exists m \exists n b_m < a_n \). Show that if \( Q \) is well founded but not wqo then there is a minimal bad \( (a_n) \in Q^\omega \).

9.5. Exercise. Use the result of the previous exercise to show that if \( Q \)
Historical Note. The wqo concept occurs in the Ph.D. thesis of Irving Kaplansky [6]. One of the earliest applications of wqo theory is the following result due independently to A.I. Malcev and B.H. Neumann. Let $K$ be a field and let $G$ be a linearly ordered group. Then the group algebra $K(G)$ is embeddable in a skew field. For proof and references see Higman [5].

Theorem of Galvin and Prikry.

Given an infinite set $A \subseteq \omega$, we denote by $[A]^\omega$ the set of infinite subsets of $A$ and $[A]^{<\omega}$ the set of finite subsets of $A$. For $s \in [\omega]^{<\omega}$ and $U \in [\omega]^\omega$ we write

$$U/s = \{ n \in U : n > i \text{ for all } i \in s \}$$

and

$$[s, U] = \{ X \in [\omega]^\omega : s \subseteq X \subseteq s \cup U \}.$$

We endow $[\omega]^\omega$ with the usual topology whose basic open sets are of the form $[s, \omega/s]$.

The following is a special case of the theorem of Galvin and Prikry.

9.6. Theorem. Let $\mathcal{O}$ be an open subset of $[\omega]^\omega$. Then there exists $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq \mathcal{O}$ or $[X]^\omega \cap \mathcal{O} = \emptyset$.

To prove this we need some special terminology. We call $[s, U]$ good if there is no $V \in [U]^\omega$ such that $[s, V] \subseteq \mathcal{O}$. We call $[s, U]$ strongly good if $[s, U]$ is good and, for all $n \in U$, $[s \cup \{n\}, U/(n)]$ is good.

9.7. Lemma. If $[s, U]$ is good then there exists $V \in [U]^\omega$ such that $[s, V]$ is strongly good.

Proof. Suppose the conclusion fails. Put $W_0 = U/s$. Assume inductively that we have chosen $n_0 < \ldots < n_{i-1} < \min(W_i)$ where $W_i \subseteq U$. Then
$[s, W_i]$ is good but not strongly good so choose $n_i \in W_i$ such that $[s \cup \{n_i\}, W_i/\{n_i\}]$ is not good. Choose $W_{i+1} \subseteq W_i/\{n_i\}$ so that $[s \cup \{n_i\}, W_{i+1}] \subseteq \mathcal{O}$. Finally put $V = \{n_i : i \in \omega\}$. Then clearly $[s, V] \subseteq \mathcal{O}$ so $[s, U]$ is not good. This proves the lemma.

We now prove the theorem. If $[U]^{\omega} \subseteq \mathcal{O}$ for some $U \in [\omega]^{\omega}$ we are done, so assume no such $U$ exists. Hence $[\mathcal{O}, \omega]$ is good. Put $U_0 = \omega$ and assume inductively that we have chosen $n_0 < \ldots < n_{i-1} < \min(U_i)$ such that $[s, U_i]$ is good for all $s \subseteq \{n_0, \ldots, n_{i-1}\}$. Apply the lemma $2^i$ times to get $V_i \subseteq U_i$ such that $[s, V_i]$ is strongly good for all $s \subseteq \{n_0, \ldots, n_{i-1}\}$. Put $n_i = \min(V_i)$ and $U_{i+1} = V_i/\{n_i\}$. Finally put $X = \{n_i : i \in \omega\}$.

We claim that $[X]^{\omega} \cap \mathcal{O} = \emptyset$. Suppose not. Let $Y$ be an element of $[X]^{\omega} \cap \mathcal{O}$. Since $\mathcal{O}$ is open, we can find $[s, W]$ such that $Y \subseteq [s, W] \subseteq \mathcal{O}$. Let $i$ be such that $s \subseteq \{n_0, \ldots, n_{i-1}\}$ and $Y/s \subseteq U_i$. Then $U_i \cap W$ is infinite and $[s, U_i \cap W] \subseteq \mathcal{O}$ contradicting the goodness of $[s, U_i]$. This completes the proof. 

9.8. Remark. The same proof shows that Theorem 9.6 and Theorem 9.9 remain true if we replace the usual topology on $[\omega]^{\omega}$ by the Ellentuck topology with basic open sets of the form $[s, U]$ (see Ellentuck [1]).

9.9. Theorem. (Galvin, Prikry [3]). Given $A \in [\omega]^{\omega}$ and a Borel set $B$ in $[A]^{\omega}$. There exists $X \in [A]^{\omega}$ such that either $[X]^{\omega} \subseteq B$ or $[X]^{\omega} \cap B = \emptyset$.

Proof. If $B$ is open in $[A]^{\omega}$ then the desired conclusion follows from Theorem 9.6 since $[A]^{\omega}$ is homeomorphic to $[\omega]^{\omega}$ by a homeomorphism $h : [\omega]^{\omega} \to [A]^{\omega}$ such that $X \subseteq Y$ if and only if $h(X) \subseteq h(Y)$. It is also clear that the theorem holds for $B$ if and only if it holds for $[A]^{\omega} - B$.

It remains to show that if the theorem holds for Borel sets of rank $< \rho$ then it holds for Borel sets of rank $\rho$. So suppose $B = \bigcup \{B_i : i \in \omega\}$ where each $B_i$ has smaller rank than $B$. Put $A_0 = A$. Having defined $A_i$ let $n_i = \min(A_i)$ and apply the induction hypothesis $2^{i+1}$ times to get
such that \([s \cup \{n_i\}, \omega] \subseteq \cup (n_i, \omega] \subseteq \bigcup_{i} \omega] \subseteq \emptyset\).  So \([s, U] \subseteq \emptyset\).

Since \(U \in \omega\) we are odd.  Put \(U_0 = \omega\) and \(n_0, \ldots, n_{i-1} < \min(U_i)\) such that \(Y \subseteq \{n_0, \ldots, n_{i-1}\}\).

\(X = \{n_i : i \in \omega\}\).

Let \(Y = \{n_i, U_i\} \subseteq \emptyset\). Then \(U_i \cap W\) is

and a Borel set \(B\) in \(\mathcal{B}\) or \([X]\omega \cap B = \emptyset\).

follows from Theorem 9.6 and Theorem \(\omega\) by the Ellentuck.

Ellentuck [1].

and a Borel set \(B\) in \(\mathcal{B}\) or \([X]\omega \cap B = \emptyset\).

It is also clear that \(\omega - B\).

for Borel sets of rank \(\omega\) \(B = \bigcup_{i} \{B_i : i \in \omega\} = A\). Having defined

\(2^{i+1}\) times to get

\(A_{i+1} \subseteq A_i/\{n_i\}\) such that for all \(s \subseteq \{n_0, \ldots, n_i\}\) either \([s, A_{i+1}] \subseteq B_i\) or \([s, A_{i+1}] \cap B_i = \emptyset\). Finally put \(Z = \{n_i : i \in \omega\}\).

For each \(Y \in [Z]^{\omega}\) we have by construction \(Y \in B_i\) if and only if \([s, A_{i+1}] \subseteq B_i\) where \(s = Y \cap \{n_0, \ldots, n_i\}\).

Hence, for each \(i\), \(B \cap [Z]^\omega\) is open (in fact clopen) in \([Z]^\omega\).

Hence by Theorem 9.6 we can find \(X \subseteq Z\) such that either \([X]^\omega \subseteq B\) or \([X]^\omega \cap B = \emptyset\).

The following consequence of the Galvin-Prikry theorem will be used at a crucial point in the proof of Theorem 9.17.

9.10. **Theorem.** (cf. Mathias [11, § 6]). Given \(A \in [\omega]^\omega\) and a Borel measurable function \(f: [A]^\omega \to X\) where \(X\) is a metric space. There exists \(B \in [A]^\omega\) such that the restriction of \(f\) to \([B]^\omega\) is continuous.

**Proof.** In order to prove the theorem we use the following lemma (but see Remark 9.12 below).

9.11. **Lemma.** The image of \(f\) is separable.

**Proof.** Suppose not. Since \(\text{im}(f)\) is a nonseparable metric space, it contains a closed discrete set \(S\) of power \(\aleph_1\). Let \(T \subseteq \mathbb{R}\) be a set of reals of power \(\aleph_1\) with no perfect subset. Let \(h: S \to T\) be a \(1 - 1\) mapping of \(S\) onto \(T\).

Define \(g: X \to \mathbb{R}\) by

\[g(x) = \begin{cases} h(x) & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}\]

Since \(S\) is closed discrete, \(g\) is Borel measurable. Hence the composition \(gf: [A]^\omega \to \mathbb{R}\) is Borel measurable. Hence the range of \(gf\) is analytic. But \(T \subseteq \text{im}(gf) \subseteq T \cup \{0\}\) so \(\text{im}(gf)\) is uncountable with no perfect subset. This contradicts the well known theorem that every uncountable analytic set in \(\mathbb{R}\) has a perfect subset.

9.12. **Remark.** Lemma 9.11 says essentially that a Borel measurable function from a complete separable metric space into a metric space has
separable image (cf. Stone [15]). We do not really need this lemma since
in the application of Theorem 9.10 to be made later, \( \text{im}(f) \) can be assumed
separable. We have included the lemma because it is interesting in its own
right and not widely known. See also Louveau-Simpson [16].

We shall now prove Theorem 9.10. By Lemma 9.11 \( \text{im}(f) \) is separable
so let \( \{U_i : i \in \omega \} \) be a countable open base for the topology of \( \text{im}(f) \).
Define \( A_0 = A \). Supposing \( A_i \) has been defined, let \( n_i = \min(A_i) \) and
apply Theorem 9.9 \( 2^{i+1} \) times to get \( A_{i+1} \subseteq A_i / \{ n_i \} \) such that for all \( s \subseteq \{n_0, \ldots, n_i\} \) either
\( [s, A_{i+1}] \subseteq f^{-1}(U_i) \) or \( [s, A_{i+1}] \cap f^{-1}(U_i) = \emptyset \). Finally
put \( B = \{ n_i : i \in \omega \} \). Then for all \( X \in [B]^\omega \) we have \( X \in f^{-1}(U_i) \)
if and only if \( [s, A_{i+1}] \subseteq f^{-1}(U_i) \) where \( s = X \cap \{n_0, \ldots, n_i\} \). Hence
\( f^{-1}(U_i) \cap [B]^\omega \) is open (in fact clopen) in \( [B]^\omega \). Hence \( f \) is continuous on
\( [B]^\omega \). []

Better quasiordering.

Let \( Q \) be a qo set. We endow \( Q \) with the discrete topology. A \( Q \)-array is
a Borel measurable function \( f : [A]^\omega \to Q \) where \( A \in [\omega]^\omega \). A \( Q \)-array
\( f : [A]^\omega \to Q \) is called bad if there is no \( X \in [A]^\omega \) such that \( f(X) \leq f(X/\{ \min(X) \}) \). A qo \( Q \) is said to be better quasiordered (bqo) if there is
no bad \( Q \)-array. This concept is due to Nash-Williams [12].

9.13. Theorem. If \( Q \) is bqo then \( Q \) is wqo.

Proof. Suppose that \( Q \) is qo but not wqo. Then there exists an \( \omega \)-sequence
\( (a_n)_{n \in \omega} \) of elements of \( Q \) such that \( a_m \not\leq a_n \) for all \( n \) and \( m < n \). Define
\( f : [\omega]^\omega \to Q \) by \( f(X) = a_m \) where \( m = \min(X) \). It is easy to check that
\( f \) is a bad \( Q \)-array. []

The reader probably finds the bqo concept somewhat mysterious. For
the reader’s edification we provide the following exercises.

(i) Show that any well ordered set is a b.q.o.

(ii) Show that if \( Q \) is the union of two b.q.o subsets then \( Q \) is b.q.o.

(iii) Show that if \( Q_1 \) and \( Q_2 \) are b.q.o then \( Q_1 \times Q_2 \) is b.q.o.

(iv) Show that a b.q.o sum of b.q.o.s is b.q.o.

9.15. Exercise. Show that the following assertions are pairwise equivalent.

(i) \( Q \) is b.q.o.

(ii) There is no bad continuous \( Q \)-array.

(iii) For every \( Q \)-array \( f : [A]^\omega \to Q \) there exists \( B \in [A]^\omega \) such that

\[
    f(X) \leq f(X / \text{min}(X))
\]

for all \( X \in [B]^\omega \).

We shall now develop a powerful technique due to Nash-Williams [12] for proving that a given qo, \( Q \), suspected to be b.q.o., is in fact b.q.o. Frequently such a \( Q \) comes equipped with some sort of ordinal ranking of its elements. We formalize this idea as follows.

9.16. Definition. (Laver [10]). Let \( Q \) be a qo whose quasiorder relation is denoted \( \leq \). A partial ranking of \( Q \) is a well founded partial ordering of \( \leq' \) of the elements of \( Q \) such that \( z \leq' y \) implies \( z \leq y \).

Now let \( Q \) be a qo which is not b.q.o and suppose we have in mind a particular partial ranking \( \leq' \) of \( Q \). We write \( z \prec y \) to mean that \( z \leq' y \) and \( z \neq y \). Let \( f : [A]^\omega \to Q \) and \( g : [B]^\omega \to Q \) be bad \( Q \)-arrays. We write \( g \prec f \) to mean that \( B \subseteq A \) and \( g(X) \leq' f(X) \) for all \( X \in [B]^\omega \). We write \( g \prec f \) to mean that \( B \subseteq A \) and \( g(X) \prec f(X) \) for all \( X \in [B]^\omega \). (Caution: \( g \prec f \) is not equivalent to the conjunction of \( g \leq f \) and \( g \neq f \).) We say that a \( Q \)-array \( f \) is minimal bad (with respect to the given partial ranking \( \leq' \)) if \( f \) is bad and there is no bad \( Q \)-array \( g \prec f \).
The following theorem is essentially due to Nash-Williams [12] although it was first enunciated explicitly by Laver [10].

9.17. Theorem. Let $Q$ be a go equipped with a partial ranking. Let $f_0: [A_0]^\omega \to Q$ be a bad $Q$-array. Then there exists a minimal bad $Q$-array $f \leq^* f_0$.

Proof. Assume not. Using this assumption we shall define an uncountable transfinite sequence of bad $Q$-arrays $f_\xi: [A_\xi]^\omega \to Q$ such that $f_\eta \leq^* f_\xi$ and $A_\eta \neq A_\xi$ for all countable ordinals $\xi < \eta < \aleph_1$. This is clearly impossible.

We begin by letting $f_0: [A_0]^\omega \to Q$ be a bad $Q$-array such that there is no minimal bad $f \leq^* f_0$.

Let $\xi$ be a countable ordinal. Assume inductively that we have defined a bad $Q$-array $f_\xi \leq^* f_\gamma$ for all $\gamma < \xi$. In particular $f_\xi \leq^* f_0$ so $f_\xi$ is not minimal bad. Let $g_\xi: [B_\xi]^\omega \to Q$ be a bad $Q$-array such that $g_\xi \leq^* f_\xi$. Use Theorem 9.10 to shrink $B_\xi$ if necessary so that $g_\xi$ is continuous. By further shrinking we may also assume that $A_\xi - B_\xi$ is infinite. By continuity of $g_\xi$, there exists a nonempty initial segment $s_\xi$ of $B_\xi$ so that $g_\xi(X) = g_\xi(B_\xi)$ for all $X \in [s_\xi, B_\xi]$. Define

$$A_{\xi+1} = B_\xi \cup \{ n \in A_\xi : n \leq \max(s_\xi) \}$$

and

$$f_{\xi+1} = \begin{cases} g_\xi(X) & \text{if } X \in [B_\xi]^\omega \\ f_\xi(X) & \text{if } X \in [A_{\xi+1}]^\omega - [B_\xi]^\omega. \end{cases}$$

Clearly $f_{\xi+1}: [A_{\xi+1}]^\omega \to Q$ is a $Q$-array, i.e. Borel measurable. Using the fact that $g_\xi(X) \leq f_\xi(X)$ for all $X \in [B_\xi]^\omega$, it is easy to check that $f_{\xi+1}$ is bad. It is also clear that $f_{\xi+1} \leq^* f_\xi$ and $A_{\xi+1} \subset A_\xi$.

Now let $\delta$ be a countable limit ordinal and assume that we have defined $f_\xi: [A_\xi]^\omega \to Q$ as above for all $\xi < \delta$. Define $A_\delta = \bigcap \{ A_\xi : \xi < \delta \}$.

We claim that $A_\delta$ is infinite. Suppose for a contradiction that $A_\delta$ is finite, say $A_\delta \subseteq m < \omega$. For each $\xi < \delta$ let $n_\xi$ be the least $n \geq m$ such
ash-Williams [12] al-

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that $n \in A_\xi$. Clearly there are infinitely many $\xi$ such that $n_\xi \notin A_{\xi + 1}$. For any such $\xi$, $n_\xi > \max(s_\xi)$ by the definition of $A_{\xi + 1}$. Hence, for any such $\xi$, $m > \max(s_\xi)$ by the definition of $n_\xi$. Hence there are infinitely many $\xi$ for which the $s_\xi$ are all the same. But if $\xi < \eta$ and $s_\xi = s_\eta$, then $B_\eta \in [s_\xi, B_\xi]$, hence

$$f_\eta(B_\eta) \leq^* f_{\xi + 1}(B_\eta) = g_\xi(B_\eta) = g_\xi(B_\xi) < f_\xi(B_\xi).$$

This contradicts the well foundedness of $<^*$ and so proves the claim.

We now define $f_\delta : [A_\delta]^\omega \to Q$ by $f_\delta(X) = \lim_{\xi < \delta} f_\xi(X)$. This limit exists since $\leq^*$ is well founded and $f_\eta(X) \leq^* f_\xi(X)$ for all $\xi < \eta < \delta$. Also $f_\delta$ is Borel measurable since it is the pointwise limit of a countable sequence of Borel measurable functions. It is also easy to check that $f_\delta$ is bad and $f_\delta \leq^* f_\xi$ for all $\xi < \delta$. This completes the proof.

Transfinite sequences.

Let $Q$ be a bqo set. A transfinite $Q$-sequence is a function $s : \alpha \to Q$ where $\alpha = \lh(s)$ is an ordinal called the length of $s$. If $0 \leq \lh(s)$ we denote by $s|\theta$ the restriction of $s$ to $\theta$, i.e., the unique $s'$ of length $\theta$ such that $s'(\xi) = s(\xi)$ for all $\xi < \theta$. The set of all transfinite $Q$-sequences is denoted $\bQ$. We quasiorder $\bQ$ by $s \leq t$ if and only if there exists a strictly order preserving $h : \lh(s) \to \lh(t)$ such that $s(\xi) \leq t(h(\xi))$ for all $\xi < \lh(s)$. We shall prove that if $Q$ is bqo then $\bQ$ is bqo.

9.18. Lemma. If $s, t \in \bQ$ and $s \not\leq t$ then there exists $\theta < \lh(s)$ such that $s|\theta \leq t$ and $s|\theta + 1 \not\leq t$.

Proof. Given $s \not\leq t$ define $h$ by induction as follows. Let $h(\xi)$ be the least $\eta < \lh(t)$ such that $s(\xi) \leq t(\eta)$ and $\eta > h(\xi')$ for all $\xi' < \xi$. Let $\theta$ be the least $\xi$ such that $h(\xi)$ is undefined. Clearly $s|\theta \leq t$ but $s|\theta + 1 \not\leq t$.

9.19. Theorem. (Nash-Williams [13]). Given a bad $\bQ$-array $(s_X : X \in [A]^{\omega})$. There exists $B \in [A]^\omega$ and a bad $Q$-array $(f(X) : X \in [B]^{\omega})$ such
that, for all \( X \in [B]^{\omega} \), \( f(X) \) is a term of the transfinite \( Q \)-sequence \( s_X \).

**Proof.** For \( s, t, \in \tilde{Q} \) define \( s \leq' t \) to mean that \( s \) is in initial segment of \( t \), i.e. \( s = t \upharpoonright \gamma \) for some \( \gamma \leq \text{lh}(t) \). Clearly \( \leq' \) is a partial ranking of \( \tilde{Q} \). Note that if \( x \in Q \) is a term of \( s \leq' t \) then \( x \) is a term of \( t \). Hence, by the minimal bad array Theorem 9.17, we may safely assume that the \( \tilde{Q} \)-array \((s_X : X \in [A]^{\omega})\) is minimal bad.

Given \( X \in [A]^{\omega} \) and \( Y = X/\{\min(X)\} \) we have \( s_X \not\leq s_Y \). By Lemma 9.18 let \( \theta_X \) be such that \( s_X \upharpoonright \theta_X \leq s_Y \) and \( s_X \upharpoonright \theta_X + 1 \not\leq s_Y \). Note that

\[
(s_X \upharpoonright \theta_X : X \in [A]^{\omega}) \prec^* (s_X : X \in [A]^{\omega}).
\]

Hence, by minimality, there is no bad \( Q \)-array \( \leq^* (s_X \upharpoonright \theta_X : X \in [A]^{\omega}) \).

Hence, by the Galvin-Prikry Theorem 9.9, there exists \( B \in [A]^{\omega} \) such that \( s_X \upharpoonright \theta_X \leq s_Y \upharpoonright \theta_Y \) for all \( X \in [B]^{\omega} \), \( Y = X/\{\min(X)\} \). Thus we have \( s_X \upharpoonright \theta_X \leq s_Y \upharpoonright \theta_Y \) but \( s_X \upharpoonright \theta_X + 1 \not\leq s_Y \upharpoonright \theta_Y + 1 \). It follows that \( s_X(\theta_X) \not\leq s_Y(\theta_Y) \). Thus \((s_X(\theta_X) : X \in [B]^{\omega})\) is a bad \( Q \)-array.

**9.20. Corollary.** (Nash-Williams [13]). If \( Q \) is bqo then \( \tilde{Q} \) is bqo.

**Proof.** Immediate from the theorem.

We mention without proof the following characterization of better quasior-dering due to Pouzet [14]: \( Q \) is bqo if and only if \( \tilde{Q} \) is wqo.

**Proof of Fraïssé's Conjecture.**

A linearly ordered set \( L \) is called scattered if it has no subset isomorphic to the rational numbers. We shall prove that the class of scattered linearly ordered sets is bqo under embeddability. In order to apply the method of minimal bad arrays (Theorem 9.17), we need an appropriate partial ranking. This will be provided by the following characterization of scattered sets due to Hausdorff.
Let $S_0$ be the class of one-point linearly ordered sets. For any ordinal $\rho > 0$ let $S_\rho$ be the class of linearly ordered sets $L$ such that $L$ is isomorphic to either a well ordered sum
\[ L_0 + L_1 + \cdots + L_\xi + \cdots \quad (\xi < \alpha) \]
or a converse well ordered sum
\[ \cdots + L_\xi + \cdots + L_1 + L_0 \quad (\xi < \alpha) \]
where each $L_\xi$ belongs to $\bigcup \{ S_\pi : \pi < \rho \}$. Let $S = \bigcup \{ S_\rho : \rho \text{ an ordinal} \}$.

**9.21. Theorem.** (Hausdorff [4]). $S$ is the class of scattered linearly ordered sets.

**Proof.** It is easy to prove by induction on $\rho$ that if $L \in S_\rho$ then $L$ is scattered. Conversely, let $L$ be scattered. For $x, y \in L$ define $x \approx y$ if and only if the nonempty interval $[x, y]$ or $[y, x]$ belongs to $S$. Clearly $\approx$ is a congruence relation on $L$. If the linearly ordered set $L/\approx$ contains more than one point, then it is densely ordered, hence $L$ is not scattered, a contradiction. So $L/\approx$ consists of a single point, i.e. $x \approx y$ for all $x, y \in L$. By considering a well ordered cofinal set and a converse well ordered coinitial set, it is now easy to see that $L \in S$. 

**9.22. Theorem.** (Laver [7]). The class $S$ of scattered linearly ordered sets is bqo under embeddability.

**Proof.** We define the rank of a scattered linearly ordered set $L$ to be the least ordinal $\rho$ such that $L \in S_\rho$. We write $L \prec M$ if and only if $L \leq M$ and $\text{rank}(L) < \text{rank}(M)$. We employ the partial ranking $\leq'$ of $S$ defined by $L \leq' M$ if and only if $L \prec M$ or $L = M$.

Suppose that $S$ is not bqo. By Theorem 9.17 let $(L_X : X \in [\mathcal{A}]^\omega)$ be a minimal bad $S$-array. Each $L_X$ is either

1. a well ordered sum of scattered sets of smaller rank, or
(2) a converse well ordered sum of scattered sets of smaller rank, or

(3) a one-point set.

By the Galvin-Prikry Theorem 9.9 we may shrink \( A \) if necessary so that all the \( L_X \)'s in the array are of the same kind: (1), (2), or (3).

Clearly Case (3) does not always hold, since the array is bad. Assume that Case (1) always holds (Case (2) is similar). Thus for each \( X \in [A]^\omega \) we have

\[
L_X = L^0_X + L^1_X + \ldots + L^\xi_X + \ldots (\xi < \alpha_X)
\]

where \( \text{rank}(L^\xi_X) < \text{rank}(L_X) \), hence \( L^\xi_X \prec' L_X \), for each \( \xi < \alpha_X \). Consider the transfinite \( S \)-sequence \( s_X = \{L^\xi_X : \xi < \alpha_X\} \in \tilde{S} \). Clearly \( (s_X : X \in [A]^\omega) \) is an \( \tilde{S} \)-array.

We claim that \( (s_X : X \in [A]^\omega) \) is bad. If not, then for some \( X \in [A]^\omega \) and \( Y = X /\{\min(X)\} \) we have \( s_X \leq s_Y \). Hence there exists a strictly order preserving map \( h : \alpha_X \to \alpha_Y \) such that \( L^\xi_X \leq L^\xi_Y \) for all \( \xi < \alpha_X \). Hence \( L_X \leq L_Y \). This contradicts the assumed badness of \( (L_X : X \in [A]^\omega) \).

Now by the transfinite sequence Theorem 9.19 there exists \( B \in [A]^\omega \) and a bad \( S \)-array \( f : [B]^\omega \to S \) such that \( f(X) \) is a term of \( s_X \) for all \( X \in [B]^\omega \). In other words, \( f(X) = L^\theta_X \) for some \( \theta_X < \alpha_X \). Thus \( (L^\theta_X : X \in [B]^\omega) \) is a bad \( S \)-array. But we also have

\[
(L^\theta_X : X \in [B]^\omega) \prec^* (L_X : X \in [A]^\omega).
\]

contradicting the assumed minimality of \( (L_X : X \in [A]^\omega) \). This completes the proof.

The following corollary is the solution to Fraïssé's conjecture.

9.23. Corollary. (Laver [7]). The class of countable linearly ordered sets is wqo under embeddability.

Proof. Every countable linearly ordered set is embeddable in the rationals. Hence, every countable linearly ordered set is either scattered or equivalent
sets of smaller rank, \( n_k A \) if necessary so 
), (2), or (3).
array is bad. Assume
or each \( X \in [A]^{\omega} \) we
\( \alpha x \)

or each \( \xi < \alpha x \).
\( \alpha x \in S \). Clearly

then for some \( X \in \]
Hence there exists a
\( L^X_x \leq L^h(\xi) \) for all
assumed badness of

there exists \( B \in [A]^{\omega} \)
a term of \( s_x \) for all
\( \theta_x < \alpha x \). Thus
\( A^{\omega} \).
\( 1^{\omega} \). This completes

's conjecture.

linearly ordered sets

able in the rationals.

tered or equivalent
to the rationals under mutual embeddability. It follows from Theorem
9.22 that the countable linearly ordered sets are bqo under embeddability.
Hence by Theorem 9.13 they are wqo under embeddability. \( \square \)

We mention without proof one further theorem of Laver [7]. Call
a linearly ordered set \( \sigma \)-scattered if it is the union of countably many
scattered subsets. Then the class of \( \sigma \)-scattered linearly ordered sets is
bqo under embeddability. The proof of this theorem uses a Hausdorff-style
characterization of the \( \sigma \)-scattered sets due to Fred Galvin (see Laver [7]).

For other major applications of bqo theory, the reader may consult
Laver [8], [10] and Nash-Williams [12].

References to Chapter 9.

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