\[ \Sigma^1_1 \text{ AND } \Pi^1_1 \text{ TRANSFINITE INDUCTION} \]

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§0. Introduction.

In this paper we explore some technical questions related to the formal system $\text{ATR}_0$ of arithmetical transfinite recursion with quantifier free induction on the natural numbers. This system, and indeed all of the formal systems considered in this paper, are subsystems of second order arithmetic and use classical logic.

The specific system $\text{ATR}_0$ was introduced by Friedman [4] and was studied in some detail by Friedman, McAloon and Simpson [6]. (A stronger system $\text{ATR}_1$ consisting of $\text{ATR}_0$ plus full induction on the natural numbers, had been introduced earlier by Friedman [3] and had been studied by Friedman [1] and Steel [13].)

The interest of $\text{ATR}_0$ has by now been well established. On the one hand, it was shown in [3], [4], [6] and [13] that $\text{ATR}_0$ is just strong enough to formalize many mathematical theorems which depend on having a good theory of countable well orderings. Indeed, many such theorems turn out to be provably equivalent to $\text{ATR}_0$ over a relatively weak theory $\text{ACA}_0$. (As an example here we may cite the theorem that every uncountable Borel set contains a perfect subset.) On the other hand, it was shown in [6] that $\text{ATR}_0$ is proof theoretically not very strong, e.g. its proof theoretic ordinal is just the Feferman/Schütte ordinal $\Gamma_0$. (From recent work of Jäger [10] and Friedman (§5 below) it follows that the proof theoretic ordinal of $\text{ATR}$ is $\Gamma_0$.)

The purpose of this paper is to study the systems $\Sigma^1_1\text{-TI}_0$ and $\Pi^1_1\text{-TI}_0$ of $\Sigma^1_1$ and $\Pi^1_1$ transfinite induction along arbitrary well orderings of the natural numbers. These systems were defined in [4]. We show in §2 that $\Sigma^1_1\text{-TI}_0$ is equivalent to $\text{ATR}_0$ plus $\Sigma^1_1$ ordinary induction, or equivalently $\text{ATR}_0$ plus $\Pi^1_1$ ordinary induction. (Here "ordinary" means "along the usual well ordering of the natural numbers"). We also show that $\Sigma^1_1\text{-TI}_0$ is properly stronger than $\text{ATR}_0$. In §4 we show that $\Pi^1_1\text{-TI}_0$ is equivalent to the system $\Sigma^1_1\text{-DC}_0$ of $\Sigma^1_1$ dependent choices with quantifier free induction on the natural numbers (denoted $\text{HDC}_0$ in [4]). These results in §§2 and 4 answer questions which were naturally suggested by

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the results of [4] and [13].

In §3 we use the results of §2 to prove a conjecture from [6] concerning partition calculus in \( \text{ATR}_0 \). It was known from [6] that \( \text{ATR}_0 \) proves that Galvin/Prikry theorem for closed sets. We now show in §3 that \( \text{ATR}_0 \) does not prove the Galvin/Prikry theorem for finite sequences of closed sets.

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§1. Preliminaries

All of the formal systems considered in this paper are in the language of second order arithmetic which consists of \(+, \cdot, 0, 1, =, \leq, <, \in\), number variables \( k, m, n, \ldots \), set variables \( X, Y, \ldots \), propositional connectives, number quantifiers, and set quantifiers. Number variables are intended to range over natural numbers and set variables are intended to range over sets of natural numbers. For general background see Kreisel [11].

A formula is said to be arithmetical if it contains no set quantifiers. The weakest system we shall consider is \( \text{ACA}_0 \) which consists of the usual ordered semiring axioms for the natural numbers, the quantifier free induction axiom

\[ \forall X \forall k (k \in X \land k + 1 \in X) \rightarrow \forall k (k \in X), \]

and arithmetical comprehension axioms

\[ \exists X \forall m (m \in X \leftrightarrow \theta(m)) \]

where \( \theta \) is arithmetical and does not contain \( X \). It is easy to see that \( \text{ACA}_0 \) is finitely axiomatizable. All systems are assumed to include \( \text{ACA}_0 \).

Within \( \text{ACA}_0 \) we have the arithmetical pairing function

\[ (m, n) = \frac{1}{2}(m + n + 1)(m + n) + m. \]

A binary relation \( R \) on the natural numbers is identified with a set \( X = \{(m, n) : m R n\} \). A well ordering is a binary relation \( \prec \) which is a linear ordering of the natural numbers such that

\[ \forall X [\exists n (X \subseteq X \land n \in X \land \forall m (m \in X \rightarrow \forall n (n \in X \lor n \prec m))]. \]

We write \( \text{WO}(\prec) \) to mean that \( \prec \) is a well ordering of the natural numbers. Thus \( \text{WO}(\prec) \) is a \( \Pi^1_1 \) formula with a free set variable \( \prec \). The scheme of transfinite induction (TI) consists of all instances of

\[ \text{WO}(\prec) \land \forall m (\forall n (m \in X \land n \in X \rightarrow n \prec m) \rightarrow \forall n \varphi(n)) \rightarrow \forall m \varphi(m) \]

where \( \varphi \) is an arbitrary formula.
A $\Sigma^1_n$ (respectively $\Pi^1_n$) formula is one consisting of $n$ set quantifiers beginning with an existential (respectively universal) one followed by an arithmetical matrix. By $\Sigma^1_n$-TI$_0$ (respectively $\Pi^1_n$-TI$_0$) we mean the system consisting of ACA$_0$ plus the transfinite induction scheme TI$_0$ restricted to $\Sigma^1_n$ (respectively $\Pi^1_n$) formulas $\varphi$. It is known that the system

$\Pi^1_n$-TI$_0$ ($= \bigcup_{n\in\omega} \Pi^1_n$-TI$_0$) is not finitely axiomatizable. (See the beginning of §4 below.) The main purpose of this paper is to study the systems $\Sigma^1_n$-TI$_0$ and $\Pi^1_n$-TI$_0$.

We shall have occasion to consider certain comprehension and choice principles. By $\Pi^1_n$-CA$_0$ we mean ACA$_0$ plus all comprehension axioms

$$\exists X \forall m (m \in X \leftrightarrow \varphi(m))$$

in which $\varphi$ is $\Pi^1_n$ and does not contain $X$. By $\Delta^1_n$-CA$_0$ we mean ACA$_0$ plus all instances of

$$\forall m \exists n (\varphi(m) \leftrightarrow \neg \psi(m)) \to \exists X \forall m (m \in X \leftrightarrow \varphi(m))$$

where $\varphi$ and $\psi$ are $\Pi^1_n$ and do not contain $X$. Write $(Y)_k = \{y : (y,k) \in Y\}$.

By $\Sigma^1_n$-AC$_0$ we mean ACA$_0$ plus all instances of the countable choice scheme

$$\forall k \exists \varphi \exists k (k, X) \to \exists Y \forall k (\varphi(k, Y))$$

where $\varphi$ is $\Sigma^1_n$ and does not contain $Y$ or bound occurrences of $k$. By $\Sigma^1_n$-DC$_0$ we mean ACA$_0$ plus all instances of the $\Sigma^1_n$ dependent choice scheme

$$\forall X \exists Y \varphi (X,Y) \to \exists Z \forall k (\varphi(k, Z, (Z)_k, (Z)_{k+1}))$$

where $\varphi$ is $\Sigma^1_n$ and does not contain $Z$ or $k$. It is well known and easy to see that $\Sigma^1_{n+1}$-DC$_0$ includes $\Sigma^1_n$-AC$_0$ which includes $\Delta^1_n$-AC$_0$ which includes $\Pi^1_n$-CA$_0$. Obviously $\Pi^1_n$-CA$_0$ implies $\Sigma^1_n$-TI$_0$ and $\Pi^1_n$-TI$_0$. It is known from [2] and [9] that $\Pi^1_n$-CA$_0$ is proof theoretically stronger than $\Pi^1_n$-TI$_0$.

An important role in this paper will be played by the system ATR$_0$. ATR$_0$ consists of ACA$_0$ plus the scheme of arithmetical transfinite recursion

$$\forall \varphi \exists \theta \exists \Psi \forall \psi \varphi (\psi, \Psi) \to \exists \theta (\theta(y, n, \psi) \equiv \theta(y, n) \in \psi)$$

where $\theta(y, x, \psi)$ is arithmetical. Intuitively, $X$ is a set obtained by iterating arithmetical comprehension along the well ordering $\prec$. It is easy to see that ATR$_0$ is finitely axiomatizable: the axioms are those of ACA$_0$ plus a $\Pi^1_2$ sentence asserting that the Turing jump operator can be iterated along any well
ordering starting at any set.

For background information on $\text{ATR}_0$ see [1], [3], [6]. One may also consult [4] and [13], but see the comment just before Lemma 2.7 below. Two important facts which we shall need are

1. $\text{ATR}_0$ proves $\Sigma^1_1$-AC$_0$;

2. $\text{ATR}_0$ proves comparability of well orderings, i.e. if $\prec_1$ and $\prec_2$ are well orderings of $\omega$ then they are isomorphic or one is isomorphic to a proper initial segment of the other. (We say that $\prec_1$ and $\prec_2$ are isomorphic if there exists a binary relation of isomorphism between them.)

Each of the systems above may be strengthened by adding the induction scheme

$$\phi(0) \& \forall k (\phi(k) \to \phi(k+1)) \to \forall k \phi(k)$$

where $\phi$ is arbitrary. This scheme formalizes the principle of proof by induction on the natural numbers. If $S_0$ denotes one of our systems with only the quantifier free induction axiom, then $S$ will denote $S_0$ plus the induction scheme. For instance, $\text{ATR} = \text{ATR}_0 +$ induction scheme, and $\Sigma^1_1$-TI = $\Sigma^1_1$-TI$_0 +$ induction scheme.

§ 2. $\Sigma^1_1$ transfinite induction.

In this section, restrictions of the ordinary induction scheme will play an important role. By $\Sigma^1_1$ (respectively $\Pi^1_1$) induction we mean the induction scheme restricted to formulas $\phi$ which are $\Sigma^1_1$ (respectively $\Pi^1_1$). Note that the induction scheme restricted to arithmetical $\phi$ is provable in $\text{ACA}_0$.

In addition to $\Sigma^1_1$ and $\Pi^1_1$ induction on the natural numbers, it will be convenient to consider the following finite form of $\Pi^1_1$-CA$_0$ which we call finite $\Pi^1_1$-CA$_0$:

$$\forall n \exists x \forall y (y \in x \leftrightarrow \phi(y))$$

where $\phi$ is a $\Pi^1_1$ formula not containing $X$.

2.1 Lemma. Over $\text{ACA}_0$ the following are pairwise equivalent:

1. $\Pi^1_1$ induction (on the natural numbers);

2. $\Sigma^1_1$ induction (on the natural numbers);

3. finite $\Pi^1_1$-CA$_0$.

Proof. 1 $\leftrightarrow$ 2: Assume $\Pi^1_1$ induction. Suppose $\forall k (\phi(k) \to \phi(k+1))$ and $\neg \phi(n)$ where $\phi$ is $\Sigma^1_1$. Prove by induction on $i \leq n$ that $\neg \phi(n-i)$. In particular $\neg \phi(0)$. This proves $\Sigma^1_1$ induction from $\Pi^1_1$ induction. The converse is similar.
$\Sigma_1^1$ and $\Pi_1^1$ transfinite induction
By hypothesis and finite $\Sigma^1_1$-CA$_0$ these two definitions of goodness are equivalent. Thus the property of goodness is $\Delta^1_1$. Trivially the empty sequence is good, and the hypothesis easily implies that each good $t \in T$ has a good immediate extension in $T$. Thus we have a failure of $\Delta^1_1$-TI$_0$ along the Kleene/Brouwer ordering of $T$. This completes the proof.

**Remark.** The scheme of weak $\Sigma^1_1$-CA$_0$ is perhaps of some independent interest. It is easy to see that $\Delta^1_1$-CA$_0$ implies weak $\Sigma^1_1$-AC$_0$ and that every $\omega$-model of weak $\Sigma^1_1$-AC$_0$ is closed under relative hyperarithmeticity. Hence the hyperarithmetical sets form the minimum $\omega$-model of weak $\Sigma^1_1$-AC$_0$. Another easy observation is that given any descending sequence of Turing degrees separated by Turing jump, the reals recursive in all degrees in the sequence form an $\omega$-model of weak $\Sigma^1_1$-AC$_0$. For more information about such sequences see Friedman [5] and Steel [12]. Van Wesep [14] has shown that there exists an $\omega$-model of weak $\Sigma^1_1$-AC$_0$ which is not a model of $\Delta^1_1$-CA$_0$.

The next lemma expresses the well known fact that number variables in $\Pi^1_1$ predicates can be uniformized, provably in ATR$_0$.

**2.3 Lemma.** Let $\varphi(n)$ be a $\Pi^1_1$ formula. There exists a $\Pi^1_1$ formula $q^*(n)$ such that ATR$_0$ proves $\forall n (\varphi(n) \rightarrow q^*(n))$ and $\exists n q(n) \rightarrow \exists m q^*(n)$.

**Proof.** Let $<_n$ be the Kleene/Brouwer ordering of the tree of unsecured sequences for $\varphi(n)$. Thus by ACA$_0$ we have that $\varphi(n)$ holds if and only if $<_n$ is a well ordering. Put $q^*(n)$ if and only if $\varphi(n)$ & $\exists p < n (\forall m (\varphi(m) \rightarrow \exists l (\varphi(f(l)) \& \varphi(f(l),f(f(l)))))$. This works because ATR$_0$ proves comparability of well orderings.

**2.4 Lemma.** Let $\psi(m)$ and $\varphi(m,n)$ be $\Pi^1_1$ formulas. Then ATR$_0$ plus general induction (on the natural numbers) proves

$$\forall m [\psi(m) \rightarrow \exists n [\psi(n) \& \varphi(m,n)]] \rightarrow$$

$$\forall m [\psi(m) \rightarrow \exists f [f(0) = m \& \forall i [\psi(f(i)) \& \varphi(f(i),f(i+1))]]].$$

**Proof.** Assume $\forall m [\psi(m) \rightarrow \exists n [\psi(n) \& \varphi(m,n)]]$. By ATR$_0$ and the previous lemma we may also assume $\forall m [\psi(m) \rightarrow \exists n [\psi(n) \& \varphi(m,n)]]$. Fix $m$ such that $\psi(m)$ holds. Let $s(k,s)$ say that $s$ encodes a finite sequence of length $k+1$ such that $s(0) = m$ and $\forall i < k [\psi(s(i)) \& \varphi(s(i),s(i+1))]$. By $\Sigma^1_1$-AC$_0$ (a consequence of ATR$_0$), the statement $\exists k \exists s \exists \theta(k,s)$ is $\Pi^1_1$ so we can use $\Pi^1_1$ induction to prove that this statement holds for all $k$. Thus we have $\forall k \exists s \exists \theta(k,s)$. Hence, by $\Delta^1_1$-CA$_0$ (a consequence of $\Sigma^1_1$-AC$_0$), there exists $f$ such that $\forall k [\theta(k,f[k+1])$, i.e. $\forall i [\psi(f(i)) \& \varphi(f(i),f(i+1))]$. This completes the proof.
2.5 Theorem. The following are pairwise equivalent:

1. ATR\(_0\) plus \(\Pi^1_1\) induction (along the natural numbers);

2. ATR\(_0\) plus \(\Sigma^1_1\) induction (along the natural numbers);

3. ATR\(_0\) plus finite \(\Pi^1_1\)-CA\(_0\);

4. \(\Sigma^1_1\)-TI\(_0\).

Proof. The pairwise equivalence of 1, 2, and 3 is by Lemma 2.1. Let \(<\) be a linear ordering of the natural numbers on which \(\Sigma^1_1\)-TI\(_0\) fails, i.e. we have a \(\Pi^1_1\) formula \(\psi(m)\) such that \(\exists m \forall n [\psi(m) \rightarrow \exists n < m\psi(n)]\). By Lemma 2.4 we obtain a function \(f\) such that \(\forall k [\psi(k) \& f(k+1) < f(k)]\), i.e. \(f\) is an infinite descending sequence through \(<\). This proves 1 \(\rightarrow\) 4.

Obviously \(\Sigma^1_1\)-TI\(_0\) includes \(\Sigma^1_1\) induction on the natural numbers so it remains only to prove that \(\Sigma^1_1\)-TI\(_0\) implies ATR\(_0\). Assume \(\Sigma^1_1\)-TI\(_0\). By Lemma 2.2 we have weak \(\Sigma^1_1\)-AC\(_0\). Let \(<\) be a well ordering and suppose we are given an arithmetical formula \(\theta(y,X)\). Let \(\phi(n,X)\) be the arithmetical formula which asserts that \(X\) is the result of iterating \(\theta\) along \(<\) up to \(n\), i.e.

\[X = \{(y,m) : m < n \& \theta(y,(x,k):k < m \& (x,k) \in X)\}\]  

It is easy to see that for each \(n\) there is at most one \(X\) such that \(\phi(n,X)\).

In order to prove ATR\(_0\) we must prove \(\forall n \exists X \phi(n,X)\). Let \(n\) be fixed. By \(\Sigma^1_1\)-TI\(_0\) we may assume \(\forall m < n \exists X \phi(m,X)\). Hence \(\forall m < n \exists X \phi(m,X)\) so by weak \(\Sigma^1_1\)-AC\(_0\) there exists \(Y\) such that \(\forall m < n \exists Y \phi(m,(y,Y),)\). Then clearly \(\phi(n,X)\) if we put \(X = \{(y,m) : m < n \& \theta(y,(y,Y),)\}\). This completes the proof.

2.6 Corollary. (Friedman [3], Steel [13]). The systems ATR and \(\Sigma^1_1\)-TI (both with full induction on the natural numbers) are equivalent.

We shall now show that \(\Sigma^1_1\)-TI\(_0\) is properly stronger than ATR\(_0\). This result contradicts a claim which was made in Theorem 8 of [4] and on page 22 of [13].

2.7 Lemma. Over \(\Sigma^1_1\)-AC\(_0\) the following are equivalent:

1. \(\Sigma^1_1\) induction (on the natural numbers);

2. \(\Pi^1_3\) soundness of ACA\(_0\), i.e. the assertion that any \(\Pi^1_3\) sentence provable in ACA\(_0\) is true.

Proof. 2 \(\rightarrow\) 1: Suppose that we have a failure of \(\Sigma^1_1\) induction, i.e. \(\phi(0)\) and \(\forall k ([\phi(k) \rightarrow \phi(k+1)]\) and \(\neg \phi(n)\) for some fixed \(n\). By \(\Pi^1_3\) soundness of ACA\(_0\) let \(M\) be a model of ACA\(_0\) plus \(\phi(0)\) plus \(\forall k ([\phi(k) \rightarrow \phi(k+1)]\) plus \(\neg \phi(n)\).
The standard integers are canonically identified with an initial segment of the integers of $M$. Let $Z = \{ m : M \models \varphi(m) \}$. Then $Z$ contains the standard integer $n$ yet has no least element. This is absurd.

1 $\to$ 2: Reasoning in $\Sigma^1_1$-AC$_0$, let $\sigma$ be a true $\Sigma^1_3$ sentence. We shall use $\Sigma^1_3$ induction to prove consistency of ACA$_0$ plus $\sigma$. Let $L$ be the language of second order arithmetic augmented by set constants $C^i$, $i \in \omega$. Write $\sigma = \exists X \forall Y \varphi(U,X,Y)$ where $\varphi$ is arithmetical and let $\varphi(U,X,Y)$ be the arithmetical formula

$$\forall I \exists j \exists k [\varphi(U,(X)_I,(Y)_I) \land \omega^X_I = (Y)_I]$$

where $\omega^X_I$ denotes the $I$th set recursively enumerable in $X$. Let $T$ be the $L$-theory consisting of RCA$_0$ (= ordered semiring axioms plus recursive comprehension plus quantifier free induction) together with axioms $\varphi(C^0_i, C^1_i, C^{i+1}_i), i \in \omega$. We shall prove consistency of $T$. Let $T_k$ be the restriction of $T$ to $C^i_k, i \leq k$. Fix a set $U_0$ such that $\forall X \exists Y \varphi(U_0^i, X, Y)$. By $\Sigma^1_1$-AC$_0$ we have $\forall X \exists Y \varphi(U_0^i, X, Y)$. Hence by $\Sigma^1_1$ induction we have

$$\forall k \exists Y \forall I (\exists Z = U_0 \land (U_0^i, Z)_I \land (Z)_{i+1})].$$

It follows by cut elimination that $\forall k(T_k)$ is consistent. Hence by the compactness theorem $T$ is consistent. But from any model of $T$ we can easily extract a model of ACA$_0$ plus $\sigma$ by throwing away all sets except those which are recursive in $C^i_k$ for some $i \in \omega$. Thus ACA$_0$ plus $\sigma$ is consistent. This completes the proof.

2.8 Theorem. $\Sigma^1_1$-TI$_0$ proves $\Pi^1_3$ soundness of ATR$_0$, i.e. the assertion that any $\Pi^1_3$ sentence provable in ATR$_0$ is true. In particular $\Sigma^1_1$-TI$_0$ proves consistency of ATR$_0$.

Proof. We reason in $\Sigma^1_1$-TI$_0$. By Theorem 2.5 we have ATR$_0$ and hence $\Sigma^1_1$-AC$_0$. Let $\sigma$ be a true $\Sigma^1_3$ sentence. We know that ATR$_0$ consists of ACA$_0$ plus a $\Pi^1_3$ sentence so we may as well assume that $\sigma$ includes this $\Pi^1_3$ sentence. Now apply Lemma 2.7 to conclude that ACA$_0$ plus $\sigma$ is consistent, i.e. ATR$_0$ plus $\sigma$ is consistent.

2.9 Corollary. ATR$_0$ does not prove $\Sigma^1_1$-TI$_0$.

Proof. Immediate from Theorem 2.8 plus Gödel's second incompleteness theorem.
2.10 Corollary. \( \text{ATR}_0 \) does not prove \( \Pi^1_1 \) induction, \( \Sigma^1_1 \) induction, or finite \( \Pi^1_1 \text{-CA}_0 \).

Proof. Immediate from the previous Corollary plus Theorem 2.5.

§3. Partition calculus in \( \Sigma^1_1 \text{-TI}_0 \).

In this section we use the notation of §3 of [6]. We study closed sets in the space \( [\omega]^\omega \) of infinite sets of natural numbers. It was shown in [6] that \( \text{ATR}_0 \) is equivalent to \( \text{ACA}_0 \) plus the Galvin/Prikry theorem for closed sets, i.e. the assertion that for every closed set \( C \subseteq [\omega]^\omega \) there exists \( A \in [\omega]^\omega \) such that either \( [A]^\omega \cap C = \emptyset \) or \( [A]^\omega \cap C = \emptyset \). The purpose of this section is to prove a similar result in which \( \text{ATR}_0 \) is replaced by the stronger theory \( \Sigma^1_1 \text{-TI}_0 \).

A set \( U \subseteq \omega \) is said to be hyperarithmetic if \( U \) is recursive in \( H_0 \) for some \( b \in \mathbb{O} \).

3.1 Lemma (\( \text{ATR}_0 \)). Let \( C_i, i < n \) be a recursively coded finite sequence of closed sets in \( [\omega]^\omega \). If there is no hyperarithmetic \( U \in [\omega]^\omega \) such that \( \exists i < n[U]^\omega \cap C_i = \emptyset \) then there exists \( A \in [\omega]^\omega \) such that \( \forall i < n[A]^\omega \subseteq C_i \).

Proof. The proof of Theorem 3.8 of [6] actually establishes this stronger result.

3.2 Theorem. Over \( \text{ACA}_0 \) the following are equivalent:

1. \( \Sigma^1_1 \text{-TI}_0 \);

2. For any finite sequence of closed sets \( C_i \subseteq [\omega]^\omega, i < n \), there exists \( A \in [\omega]^\omega \) such that for each \( i < n \) either \( [A]^\omega \subseteq C_i \) or \( [A]^\omega \cap C_i = \emptyset \).

Proof. 1 \( \rightarrow \) 2: By relativization we may safely assume that the given sequence of closed sets \( C_i, i < n \), is recursively coded.

We claim that there exists a hyperarithmetic set \( U \in [\omega]^\omega \) such that for each \( i < n \) either \( [U]^\omega \cap C_i = \emptyset \) or there is no hyperarithmetic \( V \in [U]^\omega \) such that \( [V]^\omega \cap C_i = \emptyset \). Suppose not. Let \( \psi(k) \) be the assertion that there exists a hyperarithmetic \( V \in [\omega]^\omega \) and a finite set \( s \) of cardinality \( k \) such that \( \forall i \in s (i < n \& [V]^\omega \cap C_i = \emptyset) \). Clearly \( \psi(0) \) and \( \forall k (\psi(k) \rightarrow \psi(k+1)) \). By \( \Sigma^1_1 \text{-AC}_0 \) (a consequence of \( \text{ATR}_0 \)) the formula \( \psi(k) \) is equivalent to a \( \Pi^1_1 \) formula. Hence by \( \Pi^1_1 \) induction we have \( \psi(n+1) \) which is absurd. This proves the claim.

Let \( U \) be as in the above claim. By finite \( \Pi^1_1 \text{-CA}_0 \) let \( X = \{ i < n : [U]^\omega \cap C_i = \emptyset \} \). The claim tells us that there is no hyperarithmetic \( V \in [U]^\omega \) such that \( \exists i \in X [V]^\omega \cap C_i = \emptyset \). Hence by Lemma 3.1 there exists \( A \in [U]^\omega \) such that \( \forall i \in X [A]^\omega \subseteq C_i \). Hence for each \( i < n \) either \( [A]^\omega \subseteq C_i \)
(if \(i \in X\)) or \([A]^\omega \cap C_i = \phi\) (if \(i \notin X\)).

2 \(\rightarrow\) 1: We already know (by Theorem 3.2 of [6]) that the partition theorem 3.2.2 implies \(\text{ATR}_0\). By Theorem 2.5 it remains to show that the partition theorem also implies finite \(\Pi^1_1\)-\(\text{CA}_0\). Let \(\varphi(i)\) be \(\Pi^1_1\) and let \(T_i \subseteq \omega^\omega\) be the associated tree of unsecured sequences, i.e. \(\varphi(i)\) holds if and only if there is no path through \(T_i\). For any \(X \in [\omega]^{\omega}\) let \(\pi_X \in [\omega]^{\omega}\) be the function which enumerates the elements of \(X\) in increasing order. Put \(X \subseteq C_i\) if and only if \(\pi_X\) majorizes a path through \(T_i\), i.e. \(\exists y \forall j (f(j) \preceq_{\pi_X} (j) \& f(j) \in T_i)\) or equivalently by König's lemma \(\forall k \exists t (t \in T_i \& \forall h(t) = k \& \forall j < k \ t(j) <_{\pi_X} (j))\). Clearly \(C_i\) is a closed set in \([\omega]^{\omega}\). Now given \(n\), use the partition theorem 3.2.2 to get \(A \in [\omega]^{\omega}\) such that for each \(i < n\) either \([A]^\omega \subseteq C_i\) or \([A]^\omega \cap C_i = \phi\). Then for \(i < n\) we have \(\varphi(i)\) if and only if \(\sim[A]^\omega \subseteq C_i\). The latter formula is arithmetical so by \(\text{ACA}_0\) we have \(\exists X \forall i (i \in n \leftrightarrow \varphi(i))\). This completes the proof of the theorem.

The following corollary establishes a conjecture which was stated after Theorem 3.9 in [6].

3.3 Corollary. The partition theorem 3.2.2 is not provable in \(\text{ATR}_0\).

Proof. Immediate from Theorems 3.2 and 2.9.

An argument similar to the above proof of Theorem 3.2 establishes the following result which was discovered jointly by S. Shelah and the author, long before the author's discovery of Theorem 3.2.

3.4 Theorem. Over \(\text{ACA}_0\) the following are equivalent:

1. \(\Pi^1_1\)-\(\text{CA}_0\);

2. For any infinite sequence of closed sets \(C_i \subseteq \omega^\omega\), \(i \in \omega\), there exists \(A \in [\omega]^{\omega}\) such that for each \(k \in A\) and \(i < k\) either \([A/\{k\}]^\omega \subseteq C_i\) or \([A/\{k\}]^\omega \cap C_i = \phi\). (Here \(A/\{k\} = \{n \in A : n > k\}\).)

§4. \(\Pi^1_1\) transfinite induction.

Friedman [3] has shown that over \(\text{ACA}_0\) the transfinite induction scheme
\[
\Pi^1_1\text{-}\text{TI}_0 = \bigcup_{n \in \omega} \Pi^1_1\text{-}\text{TI}_n\]

is equivalent to the \(\omega\)-model reflection scheme
\[
\Sigma^1_1\text{-}\text{RFN}_0 = \bigcup_{n \in \omega} \Sigma^1_1\text{-}\text{RFN}_n\]

Here \(\Sigma^1_1\text{-}\text{RFN}_0\) asserts that for any \(\Sigma^1_1\) sentence \(\psi(X_1, \ldots, X_m)\) with set parameters \(X_1, \ldots, X_m\) there exists a countable \(\omega\)-model \(M\) of \(\text{ACA}_0\) such that \(X_1, \ldots, X_m \in M\) and \(M \models \omega \psi(X_1, \ldots, X_m)\). It is natural to ask how much transfinite induction is equivalent to how much \(\omega\)-model reflection. As a rule, special cases of this question appear to be difficult. However, one special case
is answered by the following theorem.

4.1 Theorem. Over ACA₀ the following are pairwise equivalent:
1. \( \Pi^1_1\text{-TI}_0. \)
2. \( \Sigma^1_1\text{-DC}_0. \)
3. \( \Sigma^3_3\text{-RFN}_0. \)

Remark. The equivalence of 2 and 3 is due to Friedman [1]. The equivalence of 1 and 2 may be derived from the appendix of Howard [8] together with the reduction of BI₀ to BI₀ in Howard/Kreisel [9]. The equivalence of 1 and 2 subsumes several results which have been stated by Friedman in Theorem 4.2 of [3] and Theorem 8 of [4].

Proof of Theorem 4.1. 1 \( \implies \) 2: Similar to the proof of Lemma 2.2. Recall that \( \Sigma^1_1\text{-DC}_0 \) says \( \forall X \exists Y \exists \varphi(X,Y) \rightarrow \exists Z \forall k \forall (Z)_k (Z)_{k+1} \) where \( \varphi \) is \( \Sigma^1_1 \). By ACA₀ there exist Skolem functions for the arithmetical matrix of \( \varphi \), so to prove \( \Sigma^1_1\text{-DC}_0 \) it suffices to prove

\[
\forall f \exists g \exists \vartheta \forall n \forall (g(n), \vartheta(n)) \rightarrow \exists h \exists k \forall m \forall (h(m), k) \exists t \forall (t(n), t_{k+1}(n))
\]

where \( f, g, h \) are function variables, \( \vartheta \) is arithmetical, \( f(n) = \langle f(0), \ldots, f(n-1) \rangle \), and \( h_k(m) = h(m, k) \). Assume the hypothesis and let \( T \) be the tree of unsecured sequences for the conclusion, i.e. \( t \in T \) if and only if

\[
\forall k < \ell \exists h(t) \forall n \leq \min(\ell h(t), \ell h(t_{k+1}))(t(n), t_{k+1}(n)).
\]

If the conclusion fails then \( T \) has no path, i.e. the Kleene/Brouwer ordering of \( T \) is a well ordering. Say that \( t \in T \) is good if

\[
\forall k < \ell \exists h(t) \forall n \forall (h_k(n), h_{k+1}(n)) \land h(\ell h(t)) = t.
\]

Clearly the empty sequence is good, and the hypothesis \( \forall f \exists g \exists \vartheta \forall (f(n), g(n)) \)
implies that each good \( t \) has a good immediate extension. The property of goodness is \( \Sigma^1_1 \) so we have a failure of \( \Pi^1_1\text{-TI}_0 \) along the Kleene/Brouwer ordering of \( T \).

2 \( \implies \) 3: Similar to Lemma 2.7. Let \( \varphi(U_0) \) be a true \( \Sigma^3_3 \) sentence with a set parameter \( U_0 \). Write \( \varphi(U_0) \equiv \exists Y \forall X \exists \theta(U_0, V, X, Y) \) where \( \theta \) is arithmetical. Fix \( U_1 \) such that \( \forall X \exists Y \theta(U_0, U_1, X, Y) \). Let \( \varphi(X, Y) \) say that \( (Y)_0 = U_0 \) and \( (Y)_1 = U_1 \) and

\[
\forall i \exists X \exists \theta(U_0, U_1, X, (Y)_i) \land \psi_i^X = (Y)_i
\]

where \( \psi_i^X \) is the \( i \)-th set recursively enumerable in \( X \). By \( \Sigma^1_1\text{-DC}_0 \) there exists \( Z \) such that \( \forall k \forall (Z)_k (Z)_{k+1} \). Clearly \( N = \{(Z)_k : k \in \omega \} \) is a countable \( \omega \)-model of ACA₀ plus \( \varphi(U_0) \). This proves \( \Sigma^3_3\text{-RFN}_0 \).
3.1 Let \( \prec \) be a linear ordering of the natural numbers and assume that we have a failure of \( \Pi^1_1 \) on \( \prec \), i.e., \( \forall m \left( \exists n \left( \varphi(m) \rightarrow \varphi(n) \right) \right) \) and \( \neg \varphi(p) \) where \( \varphi \) is \( \Pi^1_1 \). By \( \Sigma^1_3 \)-RFN\(_0\) there exists a countable \( \omega \)-model \( M \) containing \( \prec \) and satisfying

\[
\forall m \left( \exists n \left( \varphi(m) \rightarrow \varphi(n) \right) \right) \& \neg \varphi(p).
\]

By ACA\(_0\) let \( Z = \{ n : M \models \varphi(n) \} \). Thus \( Z \) is nonempty and has no least element under \( \prec \). Hence \( \prec \) is not a well ordering. This completes the proof of Theorem 4.1.

4.2 Corollary. (i) \( \Pi^1_1 \)-TI\(_0\) plus ATR\(_0\) proves the existence of an \( \omega \)-model of \( \Sigma^1_1 \)-TI. (ii) ATR\(_0\) proves the existence of an \( \omega \)-model of \( \Pi^1_1 \)-TI.

Proof. The first part is immediate from Theorems 4.1 and 2.5 since ATR\(_0\) consists of ACA\(_0\) plus a \( \Pi^1_2 \) sentence. For the second part, reasoning in ATR\(_0\), the proof of Theorem 3.7 of [6] gives \( c \in \mathcal{O} \) \( \& \neg c \) and a countable \( \omega \)-model \( M \) of ACA\(_0\) satisfying "c is \( \omega \) and \( H \) exists." Since \( c \notin \mathcal{O} \) let \( A \subseteq \{ a : a \prec c \} \) be such that \( \forall b \in A \left( \exists \varphi \left( b(a) \rightarrow b(a) \right) \right) \). Put \( M \downarrow = \{ x : x \in A \models M_0 \models x \text{ is recursive in } H_1 \} \). It is not hard to see that \( M \downarrow \) is a countable \( \omega \)-model of \( \Sigma^1_1 \)-DC\(_0\) and hence of \( \Pi^1_1 \)-TI.

4.3 Corollary. Neither of \( \Sigma^1_1 \)-TI and \( \Pi^1_1 \)-TI implies the other, and there exist \( \omega \)-models for the independence.

Proof. Both directions are immediate from Corollary 4.2 plus the \( \omega \)-model form of G"odel's second incompleteness theorem (for which see Friedman [5] or Steel [12]).

Remark. The two previous corollaries are not really new since it is well known [7] that the hyperarithmetic sets satisfy \( \Sigma^1_1 \)-DC\(_0\). However, this fact is not provable in ATR\(_0\), although it is provable in \( \Sigma^1_1 \)-TI\(_0\).

§5. Remark on a system considered by J"ager.

After the main part of this paper was written, Harvey Friedman pointed out that the idea of the proofs of Theorems 2.8 and 4.1 above can be used to settle the relationship between ATR and a related system ATR\(_J\) considered by J"ager [10]. With Friedman's permission we include this result here.

Let ATR\(_J\) be just like ATR\(_0\) except that arithmetical transfinite recursion is only assumed to hold for well orderings which are primitive recursive. ATR\(_J\) is ATR\(_0\) plus full induction on the integers.

Say that \( X \subseteq \omega \) is low if \( X = \omega \) or \( X = ^\omega \omega \), i.e., any well ordering of the natural numbers which is recursive in \( X \) is isomorphic to a primitive recursive well ordering of the natural numbers.
5.1 Lemma. Let \( \phi(X,Y) \) be arithmetical with no free set variables other than \( X \) and \( Y \). Then \( \text{ATR}_0^{\Sigma_1^1} \) proves
\[
X \text{ low } \& \exists Y \phi(X,Y) \rightarrow \exists Y \text{ low } \phi(X,Y).
\]

Proof. We use the notation of §3 of [6]. In \( \text{ACA}_0 \) we can prove that for all \( X, O^X \) is complete \( \Pi^1_1 \) in \( X \) and hence not \( \Sigma^1_1 \) in \( X \). Write
\[
X \setminus Y = \{ \{ n \in X \} \cup \{ 2n+1: n \in Y \} \}.
\]
Assume now that \( X \) is low and \( \exists Y \phi(X,Y) \). By \( \text{ATR}_0^{\Sigma_1^1} \) we have that for each \( e \in O^X \) there exists \( Y \) such that \( \phi(X,Y) \) and \( \Sigma_1^1 \) \( e \) exists. Since \( O^X \) is not \( \Sigma_1^1 \) in \( X \) it follows that there exist \( Y, e, Z \) such that \( \phi(X,Y) \), \( e \in O^X \setminus \phi^X \), and \( \Sigma_1^1 \) \( e \). We claim now that \( X \oplus Y \) is low. This follows from Theorem 4 of [5] relativized to \( X \oplus Y \).

As in §4 of [6] write \( X \ll Y \) to mean that there exists \( Z \) recursive in \( Y \) such that for all \( i, X \) and the Turing jump of \( (Z)_{i+1} \) are recursive in \( (Z)_{i} \).

5.2 Lemma. \( \text{ATR}_0^{\Sigma_1^1} \) proves \( \forall \text{ low } \exists Y (X \ll Y) \).

Proof. This is a straightforward combination of the proofs of Lemma 5.1 above and Lemma 4.6 of [6].

5.3 Theorem. (Friedman). \( \text{ATR} \) and \( \text{ATR}_0^{\Sigma_1^1} \) prove the same \( \Pi^1_1 \) sentences.

Proof. Let \( M^J \) be a model of \( \text{ATR}_0^{\Sigma_1^1} \) plus \( \sigma \) where \( \sigma \) is a \( \Sigma^1_1 \) sentence. Write \( \sigma = \exists X \phi(X) \) where \( \phi \) is arithmetical. Within \( M^J \) apply Lemma 5.1 to get a low set \( X_0 \) such that \( \phi(X_0) \) holds. Consider the \( \Sigma^1_1 \) assertion
\[
\exists Z \forall k \exists X_0 (X_0 \& (Z)_k \ll (Z)_{k+1}].
\]
We would like to find \( Z \in M^J \) such that this holds in \( M^J \). Unfortunately we cannot do this, but we shall find such a \( Z \) which is first order definable over \( M^J \). By \( \text{ACA}_0 \) we can write our \( \Sigma^1_1 \) assertion in the form
\[
\exists f \forall k \exists \emptyset(f_k[m], f_{k+1}[m])
\]
where \( \emptyset \) is arithmetical. Disregarding Skolem functions, \( \emptyset \emptyset(f_k[m], f_{k+1}[m]) \) says that \( f_0 = X_0 \) and \( f_k \ll f_{k+1} \). Within \( M^J \) define a finite sequence \( t \) to be good if
\[
(\exists \text{low } f)(\forall k \exists \ell h(t) \emptyset \emptyset(f_k[m], f_{k+1}[m]) \& \ell h(t) = t).
\]
Clearly the empty sequence is good, and by Lemmas 5.1 and 5.2 each good sequence has a good immediate extension.
By induction on \( n \) we can prove that there exists a lexicographically leftmost good sequence of length \( n \). Let \( f \) be the leftmost "path" through the "tree" of good sequences. (We use quotation marks to indicate that the objects in question are not elements of \( M^J \) but merely first order definable over \( M^J \).) By Lemma 4.6 of [6] the "sets" which are recursive in \( f_k \) for some \( k \) form a model \( M \) of ATR\(^J\) with the same integers as \( M \). This model \( M \) is first order definable over \( M^J \) and therefore satisfies full induction since \( M^J \) does. Thus \( M \) is a model of ATR. Also \( M \) contains \( X_0 \) and hence satisfies \( \sigma \). This proves the theorem.

5.4 Corollary. The proof theoretic ordinal of ATR is \( \Gamma_{\varepsilon_0} \).

Proof. From Theorem 5.3 it follows that ATR has the same proof theoretic ordinal as ATR\(^J\). Jäger [10] has shown that the proof theoretic ordinal of ATR\(^J\) is \( \Gamma_{\varepsilon_0} \).

By a similar but easier argument one has:

5.5 Theorem (Friedman). ATR\(_0\) and ATR\(_0^J\) prove the same \( \Pi^1_1 \) sentences. Every model of ATR\(_0^J\) has a submodel with the same integers which is a model of ATR\(_0\).

Proof. Let \( M^J_0 \) be a model of ATR\(_0^J\) plus \( \sigma \) where \( \sigma \) is a \( \Sigma^1_1 \) sentence. Write \( \sigma \equiv \exists \phi(X) \) where \( \phi \) is arithmetical. By Lemmas 5.1 and 5.2 we can find a sequence of sets \( Z_k, k \in \omega \), such that \( M^J_0 \) satisfies \( \forall(Z_0) \) and \( Z_k \ll Z_{k+1} \). Here \( k \) ranges over standard integers. By Lemma 4.6 of [6] the sets which are recursive in \( Z_k \) for some \( k \) form a model \( M_0 \) of ATR\(_0\). This model \( M_0 \) is a submodel of \( M^J_0 \).

The next corollary was proved earlier by Friedman [6], [6].

5.6 Corollary. The proof theoretic ordinal of ATR\(_0\) is \( \Gamma_{\varepsilon_0} \).

Proof. From Theorem 5.5 it follows that ATR\(_0\) and ATR\(_0^J\) have the same proof theoretic ordinal. Jäger [10] has shown that the proof theoretic ordinal of ATR\(_0^J\) is \( \Gamma_{\varepsilon_0} \).

We do not know the proof theoretic ordinal of \( \Sigma^1_1\text{-}TI_0 \) or of \( \Sigma^1_1\text{-}TI_0 + \Pi^1_1\text{-}TI_0 \) or of \( \Sigma^1_1\text{-}TI + \Pi^1_1\text{-}TI \).

It is fairly clear that the proofs of Theorems 5.3 and 5.5 can be made to yield general results in the style of Theorems 2.8 and 4.1. We leave these general formulations to the reader.
Bibliography


