Degrees of Unsolvability

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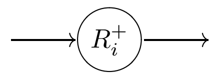
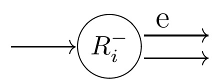
1 Oracle computations

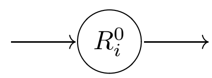
We use the register machine model of computation. Each register R_i contains a natural number r_i. There are three types of instructions:



add 1 to r_i (increment instruction)



if $r_i = 0$ go to e, otherwise subtract 1 from r_i
(decrement instruction)



if $r_i = n$ replace n by $g(n)$
(oracle instruction)

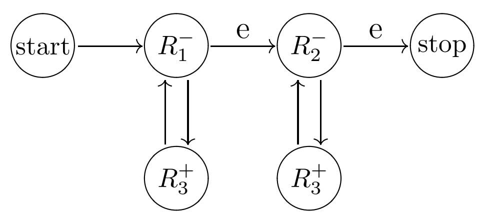
Definition 1.1. An *oracle program* is a connected flow diagram consisting of the above instructions plus (start)$\longrightarrow$ and $\longrightarrow$(stop). Each program has exactly one (start) instruction.

Definition 1.2. A (partial) function $\psi :\subseteq \mathbb{N}^k \to \mathbb{N}$ is said to be *(partial) computable* (i.e., (partial) recursive) if there exists a program $\mathcal{P}$ (with no oracle instructions) which *computes* ψ in the sense that for all $m_1, \dots, m_k \in \mathbb{N}$,

$$\psi(m_1, \dots, m_k)\downarrow \quad \Leftrightarrow \quad \mathcal{P}(m_1, \dots, m_k) \text{ halts}$$

in which case $\psi(m_1, \dots, m_k)$ appears in R_{k+1}. Here $\mathcal{P}(m_1, \dots, m_k)$ denotes the run of $\mathcal{P}$ starting with $m_1, \dots, m_k$ in $R_1, \dots, R_k$ and all other registers empty.

Example 1.3. The 2-place function $f(m_1, m_2) = m_1 + m_2$ is computable via the program



Remark 1.4. We assume familiarity with Church's Thesis for register machine programs. Thus, the functions which are "computable" in an intuitive sense are precisely the functions which are computable by register machine programs.

Definition 1.5. An *oracle* is a total function $g : \mathbb{N} \to \mathbb{N}$. The set of all such functions is denoted $\mathbb{N}^{\mathbb{N}}$. We sometimes view $\mathbb{N}^{\mathbb{N}}$ as a topological space. This space is known as the *Baire space*.

Notation 1.6. We write $\mathcal{P}^g(m_1, \ldots, m_k)$ to denote the run of the program $\mathcal{P}$ starting with $m_1, \ldots, m_k$ in $R_1, \ldots, R_k$ and all other registers empty, using g as the oracle.

Definition 1.7. Given $g \in \mathbb{N}^{\mathbb{N}}$ and $k \geq 1$, a partial function $\psi^g :\subseteq \mathbb{N}^k \to \mathbb{N}$ is said to be *(partial) g-computable* (i.e., computable relative to g) if there exists an oracle program $\mathcal{P}$ such that for all $m_1, \ldots, m_k \in \mathbb{N}$,

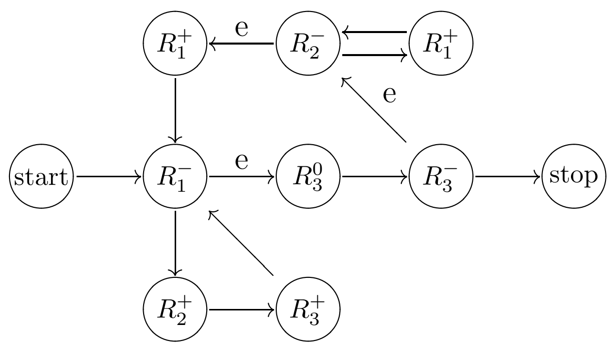
$$\psi^g(m_1, \ldots, m_k) \downarrow \quad \Leftrightarrow \quad \mathcal{P}^g(m_1, \ldots, m_k) \text{ halts}$$

in which case $\psi^g(m_1, \ldots, m_k)$ appears in R_{k+1}.

Example 1.8. Given $g \in \mathbb{N}^{\mathbb{N}}$, the partial function

$$\psi^g(m) \simeq \text{least } n > m \text{ such that } g(n) = 0$$

is partial g-computable via the program



Remark 1.9. In the above definition, we have viewed the oracle g as fixed. An alternative point of view is to regard g as variable, as in the following definition.

Definition 1.10. For $k \geq 0$, a *partial functional* $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}$ is said to be *(partial) computable*, i.e., (partial) recursive, if there exists an oracle program $\mathcal{P}$ such that for all $g \in \mathbb{N}^{\mathbb{N}}$ and all $m_1, \ldots, m_k \in \mathbb{N}$,

$$\Psi(g, m_1, \ldots, m_k) \downarrow \quad \Leftrightarrow \quad \mathcal{P}^g(m_1, \ldots, m_k) \text{ halts}$$

in which case $\Psi(g, m_1, \ldots, m_k)$ appears in R_{k+1}.

Example 1.11. The partial functional

$$\Psi(g, m) \simeq \text{least } n > m \text{ such that } g(n) = 0$$

is computable, by the same program as in Example 1.8.

Remark 1.12. By the special case $k = 0$ of Definition 1.10, we now know what it means for a partial functional from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$ to be computable. Likewise, there is an obvious way to define what it means for a partial functional from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$ to be computable. More generally, we have the following definition.

Definition 1.13. For $k \geq 0$, a partial functional $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^k \to \mathbb{N}^\mathbb{N}$ is said to be *(partial) computable*, i.e., (partial) recursive, if there exists an oracle program $\mathcal{P}$ such that for all $g, h \in \mathbb{N}^\mathbb{N}$ and all $m_1, \ldots, m_k \in \mathbb{N}$,

$$\Psi(g, m_1, \ldots, m_k) = h \quad \Leftrightarrow \quad \forall m \, (\mathcal{P}^g(m_1, \ldots, m_1, m) \text{ halts})$$

in which case $h(m)$ appears in R_{k+2}.

2 Mass problems

Definition 2.1. Given $f, g \in \mathbb{N}^\mathbb{N}$ we say that f is *Turing reducible* to g, abbreviated $f \leq_T g$, if f is g-computable. Equivalently, there exists a partial recursive functional (in the sense of Definition 1.13 with $k = 0$) $\Psi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $f = \Psi(g)$.

Remark 2.2. Some basic facts about $\leq_T$ are:

1. $f \leq_T f$.
2. $f \leq_T g$ and $g \leq_T h$ imply $f \leq_T h$.

Definition 2.3. A *mass problem* is a set $P \subseteq \mathbb{N}^\mathbb{N}$.

Definition 2.4. Let P, Q be mass problems. We say that P is *weakly reducible* to Q, abbreviated $P \leq_w Q$, if for all $g \in Q$ there exists $f \in P$ such that $f \leq_T g$.

Remark 2.5. By way of motivation, note that a mass problem is identified with its set of solutions. Thus, if P is a mass problem, a "solution" of P is an element of the set P. In these terms, weak reducibility of P to Q has the following natural interpretation: given any solution of Q, we can use it to compute some solution of P.

Remark 2.6. Some basic facts about $\leq_w$ are:

1. $P \leq_w P$.
2. $P \leq_w Q$ and $Q \leq_w R$ imply $P \leq_w R$.

Definition 2.7. The *weak degree* of P, denoted $\deg_w(P)$, is the set of mass problems which are equivalent to P with respect to weak reducibility.

Details: We define $P \equiv_w Q$ if and only if $P \leq_w Q$ and $Q \leq_w P$. Note that $\equiv_w$ is an equivalence relation on the set of all mass problems. Note that the set of all mass problems is just the powerset of $\mathbb{N}^\mathbb{N}$. For each mass problem P we define $\deg_w(P)$ to be the equivalence class of P under $\equiv_w$. We write

$$\mathcal{D}_w \;=\; \text{the set of all weak degrees} \;=\; \text{Powerset}(\mathbb{N}^\mathbb{N})/\equiv_w .$$

We partially order $\mathcal{D}_w$ by letting $\deg_w(P) \leq \deg_w(Q)$ if and only if $P \leq_w Q$.

An alternative reducibility notion for mass problems is:

Definition 2.8. Let P, Q be mass problems. We say that P is *strongly reducible* to Q, abbreviated $P \leq_s Q$, if there exists a partial recursive functional $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ (in the sense of Definition 1.13 with $k = 0$) such that $\Psi(g) \in P$ for all $g \in Q$.

Remark 2.9. Clearly $P \leq_s Q$ implies $P \leq_w Q$. To compare these reducibility notions, note that

$$P \leq_s Q \quad \Leftrightarrow \quad \exists \Psi \, (\forall g \in Q) \, (\Psi(g) \in P)$$

while

$$P \leq_w Q \quad \Leftrightarrow \quad (\forall g \in Q) \, \exists \Psi \, (\Psi(g) \in P).$$

Thus strong reducibility is a "uniform" variant of weak reducibility.

Remark 2.10. Some basic facts about $\leq_s$ are:

1. $P \leq_s P$.
2. $P \leq_s Q$ and $Q \leq_s R$ imply $P \leq_s R$.

Definition 2.11. The *strong degree* of P, denoted $\deg_s(P)$, is the equivalence class of P under the equivalence relation $\equiv_s$ defined by $P \equiv_s Q$ if and only if $P \leq_s Q$ and $Q \leq_s P$. We write

$$\mathcal{D}_s \;=\; \text{the set of all strong degrees} \;=\; \text{Powerset}(\mathbb{N}^{\mathbb{N}}) / \equiv_s$$

and $\mathcal{D}_s$ is partially ordered by letting $\deg_s(P) \leq \deg_s(Q)$ if and only if $P \leq_s Q$.

Remark 2.12. An outline of the history of these concepts is as follows.

- Kolmogorov 1932: A "calculus of problems" (non-rigorous) as an interpretation of intuitionistic propositional calculus.
- Medvedev 1955: Defined mass problems, strong reducibility (also known as Medvedev reducibility and Medvedev degrees, or "degrees of difficulty").
- Muchnik 1963: Defined weak reducibility and weak degrees (also known as Muchnik reducibility and Muchnik degrees).
- Later work by Skvortsova, Sorbi, Terwijn, ...

Our principal reason for studying $\mathcal{D}_w$ and $\mathcal{D}_s$ is to use them as a tool for classifying unsolvable (that is, algorithmically unsolvable) mathematical problems. This is different from the original intuitionistic motivation of Kolmogorov and Medvedev.

Definition 2.13. A mass problem P is said to be *solvable* (i.e., algorithmically solvable) if P contains a recursive function. In other words, $P \cap \text{REC} \neq \emptyset$. Equivalently, $\deg_w(P) = \mathbf{0} =$ the smallest degree in $\mathcal{D}_w$ and $\deg_s(P) = \mathbf{0} =$ the smallest degree in $\mathcal{D}_s$. Otherwise, the mass problem P is said to be *unsolvable.*

Examples 2.14. Some well-known examples of unsolvable problems are:

1. The Halting Problem

2. The Word Problem for Groups

3. Hilbert's 10th Problem

4. The Entscheidungsproblem, i.e., the Validity Problem for predicate calculus

5. The problem of finding a completion of first-order arithmetic

3 Some basic theorems

In this section we note that some familiar basic theorems concerning partial recursive functions can be generalized straightforwardly to partial recursive functionals.

Definition 3.1. If $\mathcal{P}$ is an oracle program, we define the *Gödel number* of $\mathcal{P}$, denoted $\#(\mathcal{P})$, as follows. View $\mathcal{P}$ as a finite sequence of numbered instructions, $I_1, \ldots, I_l$, where I_1 is the initial instruction. Then

$$\#(\mathcal{P}) = \prod_{m=1}^{l} p_m^{\#(I_m)}$$

where the Gödel numbers of the instructions are given by

$$\#\left(\circled{R_i^+} \longrightarrow I_n \right) = 3^i \cdot 5^n$$

$$\#\left(\circled{R_i^-} \begin{array}{l} \xrightarrow{\text{e}} I_{n_0} \\ \longrightarrow I_{n_1} \end{array} \right) = 2 \cdot 3^i \cdot 5^{n_0} \cdot 7^{n_1}$$

$$\#\left(\circled{R_i^0} \longrightarrow I_n \right) = 4 \cdot 3^i \cdot 5^n$$

Note that the mapping $\# : \{\text{oracle programs}\} \to \mathbb{N}$ is one-to-one.

Definition 3.2. If $e = \#(\mathcal{P})$ and $k \geq 0$ define
$\Phi_e^{(k)}(g, m_1, \ldots, m_k) \simeq \varphi_e^{(k),g}(m_1, \ldots, m_k) \simeq$ the content of R_{k+1} if and when $\mathcal{P}^g(m_1, \ldots, m_k)$ halts, otherwise undefined.

Theorem 3.3 (The Enumeration Theorem)**.** For each $k \geq 0$, $\Phi_e^{(k)}(g, m_1, \ldots, m_k)$ is a partial recursive functional of $g, e, m_1, \ldots, m_k$.

Proof. This is a straightforward generalization of the usual non-oracle proof. Define $\mathrm{State}^g(e, m_1, \ldots, m_k, s)$ by primitive recursion on s, etc. □

Theorem 3.4 (The Parametrization Theorem)**.** Let $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{k+1} \to \mathbb{N}$ be a partial recursive functional. Then, we can find a primitive recursive function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\Phi_{f(m)}^{(k)}(g, m_1, \ldots, m_k) \simeq \Psi(g, m, m_1, \ldots, m_k)$$

for all $g \in \mathbb{N}^{\mathbb{N}}$ and all $m, m_1, \ldots, m_k \in \mathbb{N}$.

Proof. We sketch the proof. Let $\mathcal{P}$ be an oracle program which computes Ψ. Define $f(m) = \#(\mathcal{P}'_m)$ where $\mathcal{P}'_m$ is $\mathcal{P}$ with m hardwired. It can be shown that f is primitive recursive. □

Theorem 3.5 (The Recursion Theorem)**.** Let $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{k+1} \to \mathbb{N}$ be a partial recursive functional. Then, we can find $e \in \mathbb{N}$ such that

$$\Phi_e^{(k)}(g, m_1, \ldots, m_k) \simeq \Psi(g, e, m_1, \ldots, m_k)$$

for all $g \in \mathbb{N}^{\mathbb{N}}$ and all $m_1, \ldots, m_k \in \mathbb{N}$.

Proof. As usual. □

4 The arithmetical hierarchy

In this section we generalize the arithmetical hierarchy from predicates on $\mathbb{N}^k$ to predicates on $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k$.

Definition 4.1. For $k \geq 0$, a predicate $R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k$ is said to be *recursive* (i.e., *computable*) if its characteristic function $\chi_R : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \to \{0, 1\}$ given by

$$\chi_R(g, m_1, \ldots, m_k) = \begin{cases} 1 & \text{if } R(g, m_1, \ldots, m_k) \\ 0 & \text{if } \neg R(g, m_1, \ldots, m_k) \end{cases}$$

is recursive (i.e., computable).

Definition 4.2. For $k \geq 0$ and $n \geq 1$, a predicate $S \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k$ is said to be Σ_n^0 if it is of the form

$$S(g, m_1, \ldots, m_k) \equiv \exists i_1 \, \forall i_2 \cdots \exists/\forall i_n \, R(g, m_1, \ldots, m_k, i_1, \ldots, i_n)$$

where $R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{k+n}$ is recursive. Similarly, S is said to be Π_n^0 if it is of the form

$$S(g, m_1, \ldots, m_k) \equiv \forall i_1 \, \exists i_2 \cdots \exists/\forall i_n \, R(g, m_1, \ldots, m_k, i_1, \ldots, i_n)$$

where R is recursive. We say that S is Δ_n^0 if it is both Σ_n^0 and Π_n^0.

Definition 4.3. Given $f, g \in \mathbb{N}^{\mathbb{N}}$, let $f \oplus g \in \mathbb{N}^{\mathbb{N}}$ be given by

$$\begin{aligned} (f \oplus g)(2n) &= f(n) \\ (f \oplus g)(2n+1) &= g(n). \end{aligned}$$

We can then define one-to-one mappings $(\mathbb{N}^{\mathbb{N}})^k \to \mathbb{N}^{\mathbb{N}}$ by letting $f_1 \oplus f_2 \oplus \cdots \oplus f_{k+1} = f_1 \oplus (f_2 \oplus \cdots \oplus f_k)$. In this way we can extend the arithmetical hierarchy to predicates $S \subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^l$.

Lemma 4.4. The arithmetical hierarchy has the following closure properties.

1. For $n \geq 1$ the class Σ^0_n is closed under conjunction ($\wedge$), disjunction ($\vee$), bounded number quantification ($\exists n < t$, $\forall n < t$), and existential number quantification ($\exists n$).

2. For $n \geq 1$ the class Π^0_n is closed under conjunction, disjunction, bounded number quantification, and universal number quantification ($\forall n$).

3. For $n \geq 1$ the classes Σ^0_n and Π^0_n are closed under substitution of total recursive functionals. In detail:

   (a) If $F : (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^l \to \mathbb{N}$ is a total recusive functional in the sense of Definitions 1.10 and 4.3, and if the predicate $S(g_1, \ldots, g_k, m, m_1, \ldots, m_l)$ is Σ^0_n, then the predicate

   $$S(g_1, \ldots, g_k, F(g_1, \ldots, g_k, m_1, \ldots, m_l), m_1, \ldots, m_l) \qquad (1)$$

   is Σ^0_n.

   (b) If $F : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}^{\mathbb{N}}$ is a total recusive functional in the sense of Definitions 1.13 and 4.3, and if the predicate $S(g_1, \ldots, g_k, g, m_1, \ldots, m_l)$ is Σ^0_n, then the predicate (1) is Σ^0_n.

   And similarly for Π^0_n.

4. S is Σ^0_n if and only if $\neg S$ is Π^0_n.

Proof. Straightforward, as usual. □

Exercises 4.5. Let $S \subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^l$.

1. Prove that S is Σ^0_1 if and only if $S = \mathrm{dom}(\Psi)$ for some partial recursive functional $\Psi :\subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^l \to \mathbb{N}$ as in Definitions 1.10 and 4.3.

2. Prove that S is Δ^0_1 if and only if S is recursive.

3. Prove that S is Π^0_2 if and only if $S = \mathrm{dom}(\Psi)$ for some partial recursive functional $\Psi :\subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^l \to \mathbb{N}^{\mathbb{N}}$ as in Definitions 1.13 and 4.3.

Theorem 4.6. For each $k \geq 0$ and $n \geq 1$, we can find a *universal* Σ^0_n *predicate* $U_{n,k}$ having the following properties:

1. $U_{n,k} \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{k+1}$ and $U_{n,k}$ is Σ^0_n.

2. Given a Σ^0_n predicate $S \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k$, we can find $e \in \mathbb{N}$ such that

$$S(g, m_1, \ldots, m_k) \equiv U_{n,k}(g, e, m_1, \ldots, m_k)$$

   for all $g \in \mathbb{N}^{\mathbb{N}}$ and all $m_1, \ldots, m_k \in \mathbb{N}$.

Proof. As usual. Namely, $U_{1,k}(g, e, m_1, \ldots, m_k) \equiv \Phi^{(k)}_e(g, m_1, \ldots, m_k) \downarrow$ and $U_{n+1,k}(g, e, m_1, \ldots, m_k) \equiv \exists m \, \neg U_{n,k+1}(g, e, m_1, \ldots, m_k, m)$. □

Theorem 4.7. For each $n \geq 1$ and $k \geq 0$, the universal Σ^0_n predicate $U_{n,k}$ is not Π^0_n.

Proof. As usual, by diagonalization. □

Corollary 4.8. For each $n \geq 1$ we can find a mass problem $S_n \subseteq \mathbb{N}^{\mathbb{N}}$ which is Σ^0_n but not Π^0_n.

Proof. For instance $S_n = \{g \mid U_{n,0}(g^-, g(0))\}$ where $g^-(m) = g(m+1)$. □

5 Mass problems in the arithmetical hierarchy

In this section we present some initial results and problems concerning the weak and strong degrees of mass problems at various levels of the arithmetical hierarchy.

Theorem 5.1. If $S \subseteq \mathbb{N}^{\mathbb{N}}$ is nonempty and Σ^0_1, then $\deg_s(S) = \deg_w(S) = \mathbf{0}$, i.e. $S \cap \mathrm{REC} \neq \emptyset$.

Before proving this theorem, we consider finite sequences.

Definition 5.2. $\mathbb{N}^{<\mathbb{N}}$ is the set of finite sequences of natural numbers. For $g \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$ we write $g \upharpoonright n = \langle g(0), \ldots, g(n-1) \rangle \in \mathbb{N}^{<\mathbb{N}}$. For $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we write $|\sigma| =$ length of σ. Note that $|g \upharpoonright n| = n$.

Remark 5.3 (The Finite Use Principle)**.** If an oracle computation $\mathcal{P}^g(m_1, \ldots, m_k)$ halts, it "uses" only a finite amount of information about the oracle g. Hence, for all sufficiently large n and all $h \in \mathbb{N}^{\mathbb{N}}$, if $h \upharpoonright n = g \upharpoonright n$ then $\mathcal{P}^h(m_1, \ldots, m_k)$ halts with the same output.

Proof of Theorem 5.1. Let $S \subseteq \mathbb{N}^{\mathbb{N}}$ be Σ^0_1. By Exercise 4.5.1 let $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ be a partial recursive functional such that $S = \mathrm{dom}(\Psi)$. Let $\mathcal{P}$ be a program which computes Ψ. For all g we have $g \in S \Leftrightarrow \Psi(g) \downarrow$. Since $S \neq \emptyset$, fix $g \in S$. Since $\Psi(g) \downarrow$, $\mathcal{P}^g(0)$ halts. By the Finite Use Principle, let n be so large that the computation $\mathcal{P}^g(0)$ uses only oracle information from $g \upharpoonright n$. Then, for any $h \in \mathbb{N}^{\mathbb{N}}$ such that $g \upharpoonright n = h \upharpoonright n$, $\mathcal{P}^h(0)$ also halts. Now let $h \in \mathbb{N}^{\mathbb{N}}$ be a recursive function such that $g \upharpoonright n = h \upharpoonright n$. For example, we can define $h(i) = g(i)$ for $i < n$ and $h(i) = 0$ for $i \geq n$. Since $\mathcal{P}^h(0)$ halts, $\Psi(h) \downarrow$, so $h \in S$. Hence $S \cap \mathrm{REC} \neq \emptyset$, i.e., S is algorithmically solvable. □

Theorem 5.4. We can find a nonempty Π^0_1 set $P \subseteq \mathbb{N}^\mathbb{N}$ such that $\deg_w(P) > \mathbf{0}$. Equivalently, $\deg_s(P) > \mathbf{0}$.

To prove this theorem, we use the following concept.

Definition 5.5. A function $g \in \mathbb{N}^\mathbb{N}$ is called *diagonally nonrecusive*, abbreviated DNR, if $g(m) \not\simeq \varphi^{(1)}_m(m)$ for all m. Here of course $\varphi^{(1)}_e$ is the partial recursive function with index e, as in the Enumeration Theorem.

Remark 5.6. Diagonally nonrecursive functions are so called, because they are nonrecursive functions obtained by diagonalization against the recursive functions. To see this, let $g \in \mathbb{N}^\mathbb{N}$ be DNR. If g were recursive, let e be an index of g, i.e., $g(m) = \varphi^{(1)}_e(m)$ for all m. Setting $m = e$ we obtain $g(e) = \varphi^{(1)}_e(e)$, a contradiction.

Proof of Theorem 5.4. Let $P = \mathrm{DNR} = \{g \in \mathbb{N}^\mathbb{N} \mid g \text{ is diagonally nonrecursive}\}$. To see that DNR is Π^0_1, note that

$$\begin{aligned} g \in \mathrm{DNR} \quad &\equiv \quad \forall n\, (\varphi^{(1)}_m(m)\downarrow \Rightarrow \varphi^{(1)}_m(m) \neq g(m)) \\ &\equiv \quad \forall m\, \forall s\, ((\mathrm{State}(m,m,s))_0 = 0 \Rightarrow (\mathrm{State}(m,m,s))_2 \neq g(m)) \end{aligned}$$

and this is Π^0_1, because the State function is primitive recursive. It is also clear that DNR is nonempty and contains only nonrecursive functions. Hence $\deg_w(\mathrm{DNR}) > \mathbf{0}$. □

We now turn to higher levels of the arithmetical hierarchy.

Theorem 5.7. Given a Σ^0_3 set $S \subseteq \mathbb{N}^\mathbb{N}$, we can find a Π^0_1 set $P \subseteq \mathbb{N}^\mathbb{N}$ such that $P \equiv_w S$, i.e., $\deg_w(P) = \deg_w(S)$.

Remark 5.8. In fact, P and S will be *Turing degree isomorphic*. This means that for all $f \in P$ there exists $g \in S$ such that $f \equiv_T g$ and vice versa. Note that Turing degree isomorphism implies weak equivalence.

In order to prove the theorem, we use the following notation.

Notation 5.9.

1. For $f, g \in \mathbb{N}^\mathbb{N}$ define $f \oplus g \in \mathbb{N}^\mathbb{N}$ by letting $(f \oplus g)(2m) = f(m)$ and $(f \oplus g)(2m+1) = g(m)$ for all m. Note that $f \oplus g$ "encodes" f and g.

2. For $i \in \mathbb{N}$ and $f \in \mathbb{N}^\mathbb{N}$, define $g = \langle i \rangle ^\frown f \in \mathbb{N}^\mathbb{N}$ by letting $g(0) = i$ and $g(m+1) = f(m)$ for all m. Note that $\langle i \rangle ^\frown f$ "encodes" i and f.

Proof of Theorem 5.7. Since S is Σ^0_3, let $R(g,i,m,n)$ be a recursive predicate, $R \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^3$, such that $g \in S \equiv \exists i\, \forall m\, \exists n\, R(g,k,m,n)$. Now let

$$P = \{\langle i \rangle ^\frown g \oplus h \mid \forall m\, (h(m) = \text{least } n \text{ such that } R(g,i,m,n))\}.$$

Given $g \in S$, let $i \in \mathbb{N}$ be such that $\forall m \, \exists n \, R(g,i,m,n)$. Define $h \in \mathbb{N}^\mathbb{N}$ by $h(m) =$ least n such that $R(g,i,m,n)$. Note that $h \leq_T g$. Thus we have $\langle i \rangle^\frown g \oplus h \in P$ and $\equiv_T g$. Conversely, every member of P arises in this way, so P is Turing degree isomorphic to S. □

Exercise 5.10. Given a Π^0_2 set $Q \subseteq \mathbb{N}^\mathbb{N}$, find a Π^0_1 set $P \subseteq \mathbb{N}^\mathbb{N}$ such that $\deg_s(P) = \deg_s(Q)$. In fact, P and Q are *recursively homeomorphic*. This means that there are recusive functionals $\Psi : Q \to P$ and $\Psi^{-1} : P \to Q$. Note that recursive homeomorphism implies strong equivalence.

Remark 5.11. A possible research problem is to investigate what happens at higher levels of the arithmetical hierarchy. For $n \geq 4$, can we find a Σ^0_n mass problem which is not weakly equivalent to any Σ^0_{n-1} mass problem?

Remark 5.12. We now consider the case $n = 4$. Consider the mass problem

$$Q \; = \; \mathbb{N}^\mathbb{N} \setminus \mathrm{REC} \; = \; \{g \in \mathbb{N}^\mathbb{N} \mid g \text{ is not recursive}\}.$$

Clearly $Q \neq \emptyset$ and $Q \cap \mathrm{REC} = \emptyset$. Moreover Q is Π^0_3. To see this, note that

$$\begin{aligned} g \in \mathrm{REC} \quad &\equiv \quad \exists e \, \forall m \, (\varphi^{(1)}_e(m) \downarrow = g(m)) \\ &\equiv \quad \exists e \, \forall m \, \exists s \, ((\mathrm{State}(e,m,s))_0 = 0 \wedge (\mathrm{State}(e,m,s))_2 = g(m)) \end{aligned}$$

so REC is Σ^0_3. Since Q is the complement of REC, Q is Π^0_3. We shall show that, as a mass problem, Q is not weakly equivalent to any Σ^0_3 mass problem. This will be a consequence of the following theorem.

Theorem 5.13. We can find $g \notin \mathrm{REC}$ with the following property: for any Σ^0_3 set $S \subseteq \mathbb{N}^\mathbb{N}$, if $\exists f \, (f \leq_T g \wedge f \in S)$ then $\exists f \, (f \text{ is recursive} \wedge f \in S)$.

The proof of this theorem will be presented in terms of strings, trees, and genericity.

6 Strings and trees

Definition 6.1. A *string* is a finite sequence of natural numbers.

Notation 6.2. We let $\sigma, \tau, \ldots$ denote strings. $\mathbb{N}^{<\mathbb{N}}$ denotes the set of all strings. $|\sigma|$ is the length of σ. If $\sigma = \langle \sigma(0), \sigma(1), \ldots, \sigma(m-1) \rangle$ where $m = |\sigma|$ and $\tau = \langle \tau(0), \tau(1), \ldots, \tau(n-1) \rangle$ where $n = |\tau|$, then the concatenation of σ and τ is

$$\sigma^\frown\tau = \langle \sigma(0), \ldots, \sigma(m-1), \tau(0), \ldots, \tau(n-1) \rangle.$$

Note that $|\sigma^\frown\tau| = |\sigma| + |\tau|$.

If $g \in \mathbb{N}^\mathbb{N}$, then $g \upharpoonright n = \langle g(0), \ldots, g(n-1) \rangle \in \mathbb{N}^{<\mathbb{N}}$. We write $\tau \subset g$ if $\tau = g \upharpoonright |\tau|$, i.e., τ is an initial segment of g. If $n \leq |\tau|$, we write $\tau \upharpoonright n = \langle \tau(0), \ldots, \tau(n-1) \rangle$. We write $\sigma \subseteq \tau$ if $\sigma = \tau \upharpoonright n$ for some $n \leq |\tau|$. We write $\sigma \subset \tau$ if $\sigma = \tau \upharpoonright n$ for some $n < |\tau|$.

Remark 6.3. If $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_n \subseteq \cdots$ and $\lim_{n\to\infty} |\tau_n| = \infty$ then

$$g = \bigcup_{n=0}^{\infty} \tau_n \in \mathbb{N}^{\mathbb{N}}$$

Thus we can use strings to approximate oracles $g \in \mathbb{N}^{\mathbb{N}}$.

We now reformulate the finite use principle in terms of strings.

Definition 6.4. $\Phi^{(k)}_{e,s}(\tau, m_1, \ldots, m_k) \simeq \varphi^{(k),\tau}_{e,s}(m_1, \ldots, m_k) \simeq n$ means: $e = \#(\mathcal{P})$ for some program $\mathcal{P}$, and $\mathcal{P}^g(m_1, \ldots, m_k)$ halts with output n in R_{k+1} in $\leq s$ steps having used only oracle information from $\tau \subset g$.

Remark 6.5. The finite use principle implies that $\varphi^{(k),g}_e(m_1, \ldots, m_k) \simeq n$ if and only if $\exists s\, (\varphi^{(k),g\restriction s}_{e,s}(m_1, \ldots, m_k) \simeq n)$.

Remark 6.6. We Gödel number strings with the one-to-one function $\# : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ defined by $\#(\sigma) = \prod_{i<|\sigma|} p_i^{\sigma(i)+1}$ where p_i is the ith prime number. We frequently identify strings with their Gödel numbers. Thus, the functions $\sigma, \tau \mapsto \sigma^\frown\tau$ and $\sigma, i \mapsto \sigma(i)$ and $\sigma \mapsto |\sigma|$, etc., are primitive recursive.

Lemma 6.7. The predicates $\varphi^{(k),\tau}_{e,s}(m_1, \ldots, m_k) \simeq n$ and $\varphi^{(k),\tau}_{e,s}(m_1, \ldots, m_k)\downarrow$ are primitive recursive.

Proof. Routine. One shows that $\mathrm{State}^\tau(e, m_1, \ldots, m_k, s)$ is a primitive recursive function, etc. □

We now consider trees.

Definition 6.8. A *tree* is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which is closed under initial segments. This means: if $\tau \in T$ then $\sigma \in T$ for all $\sigma \subset \tau$. If T is a tree, a *path* through T is any $f \in \mathbb{N}^{\mathbb{N}}$ such that $f \restriction n \in T$ for all n.

Trees are often used to represent Π^0_1 mass problems. Namely, a Π^0_1 mass problem may be viewed as the problem of finding a path through a recursive tree, as explained in the following lemma.

Lemma 6.9. Let $P \subseteq \mathbb{N}^{\mathbb{N}}$ be Π^0_1. Then we can find a recursive tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $P = \{f \mid f \text{ is a path through } T\}$.

Proof. Since P is Π^0_1, we have $P = \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n\, R(f, n)\}$ where $R \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ is a recursive predicate. Define a partial recursive functional $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by $\Psi(f) \simeq$ the least n such that $\neg R(f, n)$. Then $P = \{f \in \mathbb{N}^{\mathbb{N}} \mid \Psi(f)\uparrow\}$. Let e be an index of Ψ. Then

$$P = \{f \in \mathbb{N}^{\mathbb{N}} \mid \varphi^{(1),f}_e(0)\uparrow\} = \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall s\, \varphi^{(1),f\restriction s}_{e,s}(0)\uparrow\}$$

so let $T = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \varphi^{(1),\sigma}_{e,|\sigma|}(0)\uparrow\}$. T is recursive, because the predicate $\varphi^{(k),\tau}_{e,|\tau|}(m_1, \ldots, m_k)\downarrow$ is recursive. T is a tree, because if $\varphi^{(k),\tau}_{e,|\tau|}(m_1, \ldots, m_k)\uparrow$ and $\sigma \subseteq \tau$ then $\varphi^{(1),\sigma}_{e,|\sigma|}(m_1, \ldots, m_k)\uparrow$. Clearly $P = \{\text{paths through } T\}$. □

For convenience we shall sometimes use the following notation for partial recursive functionals.

Definition 6.10. $\widehat{\Phi}_e : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is given by

$$\widehat{\Phi}_e(\tau) = \langle \varphi^{(1),\tau}_{e,|\tau|}(0), \ldots, \varphi^{(1),\tau}_{e,|\tau|}(l-1) \rangle$$

where $l =$ the least m such that $\varphi^{(1),\tau}_{e,|\tau|}(m) \uparrow$ or $m \geq |\tau|$.

Remark 6.11. Note that $\widehat{\Phi}_e$ is a primitive recursive function from strings to strings. Moreover $\widehat{\Phi}_e$ is *monotonic* in the sense that $\sigma \subseteq \tau$ implies $\widehat{\Phi}_e(\sigma) \subseteq \widehat{\Phi}_e(\tau)$. Also, $f \leq_T g$ if and only if $f = \bigcup_{n=0}^{\infty} \widehat{\Phi}_e(g \upharpoonright n)$ for some e. Note also that, in the proof of Lemma 6.9, the tree T can be described more simply as $T = T_e = \{\sigma \mid |\widehat{\Phi}_e(\sigma)| = 0\}$.

7 Genericity

Definition 7.1. A set of strings $D \subseteq \mathbb{N}^{<\mathbb{N}}$ is said to be *dense* if $(\forall \sigma \in \mathbb{N}^{<\mathbb{N}})\,(\exists \tau \in D)\,(\sigma \subseteq \tau)$.

Remark 7.2. There is a standard topology for the Baire space. The basic open sets for this topology are $N_\sigma = \{f \in \mathbb{N}^\mathbb{N} \mid \sigma \subset f\}$ where σ is a string. Any set of strings $A \subseteq \mathbb{N}^{<\mathbb{N}}$ "encodes" an open set $U_A = \bigcup_{\sigma \in A} N_\sigma$. All open sets are of this form. A set of strings D is *dense* (in the sense defined above) if and only if U_D is topologically dense in $\mathbb{N}^\mathbb{N}$.

Definition 7.3. A set of strings $D \subseteq \mathbb{N}^{<\mathbb{N}}$ is said to be *arithmetical* if D occurs in the arithmetical hierarchy, i.e., $D \in \Sigma^0_\infty = \bigcup_{n=1}^{\infty} \Sigma^0_n$ (identifying strings with their Gödel numbers).

Definition 7.4. A point $g \in \mathbb{N}^\mathbb{N}$ is said to *meet* D if $(\exists \tau \in D)\,(\tau \subset g)$. Equivalently, $g \in U_D$.

Definition 7.5. A point $g \in \mathbb{N}^\mathbb{N}$ is called *generic* (i.e., arithmetically generic) if g meets all dense arithmetical sets of strings.

Lemma 7.6. Given a string τ, we can find a generic g such that $\tau \subset g$.

Proof. Let D_n, $n = 0, 1, 2, \ldots$ be an enumeration of the dense arithmetical sets of strings. Let $\tau_0 = \tau$. Given τ_n, let τ_{n+1} be a proper extension of τ_n, $\tau_{n+1} \supset \tau_n$, such that $\tau_{n+1} \in D_n$. Let $g = \bigcup_{n=0}^{\infty} \tau_n$. Clearly g is generic and $\tau \subset g$. □

Remark 7.7. The previous lemma may be viewed as a consequence of the Baire Category Theorem. The open sets U_{D_n}, $n = 0, 1, 2, \ldots$ are topologically dense in $\mathbb{N}^\mathbb{N}$. The Baire Category Theorem tells us that $\bigcap_{n=0}^{\infty} U_{D_n}$ is topologically dense in $\mathbb{N}^\mathbb{N}$.

Lemma 7.8. If $g \in \mathbb{N}^\mathbb{N}$ is generic, then g is not recursive (in fact, g is not arithmetical).

Proof. Given a recursive or arithmetical function $f \in \mathbb{N}^{\mathbb{N}}$, let $D_f = \{\tau \mid \tau \not\subset f\}$. Clearly D_f is dense. Moreover, since f is arithmetical (recursive), D_f is arithmetical (recursive). Hence g meets D_f, i.e., $g \upharpoonright n \neq f \upharpoonright n$ for some n. Hence $g \neq f$. □

Remark 7.9. Let g be generic. Although g is not computable, we are going to show that g cannot compute a solution of a Π^0_1 mass problem unless the problem already has a computable solution.

Lemma 7.10. Let $g \in \mathbb{N}^{\mathbb{N}}$ be generic, and let $S \subseteq \mathbb{N}^{\mathbb{N}}$ be Σ^0_3 such that $S \cap \mathrm{REC} = \emptyset$. Then $S \cap \mathrm{REC}(g) = \emptyset$.

Proof. By Theorem 5.7 we may assume that S is Π^0_1. By Lemma 6.9 let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a primitive recursive tree such that S consists of all paths through T. Given e we show that $\widehat{\Phi}_e(g)$ is not a path through T. This tells us that g does not compute a path through T.

For each $e \in \mathbb{N}$ let $D_e = \{\tau \mid \neg\,\exists\tau' \supseteq \tau \text{ such that } \widehat{\Phi}_e(\tau) \subsetneqq \widehat{\Phi}_e(\tau') \in T\}$.

We claim that D_e is dense, i.e., for all σ there exists $\tau \supseteq \sigma$ such that $\tau \in D_e$. Suppose this does not hold. Let σ be such that $\tau \notin D_e$ for all $\tau \supseteq \sigma$. Recall that $\widehat{\Phi}_e : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is primitive recursive. Form a sequence $\sigma = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_n \subset \tau_{n+1} \subset \cdots$ such that $\forall n\,(\widehat{\Phi}_e(\tau_n) \subsetneqq \widehat{\Phi}_e(\tau_{n+1}) \in T)$. This is a computable sequence of strings. Now let $h = \bigcup_{n=0}^{\infty} \tau_n$. Then $\widehat{\Phi}_e(h)$ is a recursive path through T, contradicting the assumption that T has no recursive path. This proves the claim.

Moreover D_e is arithmetical (in fact Π^0_1). Since g is generic, $\exists\tau \subset g$ such that $\tau \in D_e$. Hence $\neg\,\exists\tau' \supset \tau$ such that $\widehat{\Phi}_e(\tau) \subsetneqq \widehat{\Phi}_e(\tau') \in T$. In particular $\widehat{\Phi}_e(g)$ is not a path through T. □

Theorem 7.11. We can find a nonrecursive g such that for all Σ^0_3 sets $S \subseteq \mathbb{N}^{\mathbb{N}}$, $S \cap \mathrm{REC}(g) \neq \emptyset$ implies $S \cap \mathrm{REC} \neq \emptyset$.

Proof. This is immediate from Lemmas 7.6 and 7.8 and 7.10. □

Corollary 7.12. For all Σ^0_3 sets $S \subseteq \mathbb{N}^{\mathbb{N}}$, if $S \cap \mathrm{REC} = \emptyset$ then $\exists g \notin \mathrm{REC}$ such that $S \cap \mathrm{REC}(g) = \emptyset$.

Corollary 7.13. Let $Q = \mathbb{N}^{\mathbb{N}} \setminus \mathrm{REC}$. Then Q is Π^0_3 and there is no Σ^0_3 set $S \subseteq \mathbb{N}^{\mathbb{N}}$ such that $Q \equiv_w S$.

Proof. A Tarski/Kuratowski computation shows that Q is Π^0_3. Suppose S is Σ^0_3 and $Q \equiv_w S$. Clearly $S \cap \mathrm{REC} = \emptyset$. Hence, by the previous corollary, we can find $g \in Q$ such that $S \cap \mathrm{REC}(g) = \emptyset$. Hence $S \not\leq_w Q$ □

8 Turing degrees and the jump operator

Recall that, for $f, g \in \mathbb{N}^{\mathbb{N}}$, $f \leq_T g$ means that f is Turing computable using g as an oracle. We define $f \equiv_T g$ to mean that $f \leq_T g$, and $g \leq_T f$. Since $\leq_T$ is reflexive and transitive, $\equiv_T$ is reflexive, transitive and symmetric. Thus $\equiv_T$ is an equivalence relation.

Definition 8.1. For $f \in \mathbb{N}^{\mathbb{N}}$ we write $\deg_T(f)$ = the *Turing degree* of f = the equivalence class of f under $\equiv_T$. We let $\mathcal{D}_T$ = the set of all Turing degrees. We partially order $\mathcal{D}_T$ by letting $\deg_T(f) \leq_T \deg_T(g)$ if and only if $f \leq_T g$.

Remark 8.2. There are obvious embeddings of $\mathcal{D}_T$ into $\mathcal{D}_w$ and $\mathcal{D}_s$, namely $\deg_T(f) \mapsto \deg_w(\{f\})$ and $\deg_T(f) \mapsto \deg_s(\{f\})$. These embeddings are one-to-one and order preserving, because $f \leq_T g \Leftrightarrow \{f\} \leq_w \{g\} \Leftrightarrow \{f\} \leq_s \{g\}$. We shall routinely identify Turing degrees with the corresponding weak and strong degrees under these embeddings.

Remark 8.3. We now mention some easy lattice-theoretical properties of $\mathcal{D}_T$, $\mathcal{D}_w$, and $\mathcal{D}_s$.

1. In $\mathcal{D}_w$ and $\mathcal{D}_s$ and $\mathcal{D}_T$ there is a minimum degree $\mathbf{0}$ which is the degree of solvable problems.

2. In $\mathcal{D}_w$ and $\mathcal{D}_s$ and $\mathcal{D}_T$ there is a binary least upper bound operation. For $\mathbf{a} = \deg_w(P)$ and $\mathbf{b} = \deg_w(Q)$, the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ is $\sup(\mathbf{a}, \mathbf{b}) = \deg_w(P \times Q)$ where $P \times Q = \{f \oplus g \mid f \in P, g \in Q\}$. And similarly for $\mathcal{D}_s$. In particular, if $\mathbf{a} = \deg_T(f)$ and $\mathbf{b} = \deg_T(g)$ then $\sup(\mathbf{a}, \mathbf{b}) = \deg_T(f \oplus g)$. Moreover, our embeddings preserve the least upper bound operation. Thus $\mathcal{D}_w$ and $\mathcal{D}_s$ and $\mathcal{D}_T$ are upper semilattices.

3. $\mathcal{D}_w$ and $\mathcal{D}_s$ have a maximum element: $\infty = \deg_s(\emptyset) = \deg_w(\emptyset) = \{\emptyset\}$. However, $\mathcal{D}_T$ has no maximal element, because given a Turing degree $\mathbf{a}$ we can find a Turing degree $\mathbf{a}' > \mathbf{a}$, namely $\mathbf{a}'$ = the Turing jump of $\mathbf{a}$. The Turing jump is explained below.

4. $\mathcal{D}_w$ and $\mathcal{D}_s$ are lattices. If $\mathbf{a} = \deg_w(P)$ and $\mathbf{b} = \deg_w(Q)$, the greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{D}_w$ is $\inf(\mathbf{a}, \mathbf{b}) = \deg_w(\langle 0 \rangle^\frown P \cup \langle 1 \rangle^\frown Q)$ where $\langle i \rangle^\frown P = \{\langle i \rangle^\frown f \mid f \in P\}$. And this also works for $\mathcal{D}_s$.

   Alternatively, for $\mathbf{a} = \deg_w(P)$ and $\mathbf{b} = \deg_w(Q)$ we have the simpler characterization $\inf(\mathbf{a}, \mathbf{b}) = \deg_w(P \cup Q)$. However, this does not work for $\mathcal{D}_s$.

5. It can be shown that $\mathcal{D}_T$ is not a lattice, i.e., we can find $\mathbf{a}, \mathbf{b} \in \mathcal{D}_T$ such that there is no greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{D}_T$. Also, we can find Turing degrees $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{D}_T$ such that $\mathbf{c} = \inf(\mathbf{a}, \mathbf{b})$ in $\mathcal{D}_T$ but not in $\mathcal{D}_w$ or in $\mathcal{D}_s$. In fact, if $\mathbf{a}$ and $\mathbf{b}$ are incomparable Turing degrees, then $\inf(\mathbf{a}, \mathbf{b})$ in $\mathcal{D}_w$ or $\mathcal{D}_s$ can never be a Turing degree.

Exercises 8.4.

1. Show that $\deg_w(P \cup Q) = \inf(\deg_w(P), \deg_w(Q))$.

2. Given an example where $\deg_s(P \cup Q) \neq \inf(\deg_s(P), \deg_s(Q))$.

3. Prove the assertions in previous remark.

Let $U_{1,1}$ be a universal Σ^0_1 predicate. For instance, $U_{1,1}(g,e,m) \equiv \Phi^{(1)}_e(g,e)\downarrow$. This means that for any Σ^0_1 set $S \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}$ we can find an index e such that $S(g,m) \equiv U_{1,1}(g,e,m)$ for all $g \in \mathbb{N}^\mathbb{N}$ and all $m \in \mathbb{N}$.

Definition 8.5 (the Turing jump)**.** Given $g \in \mathbb{N}^\mathbb{N}$, define $g' \in \mathbb{N}^\mathbb{N}$ as

$$g'(3^e 5^m) = \begin{cases} 1 & \text{if } U_{1,1}(g,e,m) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, g' is the characteristic function of the Halting Problem relative to g. We refer to g' as the *Turing jump* of g.

Exercises 8.6.

1. Show that $g <_T g'$. In other words $g' \not\leq_T g$ but $g \leq_T g'$.
2. Show that the Turing jump operator is well defined and $\leq_T$-preserving on Turing degrees. In other words, $g \leq_T h$ implies $g' \leq_T h'$.
3. Given a mass problem $P \subseteq \mathbb{N}^\mathbb{N}$, we can define the Turing jump of P as $P' = \{f' \mid f \in P\}$. Clearly $P \leq_w P'$. Show that in fact $P \leq_s P'$.
4. Show that $\deg_w(P)' = \deg_w(P')$ is well defined. Show that this jump operator on weak degrees is $\leq$-preserving and extends the Turing jump operator on Turing degrees.
5. Is $\deg_s(P)' = \deg_s(P')$ well defined?
6. Give an example of a mass problem P such that $P \equiv_w P'$.
7. Can you find an example of a mass problem P such that $P \equiv_s P'$?

Definition 8.7. If $\mathbf{a} = \deg_T(g)$, we define $\mathbf{a}' = \deg_T(g')$. Then $\mathbf{a}'$ is called *jump* of $\mathbf{a}$. Note that $\mathbf{a} < \mathbf{a}'$ for all $\mathbf{a} \in \mathcal{D}_T$.

Remark 8.8. In particular, we have the Turing degrees

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}'' < \cdots < \mathbf{0}^{(n)} < \cdots. \tag{2}$$

A famous result known as Post's Theorem reads as follows:

For $n \geq 1$, $A \subseteq \mathbb{N}$ is Δ^0_n if and only if $A \leq_T 0^{(n-1)}$.

Thus we see that the sequence of Turing degrees in (2) is essentially the same as the arithmetical hierarchy of subsets of $\mathbb{N}$.

Here of course we are identifying A with it characteristic function χ_A. For $A \subseteq \mathbb{N}$ we shall routinely write $\deg_T(A) = \deg_T(\chi_A)$.

A special case of Post's Theorem says that A is Δ^0_2 if and only if $A \leq_T 0' =$ the Halting Problem.

9 The Cantor space

Definition 9.1. The Cantor space: $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}} = \{X \mid X : \mathbb{N} \to \{0,1\}\}$.

Remark 9.2. The Cantor space $\{0,1\}^{\mathbb{N}}$ is compact, while the Baire space $\mathbb{N}^{\mathbb{N}}$ is not compact.

Lemma 9.3. For each $f \in \mathbb{N}^{\mathbb{N}}$, we can find $X \in \{0,1\}^{\mathbb{N}}$ such that $f \equiv_T X$.

Proof. We may identify points in $\{0,1\}^{\mathbb{N}}$ as characteristic functions of subsets of $\mathbb{N}$. Thus $\{0,1\}^{\mathbb{N}}$ is in one to one correspondence with Powerset($\mathbb{N}$). Given $f \in \mathbb{N}^{\mathbb{N}}$, let $X = X_f =$ the characteristic function of $\{3^m 5^n \mid f(m) = n\}$. Clearly $X_f \equiv_T f$. □

Lemma 9.4. Let $S \subseteq \mathbb{N}^{\mathbb{N}}$ be Σ^0_n where $n \geq 3$ or Π^0_n where $n \geq 2$. Then we can find $S^* \subset \{0,1\}^{\mathbb{N}}$ at the same level of the arithmetical hierarchy such that S^* is recursively homeomorphic to S.

Proof. We use the mapping $f \mapsto X_f$ from the proof of the previous lemma. Let $S^* = \{X_f \mid f \in S\}$. As an exercise, check that S^* is at the same level of the arithmetical hierarchy as S. □

Definition 9.5. The space $\{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}} = \{X \mid X : \mathbb{N} \to \{0,1\}\}$ is called the *Cantor Space*.

Remark 9.6. Obviously $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, i.e., the Cantor space is a subspace of the Baire space. A significant difference between these spaces is that $2^{\mathbb{N}}$ is compact while $\mathbb{N}^{\mathbb{N}}$ is not.

Notation 9.7. We write $2^{<\mathbb{N}} = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \sigma \text{ is 01-valued}\} = \{\text{bitstrings}\}$. For $\sigma \in \{0,1\}^{<\mathbb{N}}$ let $N_\sigma = \{X \in 2^{\mathbb{N}} \mid \sigma \in X\}$. The sets N_σ, $\sigma \in \{0,1\}^{<\mathbb{N}}$ form a basis for the topology of $2^{\mathbb{N}}$. Compactness of $2^{\mathbb{N}}$ may be rephrased as follows:

If $2^{\mathbb{N}} = \bigcup_{i=0}^{\infty} N_{\sigma_i}$ then $\exists n$ such that $2^{\mathbb{N}} = \bigcup_{i=0}^{n} N_{\sigma_i}$.

We use $X, Y, \ldots$ to denote points in the Cantor Space.

We now prove two important lemmas concerning quantification over the Cantor space. In the next two lemmas, we consider mixed predicates $S \subseteq (\mathbb{N}^{\mathbb{N}})^k \times (2^{\mathbb{N}})^l \times \mathbb{N}^m$.

Lemma 9.8 (bounding principle)**.** Suppose $S(X, i, -)$ is a Σ^0_1 predicate where X ranges over $2^{\mathbb{N}}$. Then

$$\forall X \, \exists i \, S(X, i, -) \quad \equiv \quad \exists j \, \forall X \, (\exists i < j) \, S(X, i, -).$$

Proof. $\Leftarrow$: Obvious.

$\Rightarrow$: Let

$$\begin{aligned} S(X, i, -) \quad &\equiv \quad \Phi_e(X, i, -)\downarrow \\ &\equiv \quad \exists s \, \Phi_{e,s}(X \restriction s, i, -)\downarrow . \end{aligned}$$

Then

$$\begin{aligned}\forall X\,\exists i\,S(X,i,-) &\equiv \forall X\,\exists i\,\exists s\,\Phi_{e,s}(X\upharpoonright s,i,-)\downarrow\\ &\equiv \forall X\,\exists i\,\exists s\,(i<s\wedge\Phi_{e,s}(X\upharpoonright s,i,-)\downarrow)\end{aligned}$$

so we have

$$2^{\mathbb{N}}=\bigcup\{N_\sigma\mid(\exists i<|\sigma|)\,\Phi_{e,|\sigma|}(\sigma,i,-)\downarrow\}.$$

By compactness, $\exists\sigma_1,\ldots,\sigma_n$ as above such that

$$2^{\mathbb{N}}=\bigcup_{k=1}^{n}N_{\sigma_k}$$

so let $j=\max_{1\le k\le n}|\sigma_k|$. Then $\forall X\,\exists i<j\,\Phi_{e,j}(X\upharpoonright j,i,-)$, Q.E.D. □

Lemma 9.9. Suppose $S(X,-)$ be a Σ^0_1 predicate where X ranges over $2^{\mathbb{N}}$. Then, the predicate $\forall X\,S(X,-)$ is again Σ^0_1.

A restatement of this lemma is:

> The class of Σ^0_1 predicates is closed under universal quantification over the Cantor Space.

This is a useful supplement to the closure properties in Lemma 4.4.

Proof. Again, let

$$\begin{aligned}S(X,-) &\equiv \Phi_e(X,-)\downarrow\\ &\equiv \exists s\,\Phi_{e,s}(X\upharpoonright s,-)\downarrow.\end{aligned}$$

Then by the previous lemma we have

$$\begin{aligned}\forall X\,S(X,-) &\equiv \forall X\,\exists s\,\Phi_{e,s}(X\upharpoonright s,-)\downarrow\\ &\equiv \exists s\,\forall X\,\Phi_{e,s}(X\upharpoonright s,-)\\ &\equiv \underbrace{\exists s\quad \underbrace{\forall\sigma\,(|\sigma|=s}_{\text{bdd. quantifier}}\quad\Rightarrow\quad\underbrace{\Phi_{e,s}(\sigma,-)\downarrow)}_{\text{rec. predicate}}}_{\Sigma^0_1}\end{aligned}$$

and this completes the proof. □

10 Basis Theorems in the Cantor space

Recall g' = the Turing jump of g = the characteristic function of $\{3^e5^m\mid\varphi_e^{(1),g}(m)\downarrow\}$. Hence, for all $\Sigma^{0,g}_1$ sets $A\subseteq\mathbb{N}$ we have

$$\begin{aligned}A &= \mathrm{dom}(\varphi_e^{(1),g})\text{ for some }e\\ &= W_e^g.\end{aligned}$$

Hence $A\le_T g'$ because $\chi_A(m)=g'(3^e5^m)$ for all m.

Theorem 10.1 (Kleene Basis Theorem)**.** For all nonempty Π^0_1 sets $P \subseteq 2^{\mathbb{N}}$, we can find $X \in P$ such that $X \leq_T 0'$.

Proof. First consider a nonempty Π^0_1 set $P \subseteq \mathbb{N}^{\mathbb{N}}$, the Baire space. We can associate with P a canonical tree

$$T = T_P = \{f \upharpoonright n \mid f \in P \text{ and } n \in \mathbb{N}\}.$$

Clearly T_P is a tree and $P = \{\text{paths through } T\}$. Note however that, in contrast to Lemma 6.9, T is not necessarily recursive. On the other hand, our tree T is *tidy*, i.e., $(\forall \tau \in T)\, \exists i\, (\tau ^\frown \langle i \rangle \in T)$. Another name for tidy trees is "trees with no dead ends" meaning that every string in T has a proper extension in T. Because T is tidy, the *leftmost path* through T can be obtained as $g = \bigcup_{n=0}^{\infty} \tau_n$ where $\tau_0 = \langle\rangle$ (note that $\langle\rangle \in T$ because P is nonempty) and for all n, $\tau_{n+1} = \tau_n ^\frown \langle i_n \rangle$. Here $i_n =$ the least i such that $\tau_n ^\frown \langle i \rangle \in T$. Clearly $g \in P$ and $g \leq_T T$.

Now consider the case of a nonempty Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, the Cantor space. In this case our tidy tree $T = T_P$ is Π^0_1. This is because $\tau \in T_P$ if and only if $(\exists X \in 2^{\mathbb{N}})\, (X \in P \wedge \tau \subset X)$ and this is Π^0_1 in view of Lemma 9.9. Since T is a Π^0_1 subset of $2^{<\mathbb{N}}$, it follows that $T \leq_T 0'$. Hence the leftmost path through T is $\leq_T 0'$. Thus we have $\exists X\, (X \in P, X \leq_T 0')$, Q.E.D. □

Corollary 10.2. If $\mathbf{c}$ is the weak degree of a nonempty Π^0_1 subset of $2^{\mathbb{N}}$, then $\mathbf{c} \leq \mathbf{0}'$.

Later we shall prove the following refinement of the Kleene Basis Theorem.

Theorem 10.3 (Low Basis Theorem)**.** For all nonempty Π^0_1 sets $P \subseteq 2^{\mathbb{N}}$, $\exists X \in P$ such that X is *low*, i.e., $X' \leq_T 0'$.

Corollary 10.4. If $\mathbf{c}$ is the weak degree of a nonempty Π^0_1 subset of $2^{\mathbb{N}}$, then $\mathbf{c}' \leq \mathbf{0}'$.

Remark 10.5. We shall now show that the Kleene Basis Theorem fails badly for the Baire space. Namely, for each n, $0^{(n)}$ is a Π^0_2 *singleton*, i.e., $\{0^{(n)}\}$ is Π^0_2. Hence, we can find a Π^0_1 singleton $g_n \in \mathbb{N}^{\mathbb{N}}$ such that $g_n \equiv_T 0^{(n)}$.

Consequently, each of the weak degrees $\mathbf{0}^{(n)}$ for $n \geq 0$ is the weak degree of a Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$. Hence, by the Kleene Basis Theorem, the weak degrees of Π^0_1 sets in $2^{\mathbb{N}}$ are properly included in the weak degrees of Π^0_1 sets in $\mathbb{N}^{\mathbb{N}}$.

On the other hand, it can be shown that for each $n \geq 2$ the weak degrees of Π^0_n sets in $2^{\mathbb{N}}$ are the same as the weak degrees of Π^0_n sets in $\mathbb{N}^{\mathbb{N}}$.

Theorem 10.6. For each $n \geq 0$ we can find $g_n \in \mathbb{N}^{\mathbb{N}}$ such that $g_n \equiv_T 0^{(n)}$ and g_n is a Π^0_1 singleton.

Corollary 10.7. For each $n \geq 0$ we can find a Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$ whose weak degree is $\mathbf{0}^{(n)}$.

For the proof of Theorem 10.6 we need several easy lemmas.

Lemma 10.8. The predicate $J(g, h) \equiv g' = h$ is Π^0_2.

Proof. We have $J(g,h) \equiv \forall e\,\forall m\,\forall n\,(h(3^e5^m) = 1$ if $\varphi_e^{(1),g}(m)\downarrow$, $h(3^e5^m) = 0$ if $\varphi_e^{(1),g}\uparrow$, and $h(n) = 0$ otherwise). This predicate is Π^0_2 because the predicate $\varphi_e^{(1),g}(m)\downarrow$ is Σ^0_1. $\square$

Lemma 10.9. For each $n \geq 0$ we can find a Π^0_2 singleton $X_n \in 2^{\mathbb{N}}$ such that $X_n \equiv_T 0^{(n)}$.

Proof. For $g \in \mathbb{N}^{\mathbb{N}}$ let us write $g = (g)_0 \oplus (g)_1$ and note that the functionals $g \mapsto (g)_0$ and $g \mapsto (g)_1$ are recursive. Let $X_0 =$ the constant function 0, and for each n let $X_{n+1} = X_n \oplus X'_n$.

We claim that X_n is a Π^0_2 singleton, i.e., the singleton set $\{X_n\}$ is Π^0_2. We prove this by induction on n. The base step $n = 0$ is trivial. For $n+1$ we have $Z = X_{n+1}$ if and only if $(Z)_0 = X_n$ and $(Z)'_0 = (Z)_1$, i.e., $J((Z)_0, (Z)_1)$. By inductive hypothesis the predicate $Z = X_n$ is Π^0_2, and the previous lemma shows that $J(X,Y)$ is Π^0_2. Thus, the predicate $Z = X_{n+1}$ is Π^0_2. This completes the proof. $\square$

Lemma 10.10. If $Q \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_2, we can find a Π^0_1 set $P \subseteq \mathbb{N}^{\mathbb{N}}$ such that P and Q are recursively homeomorphic.

Proof. If $Q = \{g \mid \forall m\,\exists n\,R(g,m,n)\}$ where R is recursive, let

$$P = \{g \oplus h \mid \forall m\,(h(m) = \text{least } n \text{ such that } R(g,m,n))\}.$$

Clearly P is Π^0_1. Moreover, the recursive functionals $g \mapsto g \oplus h$ and $g \oplus h \mapsto g$ provide a recursive homeomorphism between P and Q. $\square$

Corollary 10.11. If f is a Π^0_2 singleton, we can find a Π^0_1 singleton g such that $f \equiv_T g$.

Proof of Theorem 10.6. We have seen that $X_n \equiv_T 0^{(n)}$ and X_n is a Π^0_2 singleton. Hence, let $g_n \equiv_T X_n$ be a Π^0_1 singleton. We then have $g_n \equiv_T X_n \equiv_T 0^{(n)}$, Q.E.D. $\square$

Exercises 10.12.

1. Show that if X is a Π^0_1 singleton and $X \in 2^{\mathbb{N}}$, then X is recursive.

2. Show that g is a Σ^0_3 singleton $\Leftrightarrow$ g is a Π^0_2 singleton.

3. Show that if g is a Π^0_2 singleton and $g \equiv_T h$ then h is a Π^0_2 singleton. Hence, $0^{(n)}$ is a Π^0_2 singleton.

4. Define

$$\begin{aligned} 0^{(\omega)} &= \text{(the characteristic function of)}\{3^n5^i \mid i \in 0^{(n)}\} \\ &= \textstyle\bigoplus_{n=0}^{\infty} 0^{(n)} \end{aligned}$$

   Show that $0^{(\omega)}$ is a Π^0_2 singleton.

5. Show that $\text{TrueSnt}_{\mathbb{N}}$ is a Π^0_2 singleton.

6. Show that $0^{(\omega)} \equiv_T \mathrm{TrueSnt}_{\mathbb{N}}$.

7. Show that if f is a Π^0_2 singleton and $g \leq_T f'$ then $f \oplus g$ is a Π^0_2 singleton.

8. Deduce that if $0^{(n)} \leq_T g \leq_T 0^{(n+1)}$ for some n, then g is a Π^0_2 singleton.

9. Show that if g is generic then g is not a Π^0_2 singleton. In fact, g does not belong to any countable Π^0_n set, for any n.

11 The Low Basis Theorem

In order to prove the Low Basis Theorem, we first develop some standard machinery.

Definition 11.1. We define a standard method of indexing Π^0_1 sets in $2^{\mathbb{N}}$, namely

$$P_e = \{X \in 2^{\mathbb{N}} \mid \varphi^{(1),X}_e(0) \uparrow\}.$$

Note that P_e is a Π^0_1 subset of $2^{\mathbb{N}}$. We say that e is an *index* of P_e.

Lemma 11.2.

1. P_e for $e = 0, 1, 2, \ldots$ is an enumeration of all Π^0_1 subsets of $2^{\mathbb{N}}$.

2. Given a Π^0_1 predicate $Q \subseteq 2^{\mathbb{N}} \times \mathbb{N}^k$, we can find a primitive recursive function $\alpha : \mathbb{N}^k \to \mathbb{N}$ such that $X \in P_{\alpha(m_1,\ldots,m_k)} \equiv Q(X, m_1, \ldots, m_k)$ for all $X \in 2^{\mathbb{N}}$ and all $m_1, \ldots, m_k \in \mathbb{N}$.

3. Given $x \in 2^{\mathbb{N}}$, let $H^X =$ the Halting Problem relative to X=$\{e \mid \varphi^{(1),X}_e(0) \downarrow\} = \{e \mid X \notin P_e\}$. Then $H^X \equiv_T X'$, the Turing jump of X.

Proof. By the Enumeration Theorem and the Parametrization Theorem, let $\widehat{\alpha} : \mathbb{N} \to \mathbb{N}$ be primitive recursive such that

$$\varphi^{(1),X}_{\widehat{\alpha}(n)}(0) \simeq \varphi^{(k),X}_{(n)_0}((n)_1, \ldots, (n)_k)$$

for all $X \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$. Here we are using the notation $p_i =$ the ith prime number, and $(n)_i =$ the exponent of p_i in n. Thus $n = \prod_{i=0}^{\infty} p_i^{(n)_i}$ for all $n \geq 1$.

Since Q is Π^0_1, let e be such that $Q(X, m_1, \ldots, m_k) \equiv \varphi^{(k),X}_e(m_1, \ldots, m_k) \uparrow$. Let $\alpha(m_1, \ldots, m_k) = \widehat{\alpha}(p_0^e p_1^{m_1} \cdots p_k^{m_k})$. Then $X \in P_{\alpha(m_1,\ldots,m_k)}$ if and only if $\varphi^{(1),X}_{\alpha(m_1,\ldots,m_k)}(0) \uparrow$, i.e., $\varphi^{(1),X}_{\widehat{\alpha}(p_0^e p_1^{m_1} \cdots p_k^{m_k})}(0) \uparrow$, i.e., $\varphi^{(1),X}_e(m_1, \ldots, m_k) \uparrow$, i.e., $Q(X, m_1, \ldots, m_k)$. This proves part 2. Part 1 is the special case where $k = 0$.

For part 3, note first that $e \in H^X \equiv X \notin P_e \equiv \varphi^{(1),X}_e(0) \downarrow \equiv 3^e 5^0 \in X'$, so $H^X \leq_T X'$. Conversely, $3^e 5^m \in X' \equiv \varphi^{(1),X}_e(m) \downarrow \equiv \varphi^{(1),X}_{\widehat{\alpha}(2^e 3^m)}(0) \downarrow \equiv \widehat{\alpha}(2^e 3^m) \in H^X$, so $X' \leq_T H^X$. ∎

Remark 11.3. Part 2 of the lemma implies that various set-theoretical operations on Π^0_1 sets are "primitive recursive in terms of the indices." This important principle is illustrated in the following examples.

Example 11.4. We can find primitive recursive functions $u, v, w, \ldots$ such that $P_{u(i,j)} = P_i \cup P_j$, $P_{v(i,j)} = P_i \cap P_j$, $P_{w(i,j)} = P_i \times P_j = \{X \oplus Y \mid X \in P_i, Y \in P_j\}$. For example, by part 2 of Lemma 11.2, let v be such that

$$X \in P_{v(i,j)} \equiv X \in P_i \wedge X \in P_j$$

for all $X \in 2^{\mathbb{N}}$ and $i, j \in \mathbb{N}$.

Lemma 11.5. Let E be the set of indices of the empty set, i.e., $E = \{e \mid P_e = \emptyset\}$. Then $E \equiv_T 0'$.

Proof. We have $e \in E \equiv P_e = \emptyset \equiv \neg (\exists X \in 2^{\mathbb{N}}) (X \in P_e)$, which is Σ^0_1. Hence, $E \leq_T 0'$. Conversely, $3^e 5^m \in 0' \equiv \varphi^{(1)}_e(m) \downarrow \equiv \varphi^{(1),X}_e(m) \downarrow \equiv \varphi^{(1),X}_{\widehat{\alpha}(2^e 3^m)}(0) \downarrow$ for all X. i.e., $\widehat{\alpha}(2^e 3^m) \in E$. So $0' \leq_T E$. □

Theorem 11.6 (Low Basis Theorem)**.** Let $Q \subseteq 2^{\mathbb{N}}$ be Π^0_1 and $\neq \emptyset$. Then, we can find $Y \in Q$ such that $Y' \leq_T 0'$, i.e., Y is *low*.

Example 11.7. Let $Q = \mathrm{DNR}_2 = \{Y \in 2^{\mathbb{N}} \mid \forall n\, Y(n) \not\simeq \varphi^{(1)}_n(n)\}$. Clearly Q is Π^0_1 and $\neq \emptyset$ and has no recursive members. This shows that the Kleene Basis Theorem cannot be improved to say "Y is recursive."

Proof of Theorem 11.6. Construct a sequence of nonempty Π^0_1 sets. $Q = Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n \supseteq \cdots$ by letting $Q_0 = Q$, $Q_{n+1} = Q_n \cap P_n$ if $Q_n \cap P_n \neq \emptyset$, and $Q_{n+1} = Q_n$ otherwise. By induction on n, Q_n is $\neq \emptyset$ and Π^0_1. By compactness, $\bigcap_{n=0}^{\infty} Q_n \neq \emptyset$, so let $Y \in \bigcap_{n=0}^{\infty} Q_n$.

We claim that Y is low. To see this, note that the decision about whether $n \in H^Y$ was made at stage $n+1$. (See part 3 of Lemma 11.2.) We shall show that the entire construction is computable using $0'$ as an oracle. We shall find a function $g(n)$, $g \leq_T 0'$ such that $Q_n = P_{g(n)}$ for all n. We start with $g(0) =$ some e such that $Q = P_e$. We define $g(n)$ inductively by letting $g(n+1) = v(g(n), n)$ if $v(g(n), n) \notin E$, and $g(n+1) = g(n)$ if $v(g(n), n) \in E$. Here v is as in Example 11.4. Clearly g is recursive in E, i.e., $g \leq_T E \equiv_T 0'$. Moreover, for all n, $n \in H^Y \Leftrightarrow v(g(n), n) \notin E$. Thus, $Y' \equiv_T H^Y \leq_T g \leq_T 0'$, i.e., Y is low. □

12 The Hyperimmune-Free Basis Theorem

We shall now prove a variant of the Low Basis Theorem known as the Hyperimmune-Free Basis Theorem.

Definition 12.1. For $f, g \in \mathbb{N}^{\mathbb{N}}$, let us say that f *is dominated by* g if $f(n) < g(n)$ for all sufficiently large n.

Definition 12.2. The *domination ordering* is defined as follows. $X \leq_{\text{dom}} Y$ if $(\forall f \leq_T X)\,(\exists g \leq_T Y)\,(f$ is dominated by $g)$. This depends only on the Turing degrees of X and Y.

Definition 12.3. X is *hyperimmune-free* if $X \leq_{\text{dom}} 0$, i.e., every function which is computable from X is dominated by some computable function.

Theorem 12.4 (Hyperimmune-free Basis Theorem)**.** Given $Q \subseteq 2^{\mathbb{N}}$ which is Π^0_1 and $\neq \emptyset$, we can find $Y \in Q$ which is hyperimmune-free.

Before proving the theorem, we first prove a standard lemma.

Lemma 12.5 (Σ^0_1 Selection Lemma)**.** Let $S(-, m, n)$ be a mixed predicate, i.e., $S \subseteq (\mathbb{N}^{\mathbb{N}})^k \times (2^{\mathbb{N}})^l \times \mathbb{N}^{i+2}$. If $S(-, m.n)$ is Σ^0_1, we can find a partial recursive functional (or function) $\Psi(-, m)$ such that

1. $\Psi(-, m)\downarrow \Leftrightarrow \exists n\, S(-, m, n)$,
2. $\Psi(-, m)\downarrow \Rightarrow S(-, m, \Psi(-, m))$

for all $-, m$. The functional Ψ is called a *selector* for S.

Proof. Intuitively, $\Psi(-, m)$ is found by dove-tailing computations. Formally, let $S(-, m, n) \equiv \exists j\, R(-, m, n, j)$ where R is a recursive predicate, and define $\Psi(-, m) =$ (the least j such that $R(-, m, (j)_1, (j)_2))_1$. Clearly this works. □

Proof of Theorem 12.4. As in the proof of the Low Basis Theorem, we construct a descending sequence $Q = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_e \supseteq Q_{e+1} \supseteq \cdots$ where each Q_e is Π^0_1 and $\neq \emptyset$. We then let $Y \in \bigcap_{e=0}^{\infty} Q_e$ and argue that Y is hyperimmune-free.

Stage 0. Let $Q_0 = Q$.

Stage $e + 1$. Given Q_e which is Π^0_1 and $\neq \emptyset$, we have two cases.

Case 1: $\exists m\, (\exists X \in Q_e)\, (\varphi_e^{(1),X}(m)\uparrow)$.

In this case, choose such an m and let $Q_{e+1} = \{X \in Q_e \mid \varphi_e^{(1),X}(m)\uparrow\}$.

Case 2: $\forall m\, (\forall X \in Q_e)\, (\varphi_e^{(1),X}(m)\downarrow)$.

In this case we have

$$\forall m\, \forall X\, \exists i\, (X \notin Q_e \vee \varphi_e^{(1),X}(m) \simeq i).$$

It follows by the Σ^0_1 Bounding Lemma that

$$\forall m\, \exists n\, \forall X\, (\exists i < n)\, (X \notin Q_e \vee \varphi_e^{(1),X}(m) \simeq i).$$

Hence, by the Σ^0_1 Selection Lemma 12.5, we can find a recursive function g such that

$$\forall m\, \forall X\, (\exists i < g(m))\, (X \notin Q_e \vee \varphi_e^{(1),X}(m) \simeq i).$$

In other words,

$$(\forall X \in Q_e)\, \forall m\, (\varphi_e^{(1),X}(m)\downarrow < g(m))$$

so in this case we let $Q_{e+1} = Q_e$.

To summarize, in case 1 $\varphi_e^{(1),X}(m)$ is not total, and in case 2 $\varphi_e^{(1),X}(m)$ is total and dominated by a recursive function $g(m)$. Thus Y is hyperimmune-free. $\square$

Remark 12.6. We cannot combine the Kleene (or Low) Basis Theorem and the Hyperimmune-Free Basis Theorem into one theorem. In fact, the following exercise implies that if $0 <_T X \leq_T 0'$ then X is not hyperimmune-free.

Exercise 12.7. Show that if X is hyperimmune-free and not recursive, then X is not a Π^0_2 singleton. Hence $X \not\leq_T 0'$.

Here is a corollary of the Low Basis Theorem.

Corollary 12.8. If $Q \subseteq 2^{\mathbb{N}}$ is Π^0_1 and $\neq \emptyset$, then $\deg_w(Q) < \mathbf{0}'$.

Proof. By the Low Basis Theorem, there is a $Y \in Q$ such that $Y <_T 0'$ and the corollary follows. $\square$

13 The lattice $\mathcal{E}_w$

From now on we are going to focus on the lattice of weak degrees of nonempty Π^0_1 sets in Cantor space:

$$\mathcal{E}_w = \{\deg_w(P) \mid P \subseteq 2^{\mathbb{N}},\, \Pi^0_1,\, \neq \emptyset\}.$$

Example 13.1. $\mathrm{DNR}_2 = \{X \in 2^{\mathbb{N}} \mid X \text{ is DNR}\}$ is Π^0_1, $\neq \emptyset$ and $\deg_w(\mathrm{DNR}_2) > \mathbf{0}$.

This example shows that $\mathcal{E}_w$ is nontrivial. Note also that $\mathcal{E}_w$ is the simplest nontrivial class of weak degrees with respect to the arithmetical hierarchy.

Remark 13.2. Some obvious facts about $\mathcal{E}_w$ are:

1. $\mathbf{0} = \deg(2^{\mathbb{N}}) \in \mathcal{E}_w$.
2. $\mathcal{E}_w$ contains degrees greater than $\mathbf{0}$.
3. $\mathcal{E}_w$ is a countable lattice.

We shall now show that $\mathcal{E}_w$ has a top degree.

Theorem 13.3. $\mathcal{E}_w$ has a top degree, denoted $\mathbf{1}$.

Proof. We use our standard enumeration of all Π^0_1 subsets of $2^{\mathbb{N}}$:

$$P_e = \{X \in 2^{\mathbb{N}} \mid \varphi_e^{(1),X}(0)\uparrow\}.$$

For $Y \in 2^{\mathbb{N}}$ let $(Y)_e \in 2^{\mathbb{N}}$ be defined by $(Y)_e(m) = Y(3^e 5^m)$. The naive idea of the proof will be to define

$$P = \prod_{e=0}^{\infty} P_e = \{Y \mid \forall e\, (Y)_e \in P_e\}.$$

and note that $P_e \leq_s P$ via $Y \mapsto (Y)_e$. This naive idea doesn't work, because $P_e = \emptyset$ for some e, hence $P = \emptyset$, and hence $\deg_w(P) \notin \mathcal{E}_w$. We shall modify this construction to use only nonempty Π^0_1 sets.

Use the tree representation of Π^0_1 sets: let $T_e = \{\tau \in 2^{<\mathbb{N}} \mid \varphi^{(1),\tau}_{e,|\tau|}(0) \uparrow\}$. Note:

1. $P_e = \{\text{paths through } T_e\}$.
2. $P_e = \emptyset \Leftrightarrow T_e$ is finite (by compactness).

Lemma 13.4 (Weak Konig's Lemma)**.** If T is a subtree of $2^{<\mathbb{N}}$, T is infinite $\Leftrightarrow$ T has an infinite path.

Proof. $\Leftarrow$: Trivial

$\Rightarrow$: Suppose T has no infinite path. $2^{\mathbb{N}} = \bigcup_{\tau \notin T} N_\tau$. By compactness, let $\tau_1, \ldots, \tau_n$ be a finite set of bitstrings such that $2^{\mathbb{N}} = \bigcup_{i=1}^n N_{\tau_i}$. $T \subseteq \{\text{initial segments of } \tau_1, \ldots, \tau_n\}$. Hence T is finite. □

For all e let

$$S_e = \{\tau \in T_e \text{ of maximal length}\}.$$

In other words, $S_e = \{\tau \in T_e \mid \neg\,(\exists \sigma \in T_e)\,(|\sigma| = |\tau| + 1)\}$. Let

$$T_e^+ \;=\; T_e \cup \{\tau^\frown \underbrace{\langle 0, \ldots, 0 \rangle}_{n} \mid \tau \in S_e, n \in \mathbb{N}\}.$$

Then T_e^+ is an infinite subtree of $2^{<\mathbb{N}}$. Let

$$P_e^+ = \{\text{paths through } T_e^+\}.$$

Note that $P_e^+ \neq \emptyset$ for all e, and $P_e^+ = P_e$ if $P_e \neq \emptyset$. Moreover, the P_e^+ for $e = 0, 1, 2, \ldots$ are uniformly Π^0_1 because T_e^+ for $e = 0, 1, 2, \ldots$ are uniformly recursive. Let

$$P^+ = \textstyle\prod_{e=0}^\infty P_e^+ = \{Y \in 2^{\mathbb{N}} \mid \forall e\, (Y)_e \in P_e^+\}.$$

Clearly P^+ is Π^0_1, and $P_e^+ \leq_s P^+$ via $Y \mapsto (Y)_e$. Thus $\deg_w(P^+)$ is the top degree in $\mathcal{E}_w$. □

Remark 13.5. Let $\mathcal{E}_s = \{\text{strong degrees of nonempty } \Pi^0_1 \text{ subsets of } 2^{\mathbb{N}}\}$. Then $\deg_s(P^+) =$ the top degree in $\mathcal{E}_s$, by the same proof.

In the next section we shall obtain two interesting characterizations of the top degree $\mathbf{1}$ in $\mathcal{E}_w$ and $\mathcal{E}_s$. Namely:

Theorem 13.6. $\deg_w(\mathrm{DNR}_2)$ (respectively $\deg_s(\mathrm{DNR}_2)$) is the top degree in $\mathcal{E}_w$ (respectively $\mathcal{E}_s$).

Definition 13.7. Let $\mathrm{CPA} = \{X \in 2^{\mathbb{N}} \mid X$ encodes a complete consistent extension of Peano Arithmetic$\}$.

Theorem 13.8. $\deg_w(\mathrm{CPA})$ (respectively $\deg_s(\mathrm{CPA})$) is the top degree in $\mathcal{E}_w$ (respectively $\mathcal{E}_s$).

14 Some mass problems of degree 1

In this section we present two interesting mass problems of degree **1**.

Definition 14.1. Let $A, B \subseteq \mathbb{N}$ be recursively enumerable. We say that $X \in 2^{\mathbb{N}}$ *separates* A and B if $X(n) = 1$ for all $n \in A$, and $X(n) = 0$ for all $n \in B$. We define

$$\mathrm{Sep}(A, B) = \{X \in 2^{\mathbb{N}} \mid X \text{ separates } A, B\}.$$

Note that $\mathrm{Sep}(A, B)$ is a Π^0_1 subset of $2^{\mathbb{N}}$: $X \in \mathrm{Sep}(A, B) \equiv \forall n\,((n \in A \Rightarrow X(n) = 1) \wedge (n \in B \Rightarrow X(n) = 0))$. Viewed as mass problems, Π^0_1 subsets of $2^{\mathbb{N}}$ of this form are called *separation problems*.

Lemma 14.2. Let $Q \subseteq 2^{\mathbb{N}}$ be nonempty Π^0_1. Then, we can find disjoint recursively enumerable sets $A, B \subseteq \mathbb{N}$ such that $Q \leq_s \mathrm{Sep}(A, B)$.

In other words, every Π^0_1 mass problem in the Cantor space is strongly reducible to a separation problem.

Proof. Consider the predicate

$$S(\sigma, i) \;\;\equiv\;\; \sigma \in 2^{<\mathbb{N}} \wedge i < 2 \wedge N_{\sigma^\frown\langle i\rangle} \cap Q = \emptyset.$$

By Lemma 9.9 this predicate is Σ^0_1. By the Σ^0_1 Selection Lemma 12.5 let $\psi(\sigma)$ be a partial recursive function which is a selector for this predicate. For $i = 0, 1$ let $A_i = \{\sigma \in 2^{<\mathbb{N}} \mid \psi(\sigma) \simeq i\}$. Clearly A_0 and A_1 are disjoint and Σ^0_1, i.e., recursively enumerable. We shall prove that $Q \leq_s \mathrm{Sep}(A_0, A_1)$.

First, suppose $\sigma \in 2^{<\mathbb{N}}$ is such that $N_\sigma \cap Q \neq \emptyset$. We claim that $N_{\sigma^\frown\langle X(\sigma)\rangle} \cap Q \neq \emptyset$ for all $X \in \mathrm{Sep}(A_0, A_1)$. To see this, note first that $N_{\sigma^\frown\langle i\rangle} \cap Q \neq \emptyset$ for at least one of $i = 0, 1$. If $N_{\sigma^\frown\langle 0\rangle} \cap Q = \emptyset$, it follows that $N_{\sigma^\frown\langle 1\rangle} \cap Q \neq \emptyset$, hence $S(\sigma, 0) \wedge \neg S(\sigma, 1)$, hence $\psi(\sigma) = 0$, hence $\sigma \in A_0$, hence $X(\sigma) = 1$, hence $N_{\sigma^\frown\langle X(\sigma)\rangle} \cap Q \neq \emptyset$. Similarly, if $N_{\sigma^\frown\langle 1\rangle} \cap Q = \emptyset$, it follows that $N_{\sigma^\frown\langle 0\rangle} \cap Q \neq \emptyset$, hence $S(\sigma, 1) \wedge \neg S(\sigma, 0)$, hence $\psi(\sigma) = 1$, hence $\sigma \in A_1$, hence $X(\sigma) = 0$, hence $N_{\sigma^\frown\langle X(\sigma)\rangle} \cap Q \neq \emptyset$. This proves our claim.

Consider the recursive functional $\Psi : \mathrm{Sep}(A_0, A_1) \to 2^{\mathbb{N}}$ defined by $\Psi(X) = Y$ where $Y(n) = X(Y \upharpoonright n)$ for all n. By the previous claim plus induction on n starting with $N_{\langle\rangle} \cap Q \neq \emptyset$, we have $N_{Y\upharpoonright n} \cap Q \neq \emptyset$ for all n and all $X \in \mathrm{Sep}(A_0, A_1)$. Since Q is closed, it follows that $Y \in Q$. Thus $\Psi : \mathrm{Sep}(A_0, A_1) \to Q$ and $Q \leq_s \mathrm{Sep}(A_0, A_1)$, Q.E.D. $\square$

Remark 14.3. Recall that DNR_2 is the set of $X \in 2^{\mathbb{N}}$ such that X is diagonally nonrecursive. Note that DNR_2 may be viewed as a separation problem, namely $\mathrm{DNR}_2 = \mathrm{Sep}(K_0, K_1)$ where $K_i = \{m \mid \varphi^{(1)}_m(m) \simeq i\}$. The following lemma says that $\mathrm{Sep}(K_0, K_1)$ is a "universal separation problem."

Lemma 14.4. If A, B are recursively enumerable and $A \cap B = \emptyset$, then $\mathrm{Sep}(A, B) \leq_s \mathrm{Sep}(K_0, K_1)$.

Proof. Consider the partial recursive function

$$\psi(m) \simeq \begin{cases} 0 & \text{if } m \in A, \\ 1 & \text{if } m \in B. \end{cases}$$

By the Parametrization Theorem, let α be a primitive recursive function such that $\varphi^{(1)}_{\alpha(m)}(n) \simeq \psi(m)$ for all m, n. Then

$$m \in A \Leftrightarrow \psi(m) = 0 \Leftrightarrow \varphi^{(1)}_{\alpha(m)}(\alpha(m)) = 0 \Leftrightarrow \alpha(m) \in K_0$$

and

$$m \in B \Leftrightarrow \psi(m) = 1 \Leftrightarrow \varphi^{(1)}_{\alpha(m)}(\alpha(m)) = 1 \Leftrightarrow \alpha(m) \in K_1$$

so $X \in \mathrm{Sep}(K_0, K_1)$ implies $X \circ \alpha \in \mathrm{Sep}(A, B)$. Thus $\mathrm{Sep}(A, B) \leq_s \mathrm{Sep}(K_0, K_1)$ via the recursive functional $X \mapsto X \circ \alpha$. □

Lemma 14.5. For any nonempty Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ we have $Q \leq_s \mathrm{DNR}_2$.

Proof. By Lemma 14.2 we have $Q \leq_s \mathrm{Sep}(A, B)$ where A, B are disjoint and recursively enumerable. By Remark 14.3 and Lemma 14.4 we have $\mathrm{Sep}(A, B) \leq_s \mathrm{DNR}_2$. It now follows that $Q \leq_s \mathrm{DNR}_2$, Q.E.D. □

Theorem 14.6. $\deg_w(\mathrm{DNR}_2) = \mathbf{1}$ in $\mathcal{E}_w$. In fact, $\deg_s(\mathrm{DNR}_2) = \mathbf{1}$ in $\mathcal{E}_s$.

Proof. Clearly DNR_2 is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$. Then, Lemma 14.5 implies that $\deg_w(\mathrm{DNR}_2) = \mathbf{1}$ in $\mathcal{E}_w$ and $\deg_s(\mathrm{DNR}_2) = \mathbf{1}$ in $\mathcal{E}_s$. □

Remark 14.7. Let $Q \subseteq 2^{\mathbb{N}}$ be nonempty Π^0_1. We have the following additional results, which will not be proved here.

1. We can find a recursive functional $\Psi : \mathrm{DNR}_2 \to Q$ which is onto Q.

2. If $\deg_s(Q) = \mathbf{1}$ in $\mathcal{E}_s$ then Q is recursively homeomorphic to DNR_2.

3. If $\deg_w(Q) = \mathbf{1}$ in $\mathcal{E}_w$ then Q is Turing degree isomorphic to DNR_2.

Results 1 and 2 are due to Simpson 2000, while result 3 is due to Simpson 2004.

We now turn to the study of mass problems associated with theories in the predicate calculus.

Definition 14.8. Let L be a recursive language, and assume a standard Gödel numbering $\# : \{L\text{-sentences}\} \to \mathbb{N}$. If T is an L-theory, let Thm_T be the set of Gödel numbers of theorems of T. In other words,

$$\mathrm{Thm}_T = \{\#(B) \mid B \text{ is an } L\text{-sentence and } T \vdash B\}.$$

We say that T is *decidable* if Thm_T is recursive.

Let S be a consistent L-theory. A *completion of* S is a complete, consistent L-theory T such that $T \supseteq S$. Define

$$\mathrm{Cmp}_S = \{\mathrm{Thm}_T \mid T \text{ is a completion of } S\}.$$

By Lindenbaum's Lemma, we have $\mathrm{Cmp}_S \neq \emptyset$. We sometimes identify Thm_T with its characteristic function. With this identification, we have $\mathrm{Cmp}_S \subseteq 2^{\mathbb{N}}$. It can be shown that Cmp_S is a closed subset of $2^{\mathbb{N}}$.

Lemma 14.9. Let L and S be as in Definition 14.8. If S is consistent and recursively axiomatizable, Cmp_S is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$.

Proof. For $X \in 2^{\mathbb{N}}$ we have $X \in \mathrm{Cmp}_S$ if and only if

$$(\forall L\text{-sentences } B)\,(X(\#(B)) = 1 \Leftrightarrow X(\#(\neg B) = 0)$$

and

$$(\forall L\text{-sentences } B)\,(\text{if } B \text{ is an axiom of } S \text{ then } X(\#(B)) = 1)$$

and .... The details are left to the reader. □

Exercise 14.10. Given a nonempty Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$, construct a consistent, recursively axiomatizable theory S such that Cmp_S is recursively homeomorphic to Q.

Remark 14.11. Exercise 14.10 is rather easy. A more difficult result due to Hanf and Peretyatkin says that, in Exercise 14.10, S can be chosen to be finitely axiomatizable.

Definition 14.12. Recall that PA is *Peano Arithmetic*, a.k.a., Z_1, a.k.a., *first-order arithmetic*. The language of PA is $L_{\mathsf{Q}} = \{+, \cdot, 0, S, =\}$ where Q is Robinson's weak theory of arithmetic. We have $\mathsf{PA} = \mathsf{Q}+$ the induction scheme consisting of the universal closure of

$$(A[x/0] \wedge \forall x\,(A \Rightarrow A[x/Sx])) \Rightarrow \forall x\, A$$

where A ranges over L_{Q}-formulas. Let $\mathrm{CPA} = \mathrm{Cmp}_{\mathsf{PA}}$.

Remark 14.13. Viewed as a mass problem, CPA is the problem of finding a completion of PA. A motivation for considering this problem is the Gödel Incompleteness Theorem, which says that PA itself is incomplete. Recall also the theorems of Gödel, Tarski, and Rosser which say that PA has no decidable completion. In other words, viewed as a mass problem, CPA is *unsolvable*, i.e., $\deg_w(\mathrm{CPA}) > \mathbf{0}$. We shall now improve this result by showing that $\deg_w(\mathrm{CPA}) = \mathbf{1}$.

Lemma 14.14. CPA is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$.

Proof. This is a special case of Lemma 14.9 since PA is consistent and recursively axiomatizable. □

Lemma 14.15. $\mathrm{DNR}_2 \leq_s \mathrm{CPA}$.

Proof. Recall that $\mathrm{DNR}_2 = \mathrm{Sep}(K_0, K_1)$. For $i = 0, 1$ let A_i be a Σ_1 formula which defines K_i over the standard model $(\mathbb{N}, +, \cdot, 0, S, =)$. It is well known that every true Σ_1 sentence is provable in PA. Thus $K_i = \{m \in \mathbb{N} \mid \mathsf{PA} \vdash A_i[x/\underline{m}]\}$ for $i = 0, 1$. Let us also assume that $\mathsf{PA} \vdash \neg(A_0[x/\underline{m}] \wedge A_1[x/\underline{m}])$ for each $m \in \mathbb{N}$. Since $K_0 \cap K_1 = \emptyset$, it is straightforward to choose the formulas A_0 and A_1 to have this additional property.

Given $X \in \mathrm{CPA}$, let $X = \mathrm{Thm}_T$ where T is a completion of PA. Define $\Psi(X) = Y$ where Y is the characteristic function of $\{m \in \mathbb{N} \mid \mathsf{PA} \vdash A_0[x/\underline{m}]\}$. Clearly $Y \in \mathrm{Sep}(K_0, K_1)$. Moreover $\Psi : \mathrm{CPA} \to \mathrm{Sep}(K_0, K_1)$ is a recursive functional, so $\mathrm{Sep}(K_0, K_1) \leq_s \mathrm{CPA}$. This completes the proof. □

Lemma 14.16. For any nonempty Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ we have $Q \leq_s \mathrm{CPA}$.

Proof. By Lemmas 14.5 and 14.15 we have $Q \leq_s \mathrm{DNR}_2$ and $\mathrm{DNR}_2 \leq_s \mathrm{CPA}$. It follows that $Q \leq_s \mathrm{CPA}$, Q.E.D. □

Theorem 14.17. $\deg_w(\mathrm{CPA}) = \mathbf{1} \in \mathcal{E}_w$. In fact, $\deg_s(\mathrm{CPA}) = \mathbf{1} \in \mathcal{E}_s$.

Proof. By Lemma 14.9 CPA is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$. Hence $\deg_w(\mathrm{CPA}) \in \mathcal{E}_w$ and $\deg_s(\mathrm{CPA}) \in \mathcal{E}_s$. It now follows by Lemma 14.16 that $\deg_w(\mathrm{CPA}) = \mathbf{1} \in \mathcal{E}_w$ and $\deg_s(\mathrm{CPA}) = \mathbf{1} \in \mathcal{E}_s$. □

Remark 14.18. Using the Rosser trick, one can modify the proof and replace PA by any consistent, recursively axiomatizable theory $S \supseteq \mathsf{Q}$. Thus $\deg_w(\mathrm{Cmp}_S) = \mathbf{1} \in \mathcal{E}_w$ and $\deg_s(\mathrm{Cmp}_S) = \mathbf{1} \in \mathcal{E}_s$ for all such S. In particular, the problem of finding a completion of S is of the same degree of unsolvability no matter whether S is Q, Z_1, Z_2, ZFC, etc.

Remark 14.19. In the literature, Turing degrees of the form $\deg_T(X)$ where $X \in \mathrm{CPA}$ are sometimes called PA-*degrees*. However, by Remark 14.18 they could equally well be called Q-*degrees*, or Z_2-*degrees*, or ZFC-*degrees*, etc.

15 Martin-Löf randomness

We now present Martin-Löf's rigorous definition of what it means for a point $X \in 2^{\mathbb{N}}$ to be *random*. Compare Turing's rigorous definition of what it means for a function to be *computable*. Although Martin-Löf's concept of randomness is not as compelling as Turing's concept of computability, it is a good attempt. The Martin-Löf concept has many nice properties which make it reasonably convincing as a rigorous explanation of randomness.

Notation 15.1. For $\sigma \in 2^{<\mathbb{N}} = \{\text{bitstrings}\}$ we write

$$N_\sigma = \{X \in 2^{\mathbb{N}} \mid \sigma \subset X\}.$$

The "fair coin" probability measure on $2^{\mathbb{N}}$ is defined by

$$\mu(N_\sigma) \ = \ \frac{1}{2^{|\sigma|}}.$$

Definition 15.2. A set $S \subseteq 2^{<\mathbb{N}}$ is said to be *prefix-free* if there are no $\sigma, \tau \in S$ such that $\sigma \subset \tau$. If S is prefix-free, we clearly have

$$\mu\left(\bigcup_{\sigma \in S} N_\sigma\right) = \sum_{\sigma \in S} \frac{1}{2^{|\sigma|}}.$$

Given a Σ^0_1 set $U \subseteq 2^{\mathbb{N}}$, let e be an *index* for U, i.e.,

$$U = U_e = \{X \in 2^{\mathbb{N}} \mid \varphi_e^{(1),X}(0)\downarrow\}.$$

Let $S = S_e = \{\sigma \in 2^{<\mathbb{N}} \mid \varphi_{e,|\sigma|}^{(1),\sigma}(0)\downarrow$ and there is no $\rho \subset \sigma$ such that $\varphi_{e,|\rho|}^{(1),\rho}(0)\downarrow\}$. Then $U = \bigcup_{\sigma \in S} N_\sigma$ and S is prefix-free and primitive recursive (uniformly in e), and

$$\mu(U) = \sum_{\sigma \in S} \frac{1}{2^{|\sigma|}}.$$

Note that the real number $\mu(U)$ is *left recursively enumerable*, i.e., it is the limit of a non-decreasing recursive sequence of rational numbers. Let

$$U_s = U_{e,s} = \{X \in 2^{\mathbb{N}} \mid \varphi_{e,s}^{(1),X\restriction s}(0)\downarrow\}.$$

Then $U_e = \bigcup_{s=0}^{\infty} U_{e,s}$ and $\mu(U_{e,s})$ is a rational number and the predicates $\mu(U_{e,s}) = r$, $\mu(U_{e,s}) < r$ are primitive recursive, and

$$U_{e,0} \subseteq U_{e,1} \subseteq \cdots \subseteq U_{e,s} \subseteq U_{e,s+1} \subseteq \cdots$$

and $\mu(U_e) = \lim_s \mu(U_{e,s})$.

Recall some measure-theoretic facts:

- μ is countably additive, etc.
- μ is *regular*. This means: if $S \subseteq 2^{\mathbb{N}}$ is measurable, then

  $$\mu(S) = \inf\{\mu(U) \mid U \text{ open}, U \supseteq S\}.$$

  In particular, if $\mu(S) = 0$ then there is a sequence of open sets $V_n \subseteq 2^{\mathbb{N}}$ such that $S \subseteq V_n$ and $\mu(V_n) \leq 1/2^n$ for all n.

Recall also: A set $U \subseteq 2^{\mathbb{N}}$ is open $\Leftrightarrow$ U is $\Sigma^{0,g}_1$ for some oracle g. Thus we think of Σ^0_1 sets $U \subseteq 2^{\mathbb{N}}$ as "effectively open" sets.

Definition 15.3. We say that $S \subseteq 2^{\mathbb{N}}$ is *effectively null* (i.e., effectively of measure 0) if we can find a uniformly Σ^0_1 sequence of sets $V_n \subseteq 2^{\mathbb{N}}$, $n \in \mathbb{N}$ such that $S \subseteq \bigcap_{n=0}^{\infty} V_n$ and $\mu(V_n) \leq 1/2^n$ for all n.

Definition 15.4 (Martin-Löf, 1966)**.** $X \in 2^{\mathbb{N}}$ is *random* if X does not belong to any effectively null set.

A *test for randomness* is a uniformly Σ^0_1 sequence of sets $V_n \subseteq 2^{\mathbb{N}}$, $n = 0, 1, 2, \ldots$, such that $\mu(V_n) = 1/2^n$ for all n. We say that X *passes the test* if $X \notin \bigcap_{n=0}^{\infty} V_n$. Thus, X is *random* $\Leftrightarrow$ X passes all tests for randomness. Let

$$R_1 = \{X \in 2^{\mathbb{N}} \mid X \text{ is random}\}.$$

Lemma 15.5. $\mu(R_1) = 1$. Hence, random points exist.

Proof. There are only countably many tests for randomness. For each such test, we have $\mu(\bigcap_{n=0}^{\infty} V_n) = 0$ because $\mu(V_n) \leq 1/2^n$. Hence, by countable additivity, $\mu(R_1) = 1$. □

Lemma 15.6. If X is random, X is not recursive.

Proof. Let $A \in 2^{\mathbb{N}}$ be recursive. Let $V_n = N_{A \restriction n}$. Clearly V_n is uniformly Σ^0_1 and $\mu(V_n) = 1/2^n$, hence the V_n's form a Martin-Löf test, hence $X \notin \bigcap_n V_n$, i.e., X passes the test. Hence $\exists n\,(X \restriction n \neq A \restriction n)$, i.e., $X \neq A$. □

Exercise 15.7. Let $X \in 2^{\mathbb{N}}$ be random. Prove that if $P \subseteq 2^{\mathbb{N}}$ is Π^0_1 of measure 0, then $X \notin P$. In other words, X does not belong to any Π^0_1 set of measure 0. An X with this property is called *weakly random*.

Exercises 15.8. Let $X \in 2^{\mathbb{N}}$ be random. Then:

1. X obeys the *Strong Law of Large Numbers*, i.e.,

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} X(i)}{n} = \frac{1}{2}.$$

2. Furthermore, we can obtain bounds on the rate of convergence, e.g.,

$$\left| \frac{\sum_{i=0}^{n-1} X(i)}{n} - \frac{1}{2} \right| < \sqrt{\frac{\log n}{n}}$$

for all but finitely many n.

3. X obeys the *Law of the Iterated Logarithm*:

$$\limsup_{n\to\infty} \frac{\left| \dfrac{\sum_{i=0}^{n-1} X(i)}{n} - \dfrac{1}{2} \right|}{\sqrt{\dfrac{n}{2} \log\log n}} = 1.$$

A typical consequence of this is that

$$\frac{\sum_{i=0}^{n-1} X(i)}{n}$$

is $> 1/2$ for infinitely many n and $< 1/2$ for infinitely many n.

The idea here is, any "reasonable" statistical property which holds for almost all $X \in 2^{\mathbb{N}}$ holds for all random X. This is because each "reasonable" statistical property is embodied by a Martin-Löf test.

An important theorem concerning randomness is the following.

Lemma 15.9 (Martin-Löf)**.** We can find a *universal* test for randomness.

This means: there is a uniformly Σ^0_1 sequence of sets $U_n \subseteq 2^{\mathbb{N}}$ such that $\mu(U_n) \leq 1/2^n$ for all n, and if $V_n, n \in \mathbb{N}$ is any other such sequence, then $\bigcap_{n=0}^{\infty} V_n \subseteq \bigcap_{n=0}^{\infty} U_n$.

In other words, the union of all effectively null sets is an effectively null set.

Proof. Let V_i, $i \in \mathbb{N}$ be a standard uniform enumeration of all Σ^0_1 subsets of $2^{\mathbb{N}}$:

$$V_i = \{X \in 2^{\mathbb{N}} \mid \varphi_i^{(1),X}(0)\downarrow\}$$

Note that $V_i = 2^{\mathbb{N}} \setminus P_i$ where P_i is our standard enumeration of the Π^0_1 subsets of $2^{\mathbb{N}}$. Let $V_{i,s} =$ "the part of V_i which is enumerated by stage s," i.e.,

$$V_{i,s} = \{X \in 2^{\mathbb{N}} \mid \varphi_{i,s}^{(1),X\restriction s}(0)\downarrow\}.$$

Note that $V_{i,s}$ is uniformly Σ^0_1 and

$$V_{i,0} \subseteq V_{i,1} \subseteq \cdots \subseteq V_{i,s} \subseteq \cdots$$

and $V_i = \bigcup_{s=0}^{\infty} V_{i,s}$. Moreover $\mu(V_{i,s})$ is a rational number and is primitive recursive as a function of i and s.

Given a rational number r, let

$$\begin{aligned} V_i[r] &= \text{"}V_i \text{ enumerated so long as its measure is } \leq r\text{"} \\ &= \bigcup_{\mu(V_{i,s})\leq r} V_{i,s} \\ &= \{X \in 2^{\mathbb{N}} \mid \exists s\, (\underbrace{\mu(V_{i,s}) \leq r}_{\text{recursive}} \wedge \underbrace{X \in V_{i,s}}_{\text{recursive}})\}. \end{aligned}$$

Note the following properties:

1. $V_i[r]$ is uniformly Σ^0_1.
2. $V_i[r] \subseteq V_i$.
3. $\mu(V_i[r]) \leq r$.
4. If $\mu(V_i) \leq r$ then $V_i[r] = V_i$.

Now, given $e, n \in \mathbb{N}$, let

$$\widetilde{U}_{e,n} = \begin{cases} V_i[1/2^n] & \text{if } \varphi_e^{(1)}(n) \simeq i, \\ \emptyset & \text{if } \varphi_e^{(1)}(n)\uparrow . \end{cases}$$

Clearly the sets $\widetilde{U}_{e,n}$ are uniformly Σ^0_1 and $\mu(\widetilde{U}_{e,n}) \leq 1/2^n$. Thus $\widetilde{U}_{e,n}$ for $n = 0, 1, 2, \ldots$ is a Martin-Löf test.

Furthermore, we claim that every Martin-Löf test arises in this way.

To prove this claim, let W_n, $n = 0, 1, 2, \ldots$, be an arbitrary Martin-Löf test. Thus W_n is uniformly Σ^0_1 and $\mu(W_n) \leq 1/2^n$. Since W_n is uniformly Σ^0_1, let Ψ be a partial recursive functional such that $X \in W_n \equiv \Psi(X, n) \downarrow$. By the Parametrization Theorem, let $\alpha(n)$ be a primitive recursive function such that $\varphi^{(1),X}_{\alpha(n)}(0) \simeq \Psi(X, n)$. Then $X \in V_{\alpha(n)} \equiv \varphi^{(1),X}_{\alpha(n)}(0) \downarrow \equiv \Psi(X, n) \downarrow \equiv X \in W_n$. Thus $V_{\alpha(n)} = W_n$ for all n. Now let e be an index of α, i.e., $\varphi^{(1)}_e(n) \simeq \alpha(n)$ for all n. Then for all n we have $V_{\alpha(n)} = W_n$, hence $\mu(V_{\alpha(n)}) \leq 1/2^n$, hence $\widetilde{U}_{e,n} = V_{\alpha(n)}[1/2^n] = V_{\alpha(n)} = W_n$. This proves the claim.

Finally let
$$U_n = \bigcup_e \widetilde{U}_{e,e+n+1}.$$
Note U_n is uniformly Σ^0_1 and
$$\mu(U_n) \leq \sum_{e=0}^{\infty} \mu(\widetilde{U}_{e,e+n+1}) \leq \sum_{e=0}^{\infty} \frac{1}{2^{e+n+1}} = \frac{1}{2^n}$$
so the U_n's form a Martin-Löf test. Furthermore, for an arbitrary Martin-Löf test $\widetilde{U}_{e,n}$ we have
$$\bigcap_n \widetilde{U}_{e,n} \subseteq \bigcap_n U_n$$
To see this, note that if $X \in \bigcap_n \widetilde{U}_{e,n}$ then for all n we have $X \in \widetilde{U}_{e,e+n+1}$, hence $X \in U_n$, so $X \in \bigcap_n U_n$. Thus U_n, $n = 0, 1, 2, \ldots$ is a universal Martin-Löf test, Q.E.D. □

Theorem 15.10. $R_1 = \{X \in 2^{\mathbb{N}} \mid X \text{ is random}\}$ is Σ^0_2 of measure 1.

Proof. Let U_n, $n \in \mathbb{N}$, be a universal Martin-Löf test. Then
$$R_1 = 2^{\mathbb{N}} \setminus \bigcap_{n=0}^{\infty} U_n.$$
Since U_n is uniformly Σ^0_1, R_1 is Σ^0_2. Since $\mu(U_n) \leq 1/2^n$, $\mu(R_1) = 1$. □

Theorem 15.11. We can find a Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, $P \neq \emptyset$, consisting entirely of random points in $2^{\mathbb{N}}$. In other words, $P \subseteq R_1$.

Proof. Fix n and let $P = 2^{\mathbb{N}} \setminus U_n$. Then $\mu(P) \geq 1 - 1/2^n$ and $P \cap U_n = \emptyset$, hence $P \cap \bigcap_n U_n = \emptyset$, hence $P \subseteq R_1$. □

Corollary 15.12. We can find random points in $2^{\mathbb{N}}$ which are low, and hyperimmune-free, but not both.

Proof. To find low points, apply the Low Basis Theorem. To find hyperimmune-free points, apply the Hyperimmune-Free Basis Theorem. Recall also that Low $\cap$ HIF = REC and $R_1 \cap \text{REC} = \emptyset$ so we cannot find $X \in R_1$ which is both low and hyperimmune-free. □

16 Randomness as a mass problem

Theorem 16.1. There is a Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ such that $P \equiv_w R_1$.

To prove this, we use the following lemma.

Lemma 16.2. If $S \subseteq 2^{\mathbb{N}}$ is Σ^0_2, we can find a Π^0_1 set $Q \subseteq 2^{\mathbb{N}}$ such that $Q \equiv_w S$. (In fact, Q is Turing degree isomorphic to S.)

Proof. Since S is Σ^0_2, we have $S = \{X \mid \exists n \, \forall m \, R(m,n,X)\}$ where R is recursive. Let $Q_n = \{X \in 2^{\mathbb{N}} \mid \forall m \, R(m,n,X)\}$. Thus $S = \bigcup_{n \in \mathbb{N}} Q_n$ and Q_n is uniformly Π^0_1. Let T_n be a recursive tree such that $Q_n = \{\text{paths through } T_n\}$. Note that T_n is uniformly recursive.

We may safely assume that T_0 has paths but no recursive paths. Let

$$\widetilde{T_0} = \{\sigma^\frown\langle i\rangle \mid \sigma \in T_0, \sigma^\frown\langle i\rangle \notin T_0, i \in \{0,1\}\}.$$

Since T_0 has paths but no recursive paths, $\widetilde{T_0}$ is infinite. Note that the strings in $\widetilde{T_0}$ are pairwise incompatible. Let $\widetilde{T_0} = \{\tau_n \mid n \in \mathbb{N}\}$ and let

$$T^* = \{\tau_n{}^\frown\tau \mid n \in \mathbb{N}, \tau \in T_n\}.$$

Then T^* is a recursive tree $\subseteq 2^{<\mathbb{N}}$. Let $Q^* = \{\text{paths through } T^*\}$.

We claim that Q^* is Turing degree isomorphic to S. If $X \in S$, then $X \in Q_n$ for some n, and then $\tau_n{}^\frown X \in Q^*$. Conversely, if $Y \in Q^*$, there are two cases. First, if Y is a path through T_0, then $Y \in Q_0$. Second, if Y is not a path through T_0, then $\tau \subset X$ for some $\tau \in \widetilde{T_0}$. Letting n be such that $\tau = \tau_n$, we have $Y = \tau_n{}^\frown X$ where $X \in Q_n$. This proves the claim. ☐

Proof of Theorem 16.1. Since R_1 is Σ^0_2, let P be Σ^0_1 such that $P \equiv_w R_1$. ☐

Definition 16.3. $\mathbf{r}_1 = \deg_w(R_1)$.

Corollary 16.4. $\mathbf{r}_1 \in \mathcal{E}_w$.

Remark 16.5. It can be shown that $\deg_s(R_1) \notin \mathcal{E}_s$, i.e., there is no Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ such that $R_1 \equiv_s P$. This result will not be proved here.

Since $\mathbf{r}_1 \in \mathcal{E}_w$ and $\mathbf{1}$ is the top degree in $\mathcal{E}_w$, we clearly have $\mathbf{r}_1 \leq \mathbf{1}$. We shall now prove that $\mathbf{r}_1 < \mathbf{1}$. In order to prove this, we return to the idea of separation problems.

Lemma 16.6. If $A, B \subseteq \mathbb{N}$ are recursively inseparable, the Turing upward closure of $\mathrm{Sep}(A,B)$ is of measure 0. In other words,

$$\mu(\{Y \in 2^{\mathbb{N}} \mid (\exists X \in 2^{\mathbb{N}})\,(X \leq_T Y, X \text{ separates } A,B\}) = 0.$$

In order to prove this lemma, we first note the following consequence of measure-theoretic regularity.

Lemma 16.7. Let $S \subseteq 2^{\mathbb{N}}$ be measurable, and let $\epsilon > 0$. Then, we can find a finite set of bitstrings $\sigma_1, \ldots, \sigma_k$ such that $\mu(S \triangle \bigcup_{i=1}^k N_{\sigma_i}) < \epsilon$. Here $\triangle$ denotes symmetric difference.

Remark 16.8. Sets of the form $\bigcup_{i=1}^{k} N_{\sigma_i}$ are exactly the clopen sets in $2^{\mathbb{N}}$.

Proof. By regularity let U be an open set such that $S \subseteq U$ and $\mu(U \setminus S) < \epsilon/2$. Let $U = \bigcup_{i=1}^{\infty} N_{\sigma_i}$. By countable additivity let k be such that $\mu(U \setminus \bigcup_{i=1}^{k} N_{\sigma_i}) < \epsilon/2$. Then $\mu(S \triangle \bigcup_{i=1}^{l} N_{\sigma_i}) < \epsilon$, Q.E.D. □

Proof of Lemma 16.6. Suppose this set is of measure > 0. By countable additivity, let e be such that

$$\mu(\{Y \in 2^{\mathbb{N}} \mid \Phi_e(Y) \in 2^{\mathbb{N}} \text{ separates } A, B\}) > 0.$$

Let $S_e = \{Y \in 2^{\mathbb{N}} \mid \Phi_e(Y) \in 2^{\mathbb{N}} \text{ separates } A, B\}$. By the previous lemma, let $\sigma_1, \ldots, \sigma_k$ be such that $\mu(S_e \triangle U) < \mu(U)/4$ where $U = \bigcup_{i=1}^{k} N_{\sigma_i}$. Clearly, $\mu(\{Y \in U \mid \Phi_e(Y) \text{ separates } A, B\}) > 3\mu(U)/4$. Consider the predicate $R(n,i) \equiv \mu(\{Y \in 2^{\mathbb{N}} \mid \Phi_e(Y,n) \simeq i\}) > \mu(U)/4$. Clearly we have $\forall n\, \exists i\, R(n,i)$. Moreover $R(n,i)$ is Σ^0_1 since $R(n,i) \equiv \exists \langle \tau_1, \ldots, \tau_l \rangle\, (\mu(\bigcup_{j=1}^{l} N_{\tau_j}) > \mu(U)/4$ and $N_{\tau_j} \subseteq U$ and $\Phi_e(\tau_j) \downarrow\simeq i$ for all $j = 1, \ldots, l)$ which is clearly Σ^0_1. By Σ^0_1 selection, let $X \in 2^{\mathbb{N}}$ be a recursive selector for R, i.e., X is recursive and $\forall n\, R(n, X(n))$. Then X separates A, B. This is a contradiction since A and B are recursively inseparable. □

Corollary 16.9. If $g \in \mathbb{N}^{\mathbb{N}}$ is nonrecursive, then $\mu(\{X \in 2^{\mathbb{N}} \mid g \leq_T X\}) = 0$.

Proof. Let $A \subset \mathbb{N}$ such that $A \equiv_T g$. Let $B = \mathbb{N} \setminus A$. Clearly A, B are recursively inseparable. $g \leq_T Y \Leftrightarrow A \leq_T Y \Leftrightarrow (\exists X \leq_T Y)(X \text{ separates } A, B)$. So the corollary follows by the lemma above. □

Theorem 16.10. $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$ in $\mathcal{E}_w$.

Proof. Since X is random $\Rightarrow$ X is not recursive,

$$\mathbf{r}_1 = \deg_w(\{X \mid X \text{ is random}\}) > \mathbf{0}.$$

Since $R_1 \equiv_w P$ for a Π^0_1 set $P \subset 2^{\mathbb{N}}$, we know $\mathbf{r}_1 \in \mathcal{E}_w$, hence $\mathbf{r}_1 \leq \mathbf{1}$. It remains to show that $\mathbf{r}_1 < \mathbf{1}$. Recall $\mathbf{1} = \deg_w(\mathrm{DNR}_2)$ and $\mathrm{DNR}_2 = \mathrm{Sep}(K_0, K_1)$ where K_0 and K_1 are recursively inseparable. By our lemma, $\mu(\{Y \mid (\exists X \leq_T Y)\, (X \text{ separates } K_0, K_1)\}) = 0$. Since $\mu(R_1) = 1$, we can find $Y \in R_1$ such that $\neg\, (\exists X \leq_T Y)\, (X \text{ separates } K_0, K_1)$. Hence $\mathrm{Sep}(K_0, K_1) \not\leq_w R_1$, i.e., $\mathbf{1} \not\leq_w \mathbf{r}_1$ in $\mathcal{E}_w$. □

Remark 16.11. $\mathbf{r}_1$ can be characterized as the maximum weak degree of a Π^0_1 subset of $2^{\mathbb{N}}$ of positive measure.

Remark 16.12. As a consequence of Remark 14.7 we have the following result due to Simpson 2004:

> Let $P \subseteq 2^{\mathbb{N}}$ be Π^0_1 and $\neq \emptyset$. If $\{\deg_T(X) \mid X \in P\} \subseteq \{\deg_T(X) \mid X \in \mathrm{DNR}_2\}$, then P is Turing degree isomorphic to DNR_2.

Moreover, Kent/Lewis 2008 have proved a similar result with DNR_2 replaced by R_1. A possible research problem is to find other Π^0_1 subsets of $2^{\mathbb{N}}$ for which a similar result holds.

17 The Embedding Lemma

Many other examples of interesting degrees in $\mathcal{E}_w$ depend on the following lemma.

Lemma 17.1 (The Embedding Lemma)**.** Let $P \subseteq 2^{\mathbb{N}}$ be Π^0_1 and $\neq \emptyset$. Let $S \subseteq \mathbb{N}^{\mathbb{N}}$ be Σ^0_3. Then, we can find $Q \subseteq 2^{\mathbb{N}}$ which is Π^0_1 and $\neq \emptyset$ such that $Q \equiv_w P \cup S$. (In fact, Q is Turing degree isomorphic to $P \cup S$.)

Proof. Let $U \subseteq 2^{<\mathbb{N}}$ be a recursive tree such that $P = \{\text{paths through } U\}$. Since $P \neq \emptyset$, U contains bitstrings of arbitrary length. For each $n \in \mathbb{N}$ let us recursively designate one bitstring $\sigma \in U$ with $|\sigma| = n$ as *canonical*.[1]

Recall Theorem 5.7: given a Σ^0_3 subset of $\mathbb{N}^{\mathbb{N}}$, we can find a Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$ which is Turing degree isomorphic to the given set. Thus, we may safely assume that our Σ^0_3 set $S \subseteq \mathbb{N}^{\mathbb{N}}$ is actually Π^0_1. Let $V \subseteq \mathbb{N}^{<\mathbb{N}}$ be a recursive tree such that $S = \{\text{paths through } V\}$. We may safely assume that $\langle\rangle \in V$.

Let $W \subseteq 3^{<\mathbb{N}}$ be the recursive tree consisting of all $\{0,1,2\}$-valued strings of the form

$$\sigma_0{}^\frown\langle 2\rangle^\frown\sigma_1{}^\frown\langle 2\rangle^\frown\cdots{}^\frown\sigma_{n-1}{}^\frown\langle 2\rangle^\frown\sigma_n$$

where

1. for each $i \leq n$, $\sigma_i \in U$,
2. for each $i < n$, σ_i is canonical,
3. the string $\langle|\sigma_0|, |\sigma_1|, \ldots, |\sigma_{n-1}|\rangle$ belongs to V.

Let $Q = \{\text{paths through } W\}$. Clearly $Q \subseteq 3^{\mathbb{N}}$ is Π^0_1.

We claim that $Q \equiv_w S \cup P$. In fact, we shall show that Q is Turing degree isomorphic to $S \cup P$.

First note that $U \subseteq W$. (This is the special case of conditions 1 and 2 where $n = 0$.) Hence $P \subseteq Q$. Given $g \in S$, consider

$$h = \sigma_0{}^\frown\langle 2\rangle^\frown\sigma_1{}^\frown\langle 2\rangle^\frown\cdots{}^\frown\sigma_n{}^\frown\langle 2\rangle^\frown\cdots \tag{3}$$

where σ_n is the canonical bitstring of length $g(n)$. Clearly h belongs to Q and is $\equiv_T g$. We have now shown that every member of $P \cup S$ is Turing equivalent to a member of Q.

Conversely, given $h \in Q$, we have two cases.

Case 1: $h(j) = 2$ for infinitely many j. Since h is a path through W, h is of the form (3) where each σ_n is canonical. Letting $g(n) = |\sigma_n|$ for all n, it is clear that g belongs to S and is and $\equiv_T h$.

Case 2: $h(j) = 2$ for only finitely many j. Then h is of the form

$$h = \sigma_0{}^\frown\langle 2\rangle^\frown\sigma_1{}^\frown\langle 2\rangle^\frown\cdots{}^\frown\sigma_{n-1}{}^\frown\langle 2\rangle^\frown X$$

[1] For instance, we could designate as canonical the lexicographically leftmost string in U of length n, for each n.

where $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ are bitstrings and $X \in 2^{\mathbb{N}}$. Since h is a path through W, we have $X \upharpoonright m \in U$ for all m, i.e., X is a path through U. Thus $X \in P$, and clearly X is Turing equivalent to h.

We have now shown that every member of Q is Turing equivalent to a member of $S \cup P$. This completes the proof of the Embedding Lemma. □

Corollary 17.2. If $\mathbf{s} = \deg_w(S)$ where S is Σ^0_3, then $\inf(\mathbf{s}, \mathbf{1}) \in \mathcal{E}_w$.

Proof. Let $P \subseteq 2^{\mathbb{N}}$ be Π^0_1 and $\neq \emptyset$ with $\deg_w(P) = \mathbf{1}$. By the Embedding Lemma, let $Q \equiv_w S \cup P$ where $Q \subseteq 2^{\mathbb{N}}$ is Π^0_1 and $\neq \emptyset$. Then clearly $\inf(\mathbf{s}, \mathbf{1}) = \deg_w(S \cup P) = \deg_w(Q) \in \mathcal{E}_w$. □

18 Consequences of the Embedding Lemma

Corollary 17.2 is useful in showing that various interesting weak degrees belong to $\mathcal{E}_w$. In this section we give several examples of this.

Our first example involves randomness relative to the Halting Problem.

Example 18.1. Let $\mathbf{r}_2 = \deg_w(R_2)$ where $R_2 = \{X \in 2^{\mathbb{N}} \mid X \text{ is 2-random}\}$. Here 2-random means random relative to $0'$. We can use the Embedding Lemma to show that $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_w$.

Details:

Definition 18.2. If g is any oracle, $g \in \mathbb{N}^{\mathbb{N}}$, we say that X is *g-random* (i.e., random relative to g) if $X \notin \bigcap_{n=0}^{\infty} U^g_n$ for any uniformly $\Sigma^{0,g}_1$ sequence of sets U^g_n, $n = 0, 1, 2, \ldots$ with $\mu(U^g_n) \leq 1/2^n$.

Lemma 18.3. There is a universal test for g-randomness, uniformly in g.

In other words, we can find U^g_n as above such that

$$R^g_1 = \{X \in 2^{\mathbb{N}} \mid X \text{ is } g\text{-random}\} = 2^{\mathbb{N}} \setminus \bigcap_{n=0}^{\infty} U^g_n$$

and the predicate $U(X, n, g) \equiv X \in U^g_n$ is Σ^0_1.

Proof. This is a straightforward relativization of Lemma 15.9. □

Corollary 18.4. R^g_1 is $\Sigma^{0,g}_2$, uniformly in g, and $\mu(R^g_1) = 1$.

We then have $R_2 = R^{0'}_1 = \{X \in 2^{\mathbb{N}} \mid X \text{ is } 0'\text{-random}\}$.

Corollary 18.5. R_2 is $\Sigma^{0,0'}_2$, hence Σ^0_3.

Proof. In fact, any predicate which is $\Sigma^{0,0'}_2$ is Σ^0_3. This is easily seen by a Tarski/Kuratowski computation. □

Theorem 18.6. We have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_w$ and $\mathbf{r}_1 \leq \inf(\mathbf{r}_2, \mathbf{1}) < \mathbf{1}$.

Later we will show that $\mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1})$.

Proof. Since R_2 is Σ^0_3, we have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_w$ by the Embedding Lemma. Trivially $\inf(\mathbf{r}_2, \mathbf{1}) \leq \mathbf{1}$. Since $\mu(R_2) = 1$, it follows by Lemma 16.6 that we can find $X \in R_2$ such that $\neg\,(\exists Y \leq_T X)\,(Y$ separates $K_0, K_1)$. Thus $\mathbf{1} \not\leq \mathbf{r}_2$, hence $\inf(\mathbf{r}_2, \mathbf{1}) < \mathbf{1}$. □

Our second example involves diagonal nonrecursiveness.

Example 18.7. Let $\mathbf{d} = \deg_w(\mathrm{DNR})$ where $\mathrm{DNR} = \{g \in \mathbb{N}^\mathbb{N} \mid g$ is diagonally non-recursive $\} = \{g \in \mathbb{N}^\mathbb{N} \mid \forall n\,(g(n) \not\simeq \varphi^{(1)}_n(n))\}$. We can use the Embedding Lemma to show that $\mathbf{d} \in \mathcal{E}_w$.

Details:

Theorem 18.8. $\mathbf{d} \in \mathcal{E}_w$.

Proof. DNR is Π^0_1, hence Σ^0_3, so by Corollary 17.2 we have $\inf(\mathbf{d}, \mathbf{1}) \in \mathcal{E}_w$. However, $\mathbf{d} \leq \mathbf{1}$ because $\mathbf{1} = \deg_w(\mathrm{DNR}_2)$ and $\mathrm{DNR}_2 \subset \mathrm{DNR}$. We now see that $\mathbf{d} = \inf(\mathbf{d}, \mathbf{1}) \in \mathcal{E}_w$. □

Theorem 18.9. $\mathbf{0} < \mathbf{d} \leq \mathbf{r}_1$.

Later we shall prove that $\mathbf{d} < \mathbf{r}_1$.

Proof. Obviously $\mathbf{0} < \mathbf{d}$, because $\mathrm{DNR} \cap \mathrm{REC} = \emptyset$. It remains to prove that $\mathbf{d} \leq \mathbf{r}_1$, i.e., $\mathrm{DNR} \leq_w R_1$. Consider the total recursive functional $\Phi : 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ defined by

$$\begin{aligned} \Phi(X)(n) &= X(0) + 2X(1) + 4X(2) + \cdots + 2^{n-1}X(n-1) \\ &= \sum_{i=0}^{n-1} X(i)\,2^i. \end{aligned}$$

Let $U_n = \{X \in 2^\mathbb{N} \mid \Phi(X)(n) \simeq \varphi^{(1)}_n(n)\}$. Clearly $\mu(U_n) \leq 1/2^n$. Letting $V_n = \bigcup_{k=n+1}^\infty U_k$ we see that $\mu(V_n) \leq \sum_{k=n+1}^\infty \mu(U_k) \leq \sum_{k=n+1}^\infty 1/2^k = 1/2^n$. Moreover V_n is uniformly Σ^0_1, so V_n, $n = 0, 1, 2, \ldots$ is a Martin-Löf test. If X is random, we have $X \notin V_n$ for some n, hence $\Phi(X)(k) \not\simeq \varphi^{(1)}_k(k)$ for all $k \geq n+1$. Hence we can find $g \in \mathrm{DNR}$ such that $g(k) = \Phi(X)(k)$ for all $k \geq n+1$. Clearly $g \leq_T X$. Thus $\mathrm{DNR} \leq_w R_1$. □

Exercise 18.10. We have shown that $\mathrm{DNR} \leq_w R_1$. Show that $\mathrm{DNR} \not\leq_s R_1$.

Here is a third example.

Example 18.11. Define

$$\mathrm{DNR}_{\mathrm{REC}} = \{\text{recursively bounded, diagonally nonrecursive functions}\}.$$

Thus $\mathrm{DNR}_{\mathrm{REC}} = \{g \in \mathrm{DNR} \mid g$ is dominated by a recursive function$\}$. Define $\mathbf{d}_{\mathrm{REC}} = \deg_w(\mathrm{DNR}_{\mathrm{REC}})$. We can use the Embedding Lemma to show that $\mathbf{d}_{\mathrm{REC}} \in \mathcal{E}_w$.

Details:

Theorem 18.12. $\mathbf{d}_{\mathrm{REC}} \in \mathcal{E}_w$ and $\mathbf{d} \leq \mathbf{d}_{\mathrm{REC}} \leq \mathbf{r}_1$.

Proof. We have

$$g \in \mathrm{DNR}_{\mathrm{REC}} \equiv \exists e \forall n\, (\varphi_n^{(1)}(n) \not\simeq g(n) \wedge \exists j\, (\varphi_e^{(1)}(n) \simeq j \wedge\ g(n) < j))$$

and a Tarski/Kuratowski computation shows that $\mathrm{DNR}_{\mathrm{REC}}$ is Σ^0_3. Moreover, by Corollary 17.2 and Theorem 14.6 we have

$$\deg_w(\mathrm{DNR}_{\mathrm{REC}}) = \deg_w(\mathrm{DNR}_{\mathrm{REC}} \cup \mathrm{DNR}_2) = \inf(\mathbf{d}_{\mathrm{REC}}, \mathbf{1}) \in \mathcal{E}_w$$

since $\mathrm{DNR}_{\mathrm{REC}} \subseteq \mathrm{DNR}$. Thus $\mathbf{d} \leq \mathbf{d}_{\mathrm{REC}} \leq \mathbf{1}$. Moreover, our proof that $\mathbf{d} \leq \mathbf{r}_1$ actually shows that $\mathbf{d}_{\mathrm{REC}} \leq \mathbf{r}_1$. Namely, we showed that if X is 1-random then $\Phi(X)(n) \not\simeq \varphi_n^{(1)}(n)$ for all but finitely many n, but clearly $\Phi(X)(n) = \sum_{i=0}^n X(i)2^i$ is recursively bounded, in fact it is bounded by 2^n. □

Later we shall prove that $\mathbf{d} < \mathbf{d}_{\mathrm{REC}} < \mathbf{r}_1$.

19 1-genericity

Definition 19.1. A function $g \in \mathbb{N}^{\mathbb{N}}$ is said to be *n-generic* if for all Σ^0_n sets $A \subseteq \mathbb{N}^{<\mathbb{N}}$ we can find $\sigma \subset g$ such that either $\sigma \in A$ or $\neg \exists \tau\, (\sigma \subseteq \tau \wedge \tau \in A)$.

Remark 19.2. Note that g is generic in the sense of Section 7 if and only if it is n-generic for all n.

Proof. $\Rightarrow$: Assume that g is generic in the sense of Section 7. Given a Σ^0_n set of strings A, let $D_A = \{\rho \in \mathbb{N}^{<\mathbb{N}} \mid (\exists \sigma \in A)\, (\sigma \subseteq \rho \vee \nexists \sigma\, (\sigma \supseteq \rho \wedge \sigma \in A)\}$. Note that D_A is arithmetical, in fact Δ^0_{n+1}. Moreover D_A is dense: Given σ, if $\neg \exists \tau (\tau \supseteq \sigma, \tau \in A)$ then $\sigma \in D_A$, otherwise we can find $\tau \supseteq \sigma$ such that $\tau \in A$ and then $\tau \in D_A$. Now, since g is generic, g meets D_A, i.e., $(\exists \sigma \subset g)\, (\sigma \in D_A)$. Thus g is n-generic.

$\Leftarrow$: Trivial. □

Definition 19.3. g is *weakly n-generic* if for all dense Σ^0_n sets $D \subseteq \mathbb{N}^{<\mathbb{N}}$ there exists $\sigma \subset g$ such that $\sigma \in D$.

Remark 19.4. Note that $D_A = A$ if and only if A is dense. Thus weakly $n+1$-generic implies n-generic which implies weakly n-generic. The converse implications do not hold in general.

Lemma 19.5. If g is 1-generic, then g is *generalized low*, i.e., $g' \equiv_T g \oplus 0'$,

Proof. It suffices to show that $g' \leq_T g \oplus 0'$. Trivially $e \in g' \equiv \varphi_e^{(1),g}(0) \downarrow \equiv (\exists \sigma \subset g)\, (\varphi_{e,|\sigma|}^{(1),\sigma}(0) \downarrow)$ which is $\Sigma^{0,g}_1$. Let $A_e = \{\sigma \mid \varphi_{e,|\sigma|}^{(1),\sigma}(0) \downarrow\}$ and let $D_e = D_{A_e} = \{\sigma \mid \sigma \in A_e \vee \neg (\exists \tau \supseteq \sigma)\, (\tau \in A_e)\}$. Clearly A_e is Σ^0_1 (it is actually recursive). Therefore, because g is 1-generic, $(\exists \sigma \subset g)\, (\sigma \in D_e)$. Thus we have $e \notin g' \equiv (\exists \sigma \subset g) \neg (\exists \tau \supseteq \sigma)\, (\tau \in A_e)$ which is $\Sigma^{0,g\oplus 0'}_1$. Thus g' is $\Delta^{0,g\oplus 0'}_1$, i.e., $g' \leq_T g \oplus 0'$, Q.E.D. □

Exercise 19.6. Generalize the previous lemma as follows. If g is n-generic then $g^{(n)} \equiv_T g \oplus 0^{(n)}$.

Theorem 19.7. If $h \geq_T 0'$ we can find a 1-generic g such that $g' \equiv_T h$.

Proof. We shall construct an increasing sequence of strings

$$\tau_0 \subset \tau_1 \subset \cdots \subset \tau_n \subset \tau_{n+1} \subset \cdots$$

and we shall define $g = \bigcup_{n=0}^{\infty} \tau_n$.

Stage 0. Begin with $\tau_0 = \langle\rangle$, the empty string.

Stage $n+1$. Given τ_n, ask the oracle $0'$ whether (Case 1) there exists $\tau \supseteq \tau_n{}^\frown\langle h(n)\rangle$ such that $\varphi^{(1),\tau}_{n,|\tau|}(0)\downarrow$. If so, let $\tau_{n+1} =$ the least such τ. Otherwise (Case 2) let $\tau_{n+1} = \tau_n{}^\frown\langle h(n)\rangle$.

We claim that g is 1-generic. Given a Σ^0_1 set $A \subseteq \mathbb{N}^{<\mathbb{N}}$, let n be such that $\varphi^{(1),g}_n(0)\downarrow \equiv (\exists\sigma \subset g)\,(\sigma \in A)$ for all g. We know that $\tau_n{}^\frown\langle h(n)\rangle \subseteq \tau_{n+1} \subset g$. At stage n, if Case 1 held, $\varphi^{(1),\tau_{n+1}}_n(0)\downarrow$, hence $\varphi^{(1),g}_n(0)\downarrow$, hence $(\exists\sigma \subset g)\,(\sigma \in A)$. If Case 2 held, we have $\neg\,(\exists\tau \supseteq \tau_n{}^\frown\langle h(n)\rangle)\,(\varphi^{(1),\tau}_{n,|\tau|}(0)\downarrow)$, hence $\neg\,(\exists\tau \supseteq \tau_n{}^\frown\langle h(n)\rangle)(\tau \in A)$. In either case g satisfies the condition for 1-genericity, thus proving our claim.

We claim that the sequence $\langle\tau_n\rangle_{n\in\mathbb{N}}$ is $\leq_T h$ and $\leq_T g \oplus 0'$. To see this, note that it is $\leq_T h$ because $0' \leq_T h$, and it is $\leq_T g \oplus 0'$ because $h(n) = g(|\tau_n|)$.

It follows from this claim that $h \equiv_T g \oplus 0'$. Hence by the previous lemma we have $h \equiv_T g'$, Q.E.D. ☐

Corollary 19.8 (Friedberg)**.** Given $h \geq_T 0'$ we can find g such that $g' \equiv_T h$. Thus, the range of the Turing jump operator $' : \mathcal{D}_T \to \mathcal{D}_T$ given by $\mathbf{a} \mapsto \mathbf{a}'$ consists precisely of all Turing degrees $\mathbf{b} \geq \mathbf{0}'$.

Lemma 19.9. If $g \in \mathbb{N}^{\mathbb{N}}$ is 1-generic, there is no DNR function $\leq_T g$.

Proof. Given e, to show $\varphi^{(1),g}_e$ is not DNR. Let

$$A = \{\tau \in \mathbb{N}^{<\mathbb{N}} \mid \exists n\,(\varphi^{(1),\tau}_e(n)\downarrow = \varphi^{(1)}_n(n)\downarrow)\}.$$

Clearly A is Σ^0_1. Because g is 1-generic, there are two cases.

Case 1: $(\exists\tau \subset g)\,(\tau \in A)$.

Case 2: $(\exists\sigma \subset g)\,\neg\,(\exists\tau \supseteq \sigma)\,(\tau \in A)$.

In Case 1 we have $\tau \subset g$, hence $\varphi^{(1),g}_e(n)\downarrow = \varphi^{(1)}_n(n)\downarrow$, hence $\varphi^{(1),g}_e$ is not DNR. In Case 2 we have $\sigma \subset g$ and there is no $\tau \supseteq \sigma$ such that $\tau \in A$. Define a partial recursive function $\theta(m)$ as follows. Given m, search for $\tau \supseteq \sigma$ such that $\varphi^{(1),\tau}_e(m)\downarrow$. If we find such a τ let $\theta(m) = \varphi^{(1),\tau}_e(m)$, otherwise $\theta(n)\uparrow$. Now let n be an index of θ, i.e., $\varphi^{(1)}_n(m) \simeq \theta(m)$ for all m.

We claim that $\varphi^{(1),\tau}_e(n)\uparrow$ for all $\tau \supseteq \sigma$. Otherwise, let $\tau \supseteq \sigma$ be such that $\varphi^{(1),\tau}_e(n)\downarrow = \theta(n)\downarrow$. Such a τ would exist by definition of θ. Then $\varphi^{(1),\tau}_e(n)\downarrow = \theta(n)\downarrow = \varphi^{(1)}_n(n)\downarrow$ since n is an index of θ. Hence $\tau \in A$ contradicting the case hypothesis. This proves the claim.

Since $\sigma \subset g$, our claim implies that $\varphi_e^{(1),g}(n) \uparrow$. Hence $\varphi_e^{(1),g}$ is not DNR. This completes the proof. $\square$

Combining Theorem 19.7 and Lemma 19.9 we get:

Theorem 19.10. Given $B \geq_T 0'$ we can find A such that $A' \equiv_T B$ and there is no DNR function $\leq_T A$.

20 Kolmogorov complexity

Given a bitstring $\tau \in 2^{<\mathbb{N}}$, we would like to measure the "intrinsic complexity" or "amount of information" which is inherent in τ. Roughly speaking, the "complexity" of τ will be the length of the shortest description of τ. However, in view of Berry's Paradox, we need to restrict the allowable descriptions. Berry's Paradox refers to "the least n such that n cannot be described in $< 10^6$ bits." We shall consider only the following kind of description.

Definition 20.1. A *description* of τ is a program which outputs τ.

Definition 20.2. A *machine* is a partial recursive function $M :\subseteq 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ (from bitstrings to bitstrings). A machine U is said to be *universal* if for all machines M we can find a fixed bitstring ρ (depending on M) such that $M(\sigma) \simeq U(\rho^\frown\sigma)$ for all σ.

Definition 20.3 (Kolmogorov)**.** Fix a universal machine U. Given a bitstring τ, we define $C(\tau) = \min\{|\sigma| \mid U(\sigma) \simeq \tau\}$ = the complexity of τ = the minimum number of bits needed to describe τ via U.

Remark 20.4. Note that $C(\tau)$ is not computable as a function of τ. However, $C(\tau) \downarrow$ for all τ. The following lemma says that $C(\tau)$ is well-defined up to $\pm O(1)$. Thus $C(\tau)$ depends only slightly on our choice of a universal machine.

Lemma 20.5. Let U_1 and U_2 be universal machines and define $C_i(\tau) = \min\{|\sigma| \mid U_i(\sigma) \simeq \tau\}$ for $i = 1, 2$. Then $|C_1(\tau) - C_2(\tau)| \leq O(1)$. In other words, we can find a constant k depending only on U_1 and U_2 such that $|C_1(\tau) - C_2(\tau)| \leq k$ for all τ.

Proof. Since U_2 is universal, let ρ_1 be such that $U_1(\sigma) \simeq U_2(\rho^\frown\sigma)$ for all σ. We claim that $C_2(\tau) \leq C_1(\tau) + |\rho_1|$ for all τ. To see this, let σ be of minimum length such that $U_1(\sigma) = \tau$. Then $|\sigma| = C_1(\tau)$. We have $U_2(\rho_1{}^\frown\sigma) \simeq U_1(\sigma) \simeq \tau$, hence $C_2(\tau) \leq |\rho_1{}^\frown\sigma| = |\rho_1| + |\sigma| = C_1(\tau) + |\rho_1|$. Thus we see that $C_2(\tau) - C_1(\tau) \leq O(1)$, namely $|\rho_1|$. Similarly, $C_1(\tau) - C_2(\tau) \leq O(1)$. $\square$

Lemma 20.6. For any $\tau \in 2^{<\mathbb{N}}$ we have $C(\tau) \leq |\tau| + O(1)$.

Proof. Consider the identity machine, $M(\sigma) = \sigma$. Let ρ be such that $M(\sigma) \simeq U(\rho^\frown\sigma)$ for all σ. Then $M(\tau) = U(\rho^\frown\tau) = \tau$, so $C(\tau) \leq |\rho^\frown\tau| = |\rho| + |\tau| = |\tau| + O(1)$. $\square$

Remark 20.7. More generally, if τ is any finite combinatorial object, let

$$C(\tau) = \text{the complexity of } \tau = \min\{|\sigma| \mid U(\sigma) \simeq \#(\tau)\}$$

where $U :\subseteq 2^{<\mathbb{N}} \to \mathbb{N}$ is a fixed universal machine. Here $\#(\tau)$ is the Gödel number of τ. As above we can show that $C(\tau)$ is well-defined up to $\pm O(1)$.

We now consider complex points in the Cantor space.

Definition 20.8. A point $X \in 2^{\mathbb{N}}$ is said to be *complex* if there exists a total recursive function $p(n)$ such that $\sup p(n) = \infty$ and $C(X \upharpoonright n) \geq p(n)$ for all n. Let $\text{COMPLEX} = \{X \in 2^{\mathbb{N}} \mid X \text{ is complex}\}$.

Remark 20.9. The following theorem implies that if X is complex then X is nonrecursive. Indeed, complexity may be viewed as a weak form of randomness.

Theorem 20.10 (Kjos-Hanssen/Merkle/Stephan 2006)**.** We have

$$\text{COMPLEX} \ \equiv_w \ \text{DNR}_{\text{REC}}.$$

More precisely, COMPLEX is Turing degree isomorphic to DNR_{REC}.

Notation 20.11. Given a total recursive function p such that $\sup p(n) = \infty$, define the inverse function p^{-1} by letting $p^{-1}(m) =$ the least n such that $p(n) \geq m$. Note that p^{-1} is again a total recursive function and $\sup p^{-1}(m) = \infty$. Moreover p^{-1} is *monotone*, i.e., $p^{-1}(m) \leq p^{-1}(m+1)$ for all m.

Examples 20.12. If $p(n) = 2^n$ then $p^{-1}(n) \approx \log_2 n$. If $p(n) \approx \log_2 n$ then $p^{-1}(n) \approx 2^n$.

Proof of Theorem 20.10. First, given a complex $X \in 2^{\mathbb{N}}$, to find $f \in \text{DNR}_{\text{REC}}$ such that $f \equiv_T X$. Let $p(n)$ be a recursive function such that $\sup p(n) = \infty$ and $C(X \upharpoonright n) \geq p(n)$ for all n. In particular $C(X \upharpoonright p^{-1}(m)) \geq p(p^{-1}(m)) \geq m$ for all m. Let

$$f(m) = \#(X \upharpoonright p^{-1}(m)) = \textstyle\sum_{i<p^{-1}(m)} X(i)2^i$$

and note that $f \equiv_T X$ and $f(m) < 2^{p^{-1}(m)}$, a recursive function. We wish to show that f is DNR. By definition of $f(m)$ we have $C(f(m)) \geq m - O(1)$ for all m. But $C(\varphi_m^{(1)}(m)) \leq C(m) + O(1) \leq \log_2 m + O(1) < m$ for all but finitely many m. Thus $f(m) \not\simeq \varphi_m^{(1)}(m)$ for all but finitely many m. In other words, f is DNR with finitely many exceptions, and this suffices.

Conversely, assume that $f \in \text{DNR}_{\text{REC}}$, i.e., f is DNR and recursively bounded. Let $p(m)$ be a total recursive function such that $f(m) < 2^{p(m)}$ for all m. Let $q(m) = \sum_{j<m} p(j)$ and define $X \in 2^{\mathbb{N}}$ by the condition $f(m) = \sum_{i<p(m)} X(q(m)+i)2^i$ for all m. Clearly $X \equiv_T f$. We shall show that X is complex. By the Parametrization Theorem, let r be a primitive recursive function such that $\varphi_{r(\sigma)}^{(1)}(m) \simeq \sum_{i<p(m)} U(\sigma)(q(m)+i)2^i$ for all bitstrings σ. In particular

$$\varphi_{r(\sigma)}^{(1)}(r(\sigma)) \ \simeq \sum_{i<p(r(\sigma))} U(\sigma)(q(r(\sigma))+i)2^i$$

but on the other hand

$$f(r(\sigma)) \;=\; \sum_{i<p(r(\sigma))} X(q(r(\sigma))+i)2^i$$

and of course $\varphi^{(1)}_{r(\sigma)}(r(\sigma)) \not\simeq f(r(\sigma))$ for all bitstrings σ. Thus $U(\sigma) \not\simeq X \upharpoonright q(m)$ whenever σ and m are such that $r(\sigma) < m$. It follows that $C(X \upharpoonright q(m)) \geq$ the least n such that $r(\sigma) \geq m$ for some σ of length n. In other words, $C(X \upharpoonright q(m)) \geq s^{-1}(m)$ where $s(n) = \max\{r(\sigma) \mid |\sigma| = n\}$. Moreover $C(X \upharpoonright q(m)) \leq C(X \upharpoonright i) + O(1)$ whenever $q(m) \leq i < q(m+1)$, i.e., $m = q^{-1}(i+1) - 1$. Thus for all i we have

$$C(X \upharpoonright i) = C(X \upharpoonright q(q^{-1}(i+1)-1)) - O(1) \geq s^{-1}(q^{-1}(i+1)-1) - O(1)$$

so X is complex, Q.E.D. □

Remark 20.13. Define an *order function* to be a total recursive function p such that p is monotone and $\lim p(n) = \infty$. The proof of Theorem 20.10 shows that X is complex if and only if there exists an order function p such that $C(X \upharpoonright n) \geq p(n)$ for all n.

Definition 20.14. Define the inverse of an order function p to be $p^{-1}(n) =$ least m such that $p(m) \geq n$.

Note that if p is an order function, so is p^{-1}. But their growth rates are very different. If p is fast-growing, then p^{-1} is slow-growing and vice versa.

Theorem 20.15. $\mathrm{DNR}_{\mathrm{REC}}$ is Turing degree isomorphic to COMPLEX.

Recall that

$$\mathrm{DNR}_{\mathrm{REC}} = \{f \in \mathrm{DNR} \mid \exists \text{ order function } p \text{ such that } \forall n\, f(n) < p(n)\}$$

and

$$\mathrm{COMPLEX} = \{X \in 2^{\mathbb{N}} \mid \exists \text{ order function } p \text{ such that } \forall n\, C(X \upharpoonright n) \geq p^{-1}(n)\}.$$

Remark 20.16. Theorem 20.10 can be refined as follows. Let C be a "nice" class of recursive functions, e.g., the primitive recursive functions. Define

$$\mathrm{DNR}_C = \{f \in \mathrm{DNR} \mid \exists \text{ order function } p \in C \text{ such that } \forall n\, f(n) < p(n)\}$$

and

$$\mathrm{COMPLEX}_C = \{X \in 2^{\mathbb{N}} \mid \exists \text{ order function } p \in C \text{ such that } \forall n\, C(X \upharpoonright n) \geq p^{-1}(n)\}.$$

Then DNR_C and $\mathrm{COMPLEX}_C$ are Turing degree isomorphic.

Moreover, we shall see later that if C and C' are "nice" classes as above and $C \subset C'$ and C' contains a function which grows much faster than all of the functions in C, then $\mathbf{d}_{C'} < \mathbf{d}_C$.

21 Distinguishing $\mathbf{r}_1$ from $\inf(\mathbf{r}_2, \mathbf{1})$

We already know that $\mathbf{r}_1 \leq \inf(\mathbf{r}_2, \mathbf{1})$. In this section we shall prove that the inequality is strict.

Theorem 21.1 (Stephan)**.** If $X \in 2^{\mathbb{N}}$ is random and CPA $\leq_w \{X\}$, then $0' \leq_T X$.

Corollary 21.2. $\mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1})$.

Proof. Let $X \in R_1$ be $<_T 0'$ (we can find such an X by the Low Basis Theorem). Because $X \leq_T 0'$ there is no 2-random $Y \leq_T X$. Moreover by Stephan's Theorem we have CPA $\not\leq_w \{X\}$. Combining these two observations we have $R_2 \cup \text{CPA} \not\leq_w \{X\}$. Hence $\inf(\mathbf{r}_2, \mathbf{1}) \not\leq \mathbf{r}_1$. But we already know by Theorem 18.6 that $\mathbf{r}_1 \leq \inf(\mathbf{r}_2, \mathbf{1})$. The corollary follows. □

In order to prove Stephan's Theorem, the following lemma will be useful.

Definition 21.3. A set $S \subseteq \mathbb{N}^{\mathbb{N}}$ is *recursively bounded* if there exists a recursive function h such that $f(n) < h(n)$ for all $f \in S$ and $n \in \mathbb{N}$.

Lemma 21.4. If $P \subseteq \mathbb{N}^{\mathbb{N}}$ is recursively bounded and Π^0_1, then we can find a Π^0_1 set $P^* \subseteq 2^{\mathbb{N}}$ such that $P \equiv_s P^*$. In fact, P and P^* are recursively homeomorphic.

Proof. Define a total recursive functional $\Phi : \mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$\Phi(f) = \text{the characteristic function of the graph of } f.$$

Clearly Φ is injective and Φ^{-1} is partial recursive. Let $P^* = \{\Phi(f) \mid f \in P\}$. Clearly P is recursively homeomorphic to P^*. We would like to show that P^* is Π^0_1. To do this, let h be a recursive function which bounds P. Then for all $X \in 2^{\mathbb{N}}$ we have $X \in P^* \equiv (\forall m\, (\exists i < h(m))\, (X(3^m 5^i) = 1 \wedge \forall j\, (X(3^m 5^j) = 1 \Rightarrow i = j)) \wedge \Phi^{-1}(X) \in P$. Clearly this is Π^0_1. □

Corollary 21.5. If $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^0_1 and $\neq \emptyset$ and recursively bounded, then $P \leq_s$ CPA.

Proof. This is immediate from Lemma 21.4 and Theorem 14.17. □

We now prove Stephan's Theorem.

Proof of Theorem 21.1. We know that $0' = H = \{n \mid \varphi^{(1)}_n(0)\downarrow\}$. Let $H_s = \{n \mid \varphi^{(1)}_{n,s}(0)\downarrow\}$. Clearly $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_s \subseteq \cdots$ and $H = \bigcup_{s=0}^{\infty} H_s$. Define a 1-place partial recursive function ψ as follows: Given e and n, search for the least s such that $n \in H_s$. If and when s is found, for each $i \leq 2^n$ compute the rational number

$$r_{e,n,i} = \mu\{X \in 2^{\mathbb{N}} \mid \varphi^{(1),X}_{e,s}(2^e 3^n)\downarrow = i\}$$

and define $\psi(2^e 3^n) = i < 2^n$ chosen so that $r_{e,n,i} < 1/2^n$. Otherwise $\psi(m)\uparrow$. Note that ψ is partial recursive and that $\psi(2^e 3^n)\downarrow \Leftrightarrow n \in H$.

We claim that if X is random and $X \geq_T$ a total extension of ψ, then $X \geq_T 0'$.

To prove the claim, let

$$V_{e,n} = \{X \in 2^{\mathbb{N}} \mid \exists s\, (n \in H_s \setminus H_{s-1} \wedge \varphi_{e,s}^{(1),X}(2^e 3^n) \simeq \psi(2^e 3^n))\}.$$

Note that if $n \in H$ then $\mu(V_{e,n}) = r_{e,n,\psi(2^e 3^n)} \leq 1/2^n$. Moreover $V_{e,n}$ is uniformly Σ^0_1 and thus the sets $V_{e,n}$, $n \in \mathbb{N}$ form a test for randomness. Now suppose X is random and $X \geq_T$ a total extension of ψ. Let e be such that $\varphi_e^{(1),X}$ is total and extends ψ. Define $f(n) =$ least s such that $\varphi_{e,s}^{(1),X}(2^e 3^n)\downarrow$. Since $\varphi_e^{(1),X}$ is total, f is total and $f \leq_T X$. Since X is random, $X \in V_{e,n}$ for only finitely many n. Hence, for all but finitely many n, $n \in H_s \setminus H_{s-1}$ implies $\varphi_{e,s}^{(1),X}(2^e 3^n) \not\simeq \psi(2^e 3^n)$, hence $\varphi_{e,s}^{(1),X}(2^e 3^n)\uparrow$, hence $s < f(n)$. Thus, for all but finitely many n, $n \in H$ implies $n \in H_{f(n)}$. Thus $H \leq_T f \leq_T X$ and our claim is proved.

Let $P = \{g \in \mathbb{N}^{\mathbb{N}} \mid g$ is a total extension of ψ and $\forall m\, g(m) < 2^m\}$. Since $\psi(m) < 2^m$ whenever $\psi(m)\downarrow$, we see that $P \neq \emptyset$. Since P is a recursively bounded nonempty Π^0_1 set, it follows by Corollary 21.5 that $P \leq_s \text{CPA}$.

Finally, let X be random such that $\text{CPA} \leq_w \{X\}$. Since $\text{CPA} \leq_w \{X\}$, we have $P \leq_w \{X\}$. Since X is random, our claim above implies that $X \geq_T 0'$, Q.E.D. □

Remark 21.6. Another theorem, the Kučera/Gács Theorem, says that every Turing degree $\geq \mathbf{0}'$ is random, i.e., it contains a random X. Combining this with Stephan's Theorem, we see see that a random Turing degree is a PA-degree if and only if it is $\geq \mathbf{0}'$.

22 Distinguishing $\mathbf{d}$, $\mathbf{d}_{\text{REC}}$, and $\mathbf{d}_C$

The weak degrees $\mathbf{d} = \deg_w(\text{DNR})$ and $\mathbf{d}_{\text{REC}} = \deg_w(\text{DNR}_{\text{REC}})$, and $\mathbf{d}_C = \deg_w(\text{DNR}_C)$ have been introduced previously in Sections 18 and 20. Here DNR is the class of diagonally recursive functions, DNR_{REC} is the class of recursively bounded DNR functions, and DNR_C is the class of C-bounded DNR functions where C is a "nice" subclass of REC such as the primitive recursive functions.

The purpose of this section is to locate these degrees within $\mathcal{E}_w$ by proving the inequalities

$$\mathbf{0} \ < \ \mathbf{d} \ < \ \mathbf{d}_{\text{REC}} \ < \ \mathbf{d}_C \ < \ \mathbf{r}_1. \tag{4}$$

See also Remark 22.18 below.

We begin with the following theorem.

Definition 22.1. Given a total recursive function $p(n)$, let

$$\begin{aligned} \text{DNR}_p &= \{p\text{-bounded DNR functions}\} \\ &= \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n\, (f(n) < p(n) \wedge f(n) \not\simeq \varphi_n^{(1)}(n))\}. \end{aligned}$$

Theorem 22.2. For all recursive functions p, DNR $<_w$ DNR_p.

Proof of Theorem 22.2, Preliminary Outline. Given a recursive function p, we shall define an increasing sequence of strings

$$\tau_0 \subset \tau_1 \subset \cdots \subset \tau_e \subset \tau_{e+1} \subset \cdots$$

such that $g = \bigcup_{e=0}^{\infty} \tau_e$ is DNR and $\mathrm{DNR}_p \not\leq_w \{g\}$. Clearly this will suffice to prove the theorem. We may safely assume that $p(n) > 0$ for all n.

We begin the construction at stage 0 with $\tau_0 = \langle\rangle =$ the empty string. At stage $e+1$ we define $\tau_{e+1} \supset \tau_e$ so as to address the eth requirement, $\varphi_e^{(1),g} \notin \mathrm{DNR}_p$. Note that there are three possible methods of satisfying this requirement:

1. $\varphi_e^{(1),g}(n) \uparrow$ for some n.

2. $\varphi_e^{(1),g}(n) \downarrow= \varphi_n^{(1)}(n)$ for some n.

3. $\varphi_e^{(1),g}(n) \downarrow\geq p(n)$ for some n.

If we can find a string $\tau \supset \tau_e$ such that $\varphi_e^{(1),\tau}(n) \downarrow= \varphi_n^{(1)}(n)$ or $\varphi_e^{(1),\tau}(n) \downarrow\geq p(n)$ for some n, then we can let $\tau_{e+1} = \tau$ and this will satisfy the eth requirement by method 2 or method 3 respectively. Otherwise we will have to resort to method 1, which means paying attention to this requirement for the rest of the construction.

In parallel with the definition of $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_e \subset \tau_{e+1} \subset \cdots$, we shall define finite sets of ordered pairs

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_e \subseteq F_{e+1} \subseteq \cdots$$

where each pair $\langle i, n\rangle \in F_e$ represents a commitment to insure that $\varphi_i^{(1),g}(n) \uparrow$. In this way the ith requirement will be satisfied via method 1. The details of how to fulfill these commitments are the heart of the proof.

Associated with the finite set F_e will be the set

$$A_e = \bigcup_{\substack{\langle i,n\rangle \in F_e \\ j<p(n)}} A_{i,n,j} \qquad \text{where } A_{i,n,j} = \{\tau \mid \varphi_i^{(1),\tau}(n) \simeq j\}.$$

In order to fulfill the commitments embodied in F_e, we shall arrange that A_e is in certain sense "small" above τ_e. The appropriate concept of "smallness" is in Definitions 22.4 and 22.6 below.

The remaining details of the proof will be presented below. □

We now develop some combinatorial machinery which is needed for the proof.

Notation 22.3. If ρ is a string and T is a finite set of strings, we write

$$[\rho, T] = \{\sigma \mid (\exists \tau \in T)\,(\rho \subseteq \sigma \subseteq \tau)\}.$$

Definition 22.4. Let ρ and T be as above, and let k be a positive integer. We say that T is *k-good above ρ* if $\rho \in [\rho, T]$ and for all $\sigma \in [\rho, T]$ either $\{i \mid \sigma^\frown\langle i\rangle \in [\rho, T]\} = \emptyset$ or $|\{i \mid \sigma^\frown\langle i\rangle \in [\rho, T]\}| > k$.

Remark 22.5. A set $[\rho, T]$ as in Definition 22.4 is sometimes called a *k-bushy tree*. The method of bushy trees is due to Kumabe.

Definition 22.6. Let C be a set of strings. If some finite set $T \subseteq C$ is k-good above ρ, we say that C is *k-large above ρ*. Otherwise we say that C is *k-small above ρ*.

Remark 22.7. Every superset of a set which is k-large above ρ is k-large above ρ. Every subset of a set which is k-small above ρ is k-small above ρ. Every set which contains ρ is k-large above ρ for all k.

Lemma 22.8. If C is k-small above ρ and D is k-large above ρ, then C is k-small above some $\tau \in D$.

Proof. Since D is k-large above ρ, let $T \subseteq D$ be k-good above ρ. We claim that C is k-small above some $\tau \in T$. Otherwise, for each $\tau \in T$ let $S_\tau \subseteq C$ be k-good above τ. Then clearly $\bigcup_{\tau \in T} S_\tau$ is k-good above ρ. Hence C is k-large above ρ. This contradiction proves the lemma. □

Lemma 22.9. If each of $C_1, \ldots, C_l$ is k-small above ρ, then $C_1 \cup \cdots \cup C_l$ is kl-small above ρ.

Proof. Suppose $C_1 \cup \cdots \cup C_l$ is kl-large above ρ. Let $T \subseteq C_1 \cup \cdots \cup C_l$ be kl-good above ρ. For each $\sigma \in [\rho, T]$ define $f(\sigma) \in \{1, \ldots, l\}$ by downward induction on the length of σ. If $\{i \mid \sigma^\frown\langle i\rangle \in [\rho, T]\} = \emptyset$ define $f(\sigma) =$ the least j such that $\sigma \in C_j$. If $|\{i \mid \sigma^\frown\langle i\rangle \in [\rho, T]\}| > kl$ define $f(\sigma) =$ the least j such that $|\{i \mid f(\sigma^\frown\langle i\rangle) = j\}| > k$. Finally, letting $j = f(\rho)$, it is clear that C_j is k-large above ρ. This contradiction completes the proof. □

Exercise 22.10. Prove that if C_j is k_j-small above ρ for each $j = 1, \ldots, l$, then $\bigcup_{j=1}^{l} C_j$ is $\sum_{j=1}^{l} k_j$-small above ρ.

We now return to the proof of Theorem 22.2.

Proof of Theorem 22.2. We continue to use the notations τ_e, F_e, A_e and $A_{i,n,j}$ as in our preliminary outline above. At stage e we shall define F_e and τ_e and a positive integer k_e such that A_e is k_e-small above τ_e.

Stage 0. Let $F_0 = \emptyset$ and $\tau_0 = \langle\rangle$ and $k_0 = 1$. Note that A_0 is k_0-small above τ_0 (because $A_0 = \emptyset$).

Stage $e+1$. Assume inductively that F_e and τ_e and k_e have been defined and that A_e is k_e-small above τ_e. Consider a partial recursive function $\theta_e(n) \simeq \theta(n)$ defined as follows. Given n, search for a $j < p(n)$ such that $A_{e,n,j}$ is k_e-large above τ_e. We can perform this search effectively, because $p(n)$ is a recursive function and the sets $A_{e,n,j}$ are uniformly Σ^0_1. If such a j is found, let $\theta(n) \simeq j$.

Otherwise let $\theta(n)\uparrow$. By the Recursion Theorem, let $n = n_e$ be such that $\varphi_n^{(1)}(n) \simeq \theta(n)$. With this n we consider two cases.

Case 1: $\theta(n)\downarrow$. Since A_e is k_e-small above τ_e and $A_{e,n,\theta(n)}$ is k_e-large above τ_e, apply Lemma 22.8 to find $\tau'_e \supseteq \tau_e$ such that $\tau'_e \in A_{e,n,\theta(n)}$ and A_e is k_e-small above τ'_e. Let $F_{e+1} = F_e$ and $k_{e+1} = k_e$ and note that $A_{e+1} = A_e$ is k_{e+1}-small above τ'_e. Since $\tau'_e \in A_{e,n,\theta(n)}$ we have $\varphi_e^{(1),\tau'_e}(n)\downarrow\simeq \theta(n) \simeq \varphi_n^{(1)}(n)$ thus satisfying the eth requirement via method 2.

Case 2: $\theta(n)\uparrow$. In this case let $F_{e+1} = F_e \cup \{\langle e,n\rangle\}$ and $k_{e+1} = k_e(p(n)+1)$. Since A_e and $A_{e,n,j}$ for each $j < p(n)$ are k_e-small above τ_e, it follows by Lemma 22.9 that

$$A_{e+1} \;=\; A_e \cup \bigcup_{j<p(n)} A_{e,n,j}$$

is k_{e+1}-small above τ_e. Let $\tau'_e = \tau_e$.

Note that in either case we have $\tau_e \subseteq \tau'_e$ and A_{e+1} is k_{e+1}-small above τ'_e. Let $T = T_e = \{\tau'_e{}^\frown\langle i\rangle \mid i \le k_{e+1}\}$. Clearly T is k_{e+1}-large above τ'_e, so by Lemma 22.8 let $\tau_{e+1} \in T$ be such that A_{e+1} is k_{e+1}-small above τ_{e+1}. Thus $\tau_e \subseteq \tau'_e \subset \tau_{e+1}$. Since this holds for all e we have $g = \bigcup_{e=0}^\infty \tau_e = \bigcup_{e=0}^\infty \tau'_e \in \mathbb{N}^\mathbb{N}$.

We claim that $\mathrm{DNR}_p \not\le_w \{g\}$. To see this, it suffices to show that the eth requirement $\varphi_e^{(1),g} \notin \mathrm{DNR}_p$ is satisfied for all e. Fix e and let $n = n_e$. If Case 1 holds at stage $e+1$, then we have already seen that the eth requirement is satisfied via method 2. If Case 2 holds at stage $e+1$, then $\langle e,n\rangle \in F_{e+1}$, hence by construction $\langle e,n\rangle \in F_i$ for all $i \ge e+1$, hence $\bigcup_{j<p(n)} A_{e,n,j} \subseteq A_i$ is k_i-small above τ_i, hence $\tau_i \notin \bigcup_{j<p(n)} A_{e,n,j}$. Thus $\varphi_e^{(1),g}(n)\uparrow$ or $\ge p(n)$ so the eth requirement is satisfied via method 1 or 3. This proves our claim.

To finish the proof of Theorem 22.2, it remains to prove that g is DNR. To this end, consider the set of "non-DNR strings," i.e.,

$$B \;=\; \{\tau \in \mathbb{N}^{<\mathbb{N}} \mid (\exists n < |\tau|)\,(\tau(n) \simeq \varphi_n^{(1)}(n))\}.$$

We claim that B is 1-small above $\langle\rangle$. To see this, let T be 1-good above $\langle\rangle$. We shall construct an increasing sequence of strings $\sigma_n \in [\langle\rangle, T] \setminus B$, $n = 0,1,2,\ldots$ with $|\sigma_n| = n$. Clearly $\langle\rangle \notin B$ so we can start with $\sigma_0 = \langle\rangle$. Given $\sigma_n \in [\langle\rangle, T] \setminus B$, if $\sigma_n \notin T$ there are at least two integers i such that $\sigma_n{}^\frown\langle i\rangle \in [\langle\rangle, T]$ so choose $i \not\simeq \varphi_n^{(1)}(n)$. Then clearly $\sigma_{n+1} = \sigma_n{}^\frown\langle i\rangle \notin B$. Eventually for some n we have $\sigma_n \in T \setminus B$, so $T \not\subseteq B$. This proves our claim.

Finally we are ready to prove that g is DNR. Consider the partial recursive functional $\Psi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ given by

$$\Psi(g)(n) \;\simeq\; \begin{cases} 0 & \text{if } g \notin \mathrm{DNR}, \\ \uparrow & \text{if } g \in \mathrm{DNR}. \end{cases}$$

We may safely assume that 0 is an index of Ψ. (Here we are deviating from the standard indexing of partial recursive functionals, but this deviation is harmless.) Thus for all n and j we have $A_{0,n,j} \subseteq B$. Recall that $F_0 = \emptyset$ and $\tau_0 = \langle\rangle$ and $k_0 = 1$. Since B is k_0-small above τ_0, so are $A_{0,n,j}$ for all n and j. Thus at stage 1 we have $\theta_0(n_0)\uparrow$, i.e., Case 2 holds. Hence the 0th requirement is satisfied via method 1 or 3. But then since $0 < p(n)$ we must have $\varphi_0^{(1),g}(n)\uparrow$, i.e., $g \in \mathrm{DNR}$. This completes the proof of Theorem 22.2. □

We now improve Theorem 22.2 as follows.

Definition 22.11. Let p and q be 1-place recursive functions. We say that q is *primitive recursive relative to* p if q belongs to the smallest class of functions which contains the initial functions and p and is closed under composition and primitive recursion.

Theorem 22.12. Given a total recursive function p, we can find a total recursive function q such that $\mathrm{DNR}_q <_w \mathrm{DNR}_p$. Moreover, we can take q to be primitive recursive relative to p.

Proof. Given a total recursive function q, define a string τ to be *q-bounded* if $\tau(m) < q(m)$ for all $m < |\tau|$. The idea of the proof is to let q be sufficiently fast-growing so that the construction for Theorem 22.2 can be carried out using q-bounded strings instead of arbitrary strings. Accordingly, we modify the definition of $A_{i,n,j}$ as follows:

$$A_{i,n,j} \;=\; \{\tau \mid \varphi_i^{(1),\tau}(n) \simeq j \wedge \tau \text{ is } q\text{-bounded}\}.$$

Apart from this change, the construction is exactly as before.

The only difficulty occurs in the transition from τ'_e to τ_{e+1}. In order to know that all of the strings in $T_e = \{\tau'_e{}^\frown\langle i\rangle \mid i \leq k_{e+1}\}$ are q-bounded, we need to define q so as to insure that $k_{e+1} < q(|\tau'_e|)$.

Given m, we wish to define $q(m)$. By the Recursion Theorem we may safely assume that we already know an index of the recursive function q. By the Uniform Recursion Theorem, we know that n_e (= the n chosen in stage $e+1$) is primitive recursive as a function of F_e, τ_e, k_e. From this it follows that F_{e+1} and k_{e+1} can be found primitive recursively from the finite sequences $\tau_0, \tau'_0, \tau_1, \tau'_1, \ldots, \tau_e, \tau'_e$ and $a_0, a_1, \ldots, a_e$ where

$$a_e \;=\; \begin{cases} 1 & \text{if } \theta_e(n_e)\downarrow \\ 2 & \text{if } \theta_e(n_e)\uparrow . \end{cases}$$

Namely we have $F_0 = \langle\rangle$, $k_0 = 1$, $F_{i+1} = F_i$ and $k_{i+1} = k_i$ if $a_i = 1$, $F_{i+1} = F_i \cup \{\langle i, n\rangle\}$ and $k_{i+1} = k_i(p(n_i)+1)$ if $a_i = 2$. Let us indicate this dependence as

$$k_{e+1} \;=\; f(\#(\langle \tau_0, \tau'_0, \ldots, \tau_e, \tau'_e\rangle), \#(\langle a_0, \ldots, a_e\rangle))$$

where f is a primitive recursive function. Then we let

$$q(m) \;=\; \max f(\#(\langle \tau_0, \tau'_0, \ldots, \tau_e, \tau'_e\rangle), \#(\langle a_0, \ldots, a_e\rangle))$$

where $\tau_0, \tau'_0, \ldots, \tau_e, \tau'_e$ and $a_0, \ldots, a_e$ range over all finite sequences such that

$$\tau_0 \subseteq \tau'_0 \subset \tau_1 \subseteq \tau'_1 \subset \cdots \subset \tau_e \subseteq \tau'_e$$

and $a_0, a_1, \ldots, a_e \in \{1, 2\}$ and τ'_e is q-bounded and $|\tau'_e| = m$. Clearly q is primitive recursive and the construction works. □

Definition 22.13. If C is a "nice" class of recursive functions, we define $\mathbf{d}_C = \deg_w(\mathrm{DNR}_C)$, where

$$\mathrm{DNR}_C \;=\; \bigcup_{p\in C} \mathrm{DNR}_p = \{C\text{-bounded DNR functions}\}.$$

Note that Corollary 17.2 implies that DNR_C is Σ^0_3, hence $\inf(\mathbf{d}_C, \mathbf{1}) \in \mathcal{E}_w$.

Examples 22.14. The following classes are considered "nice".

$C = \{\text{primitive recursive functions}\}$,

$C = \mathrm{ER} = \{\text{elementary recursive functions}\}$,

$C = \mathrm{PTIME} = \{\text{polynomial-time computable functions}\}$,

$C = \mathrm{EXPTIME} = \{\text{exponential-time computable functions}\}$,

and other computational complexity classes as defined in resource-bounded computational complexity. Also, for each ordinal $\alpha \le \epsilon_0$ there is a "nice" class of recursive functions C_α consisting of the functions at level $\le \alpha$ of the Wainer hierarchy. In particular C_0 is the class of primitive recursive functions, C_1 is the class of functions which are primitive recursive in the Ackermann function, and C_{ϵ_0} is the class of recursive functions which are provably total in PA.

Corollary 22.15. Given a "nice" class of recursive functions C, we can find a recursive function q such that $\mathrm{DNR}_q <_w \mathrm{DNR}_C$.

Proof. We have not defined precisely what we mean by saying that a class of recursive functions $C \subseteq \mathrm{REC}$ is "nice". However, one of our requirements for niceness is that the class C should be *recursively enumerable*. This means that there is an enumeration $C = \{p_i \mid i \in \mathbb{N}\}$ such that $p_i(n)$ is recursive as a function of i and n.

If C is "nice" in this sense, consider the recursive function

$$p(n) = \max(2, p_0(0), p_1(1), \ldots, p_n(n)).$$

We claim that $\mathrm{DNR}_p \le_w \mathrm{DNR}_C$. To see this, consider a C-bounded DNR function $g \in \mathrm{DNR}_C$. Let i be such that $g(n) < p_i(n)$ for all n. Then $g(n) < p(n)$ for all $n \ge i$. Since $p(n) \ge 2$ for all n, we can easily find a p-bounded DNR function f such that $f(n) = g(n)$ for all $n \ge i$. Then $f \le_T g$ and our claim is proved. Now apply Theorem 22.12 to obtain a recursive function q such that $\mathrm{DNR}_q <_w \mathrm{DNR}_p$. Then $\mathrm{DNR}_q <_w \mathrm{DNR}_C$ and we have our corollary. □

Remark 22.16. One can consider the following research problems.

1. Refine Theorem 22.12 to calculate better estimates for q given p. The idea here is that, as n goes to infinity, the growth rate of $q(n)$ should be only slightly faster than the growth rate of $p(n)$.

2. Redo the proofs and calculations for Theorem 22.12 and Corollary 22.15 in terms of Kolmogorov complexity. The close relationship between recursively bounded DNR functions and Kolmogorov complexity has been noted earlier, in Theorem 20.10 and Remark 20.16.

We now end this section by improving Theorem 22.2 in another direction.

Theorem 22.17. $\mathrm{DNR} <_w \mathrm{DNR}_{\mathrm{REC}}$.

Proof. It suffices to construct a DNR function g such that $\mathrm{DNR}_{\mathrm{REC}} \not\leq_w \{g\}$. Let p_s, e_s, $s = 0, 1, 2, \ldots$ be a (non-uniform) enumeration of $\mathrm{REC} \times \mathbb{N}$, i.e., all ordered pairs p, e such that p is a total recursive function and $e \in \mathbb{N}$. We proceed as in the proof of Theorem 22.2 by defining $\tau_0 \subset \tau_1 \subset \cdots \tau_s \subset \tau_{s+1} \subset \cdots$ and $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_s \subseteq F_{s+1} \subseteq \cdots$ and $k_0 \leq k_1 \leq \cdots \leq k_s \leq k_{s+1} \leq \cdots$ and $g = \bigcup_{s=0}^{\infty} \tau_s$. At stage $s+1$ we address the sth requirement, $\varphi_{e_s}^{(1),g} \notin \mathrm{DNR}_{p_s}$. Associated with F_s is the set

$$A_s = \bigcup_{\substack{\langle r,n\rangle \in F_s \\ j < p_r(n)}} A_{e_r,n,j} \qquad \text{where } A_{e,n,j} = \{\tau \mid \varphi_e^{(1),\tau}(n) \simeq j\}.$$

Our inductive hypothesis says that A_s is k_s-small above τ_s. The remaining details are as in the proof of Theorem 22.2. □

Remark 22.18. Note that the inequalities in (4) on page 46 follow immediately from Theorems 18.12 and 22.2 and 22.17 and Corollary 22.15.

23 Bounded limit-recursiveness

Definition 23.1. A function $g : \mathbb{N} \to \mathbb{N}$ is said to be *limit-recursive* if there exists a recursive function $\widetilde{g} : \mathbb{N}^2 \to \mathbb{N}$ such that $g(n) = \lim_s \widetilde{g}(n, s)$ for all n.

Remark 23.2. Here $\widetilde{g}$ is called an *approximating function*. Think of $\widetilde{g}(n, s)$, $s = 0, 1, 2, \ldots$ as a sequence of guesses for the true value of $g(n)$ such that eventually these guesses are correct.

Theorem 23.3 (The Limit Lemma)**.** $g \leq_T 0' \Leftrightarrow g$ is limit-recursive.

Proof. $\Rightarrow$: Consider the predicate $P(n, t) \equiv (\forall s \geq t)\,(\widetilde{g}(n, s) = \widetilde{g}(n, s+1))$. Let $f(n) =$ the least t such that $P(n, t)$ holds. Clearly $f \leq_T P \leq_T 0'$. But $g(n) = \widetilde{g}(n, f(n))$, hence $g \leq_T 0'$, Q.E.D.

($\Leftarrow$) Assume $g \leq 0'$, say $g(n) = \varphi_e^{(1),0'}(n)$ for all n. Define $\widetilde{g}(n, s) = \varphi_{e,s}^{(1),0'_s \upharpoonright s}(n)$ if this is defined, and 0 otherwise, where

$$0'_s = H_s = \{i \mid \varphi_{i,s}^{(1)}(0)\downarrow\}.$$

Thus $\widetilde{g}(n, s)$ is recursive, and clearly $g(n) = \lim_s \widetilde{g}(n, s)$. □

By relativizing we obtain the following.

Theorem 23.4. For any oracle X we have $g \leq_T X' \Leftrightarrow g$ is *limit-recursive in* X, i.e., $g(n) = \lim \widetilde{g}(n,s)$ where $\widetilde{g}$ is X-recursive.

We now consider a variant of limit-recursiveness in X.

Definition 23.5. We say that g is *boundedly limit-recursive in* X if there exists an X-recursive function $\widetilde{g}(n,s)$ and a recursive function $\widehat{g}(n)$ such that $g(n) = \lim \widetilde{g}(n,s)$ and $|\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}| < \widehat{g}(n)$ for all n. Here $\widehat{g}$ is called a *bounding function.*

Remark 23.6. The previous definition is not a straightforward relativization to X. This is because the bounding function $\widehat{g}$ is assumed to be recursive (not only X-recursive).

Definition 23.7. We write

$$\mathrm{BLR}(X) = \{g \in \mathbb{N}^{\mathbb{N}} \mid g \text{ is boundedly limit-recursive in } X\}.$$

Exercise 23.8. For $A \subseteq \mathbb{N}$ prove that $\chi_A \in \mathrm{BLR}(X)$ with a constant bounding function $\Leftrightarrow A$ is a Boolean combination of sets which are $\Sigma^{0,X}_1$.

Remark 23.9. Note that $g \leq_T X$ implies $g \in \mathrm{BLR}(X)$ which implies that g is limit-recursive relative to X. The converses do not hold. By the Limit Lemma, we may rewrite this as

$$\mathrm{REC}(X) \subseteq \mathrm{BLR}(X) \subseteq \mathrm{REC}(X').$$

The converse inclusions do not hold.

The following lemma makes this relationship more precise.

Lemma 23.10. $g \in \mathrm{BLR}(X) \Leftrightarrow g \leq_T X'$ with recursively bounded use of X' and unbounded use of X.

Proof. $\Leftarrow$: Assume $g \leq_T X'$ with recursively bounded use of X' and (possibly) unbounded use of X. In other words, $g(n) = \varphi^{(2),X}_e(X' \upharpoonright b(n), n)$, where $b(n)$ is a total recursive function. Let $\widetilde{g}(n,s) = \varphi^{(2),X\upharpoonright s}_{e,s}(X'_s \upharpoonright b(n), n)$ if $\downarrow$ and 0 otherwise. Clearly $\widetilde{g} \leq_T X$ and $g(n) = \lim \widetilde{g}(n,s)$. Moreover $|\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}| \leq 2b(n)+1$. So $g \in \mathrm{BLR}(X)$ via $\widehat{g}(n) = 2b(n)+2$.

$\Rightarrow$: Assume $g \in \mathrm{BLR}(X)$ via $\widetilde{g}$ and $\widehat{g}$. We describe a method of computing $g(n)$ using an oracle for X'. For each $i < \widehat{g}(n)$ ask the X'-oracle whether $|\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}| > i$. From the answers to these questions, we can immediately read off

$$k(n) = |\{s \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}|$$

and then $g(n) = \widetilde{g}(n, t(n))$ where $t(n) =$ the least t such that $|\{s < t \mid \widetilde{g}(n,s) \neq \widetilde{g}(n,s+1)\}| = k(n)$. Note that we have used X' only to compute $k(n)$, and thus our use of X' was bounded by $b(n) = \max\{\alpha(n,i) \mid i < \widehat{g}(n))\}$ where $\alpha(n,i)$ is a fixed primitive recursive function. Since $\widehat{g}$ is recursive, so is $b(n)$. $\square$

Remark 23.11. We shall be interested in the binary relation of BLR-*reducibility* defined by $X \leq_{\mathrm{BLR}} Y$ if and only if $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$. Obviously $X \leq_T Y$ implies $X \leq_{\mathrm{BLR}} Y$ which implies $X' \leq_T Y'$. The converse implications do not hold. We need the following technical lemma concerning BLR-reducibility.

Lemma 23.12. The following are pairwise equivalent.

1. $\mathrm{BLR}(X) \subseteq \mathrm{BLR}(Y)$.
2. Given a partial X-recursive function $\psi(n)$, we have $g \in \mathrm{BLR}(Y)$ where
$$g(n) = \begin{cases} \psi(n)+1 & \text{if } \psi(n)\downarrow, \\ 0 & \text{otherwise.} \end{cases}$$
3. We can find $h \in \mathrm{BLR}(Y)$ such that $\forall n\,(\varphi_n^{(1),X}(n)\downarrow \Rightarrow h(n) = \varphi_n^{(1),X}(n))$.

Proof. $1 \Rightarrow 2$ is easy. Let e be an index for ψ, i.e., $\psi(n) \simeq \varphi_e^{(1),X}(n)$ for all n. Let
$$\widetilde{g}(n,s) = \begin{cases} \varphi_{e,s}^{(1),X\restriction s}(n)+1 & \text{if } \downarrow, \\ 0 & \text{if } \uparrow, \end{cases}$$
and let $\widehat{g}(n) = 2$ for all n. Clearly $g \in \mathrm{BLR}(X)$ via $\widetilde{g}$ and $\widehat{g}$. Hence $g \in \mathrm{BLR}(Y)$ in view of 1.

$2 \Rightarrow 3$ is trivial. Namely, apply 2 to the function $\psi(n) \simeq \varphi_n^{(1),X}(n)$ to get $g(n)$, and then let $h(n) = g(n) \dot{-} 1$.

$3 \Rightarrow 2$: As before, let e be an index of ψ. Let $\theta(n) \simeq$ the least $\sigma \subset X$ such that $\varphi_{e,|\sigma|}^{(1),\sigma}(n)\downarrow$. Clearly $\theta(n)$ is partial X-recursive. Let h be as in 3. By the Parametrization Theorem let $p(n)$ be primitive recursive such that $\theta(n) \simeq \varphi_{p(n)}^{(1),X}(p(n))$. Let
$$g(n) \simeq \begin{cases} \varphi_{e,|h(p(n))|}^{(1),h(p(n))}(n)+1 & \text{if } \downarrow \\ 0 & \text{otherwise} \end{cases}$$
Then g satisfies 2.

$2 \Rightarrow 1$: Given $f \in \mathrm{BLR}(X)$, to show $f \in \mathrm{BLR}(Y)$. For all n and all $i \leq \widehat{f}(n)+1$, let $\psi_i(n) \simeq$ the ith successive value of $\widetilde{f}(n,s)$ as $s \to \infty$. The functions $\psi_i(n)$ are uniformly partial X-recursive. By 2 the functions
$$g_i(n) = \begin{cases} \psi_i(n)+1 & \text{if } \psi_i(n)\downarrow, \\ 0 & \text{otherwise} \end{cases}$$
are uniformly $\mathrm{BLR}(Y)$. Let $\widehat{g}_i(n)$ be the uniformly recursive bounding functions for $g_i(n)$. Let $g(n) = g_i(n) - 1$ for the least i such that $g_{i+1}(n) = 0$. Clearly $f = g \in \mathrm{BLR}(Y)$ via the recursive bounding function
$$\widehat{g}(n) = \sum_{i<\widehat{f}(n)} \widehat{g}_i(n).$$
This completes the proof. □

Theorem 23.13. The binary relation $\leq_{\mathrm{BLR}}$ is Σ^0_3.

Proof. By Lemma 23.12 we have

$$X \leq_{\mathrm{BLR}} Y \equiv (\exists h \in \mathrm{BLR}(Y))\, \forall n\, (\varphi^{(1),X}_n(n) \downarrow \Rightarrow h(n) = \varphi^{(1),X}_n(n)).$$

Stating this in terms of $\widetilde{h}$ and $\widehat{h}$ we have

$$\begin{aligned} &\forall n\, (|\{s \mid \widetilde{h}(n,s) \neq \widetilde{h}(n,s+1)\}| < \widehat{h}(n) \\ &\wedge (\varphi^{(1),X}_n(n) \downarrow \Rightarrow \forall s\, (\exists t > s)\, \widetilde{h}(n,t) = \varphi^{(1),X}_n(n))). \end{aligned}$$

In terms of indices for $\widetilde{h}$ and $\widehat{h}$ this becomes

$$\begin{aligned} &\exists \widetilde{e}\, \exists \widehat{e}\, \forall n\, (\varphi^{(1)}_{\widehat{e}}(n) \downarrow \wedge \forall s\, \varphi^{(2),Y}_{\widetilde{e}}(n,s) \downarrow \\ &\wedge \forall F\, (\forall s \in F)\, (\varphi^{(2),Y}_{\widetilde{e}}(n,s) \neq \varphi^{(2),Y}_{\widetilde{e}}(n,s+1) \Rightarrow |F| < \varphi^{(1)}_{\widehat{e}}(n)) \\ &\wedge (\varphi^{(1),X}_n(n) \downarrow \Rightarrow \forall s\, (\exists t > s)\, (\varphi^{(2),Y}_{\widetilde{e}}(n,t) \downarrow = \varphi^{(1),X}_n(n))) \end{aligned}$$

and a Tarski/Kuratowski computation shows that this is Σ^0_3. □

Definition 23.14. Let S be a set of oracles. We define

$$S^* = \{Y \mid \exists X\, (X \in S \wedge X \leq_{\mathrm{BLR}} Y)\} = \text{the BLR-upward closure of } S.$$

Theorem 23.15. If S is Σ^0_3, then so is S^*.

Proof. By Theorem 5.7 we may assume that $S = P$ is a Π^0_1 subset of $\mathbb{N}^{\mathbb{N}}$. We have $Y \in P^* \equiv (\exists f \in P)\, (\exists g \in \mathrm{BLR}(Y))\, \forall i\, (g(2i) = f(i) \wedge \forall n\, (\varphi^{(1),f}_n(n) \downarrow\Rightarrow g(2n+1) = \varphi^{(1),f}_n(n)))$. The idea here is that $g = f \oplus h$ where h is a total function extending $\varphi^{(1),f}_n(n)$ as in the previous lemma. A Tarski/Kuratowski computation shows that P^* is Σ^0_3. □

Remark 23.16. Clearly $S_1 \leq_w S_2$ implies $S^*_1 \leq_w S^*_2$. This is trivial, because $\leq_T$ implies $\leq_{\mathrm{BLR}}$.

Definition 23.17. If $\mathbf{s} = \deg_w(S)$ let $\mathbf{s}^* = \deg_w(S^*)$.

Remark 23.18. The previous remark shows that $\mathbf{s}^*$ is well defined. Moreover $\mathbf{s}_1 \leq \mathbf{s}_2$ implies $\mathbf{s}^*_1 \leq \mathbf{s}^*_2$. We also have $\mathbf{s}^{**} = \mathbf{s}^* \leq \mathbf{s}$ and $\inf(\mathbf{s}_1, \mathbf{s}_2)^* = \inf(\mathbf{s}^*_1, \mathbf{s}^*_2)$.

Remark 23.19. From Theorem 23.15 plus the Embedding Lemma, we know that $\mathbf{s} \in \mathcal{E}_w$ implies $\mathbf{s}^* \in \mathcal{E}_w$. As a research problem, it would be interesting to investigate degrees in $\mathcal{E}_w$ such as $\mathbf{1}^*$, $\mathbf{r}^*_1$, $\inf(\mathbf{r}_2, \mathbf{1})^*$, $\mathbf{d}^*_{\mathrm{REC}}$, $\mathbf{d}^*$, etc. It can be shown that $\mathbf{d}^* \neq \mathbf{0}$, hence none of the mentioned degrees is $\mathbf{0}$.

24 Embedding hyperarithmeticity into $\mathcal{E}_w$

Definition 24.1. For each $\alpha < \omega_1^{\mathrm{CK}}$ let $0^{(\alpha)}$ = the αth Turing jump of 0. This is the hyperarithmetical hierarchy. There is a theorem of Spector saying that $0^{(\alpha)}$ is well defined up to Turing degree. In particular we have the initial segment $0, 0', 0'', \ldots, 0^{(n)}, 0^{(n+1)}, \ldots$ for $n \in \mathbb{N}$. By Post's Theorem, a set $A \subseteq \mathbb{N}$ is Δ^0_{n+1} if and only if it is $\leq_T 0^{(n)}$, so this is equivalent to the arithmetical hierarchy. In fact, A is Σ^0_{n+1} if and only if A is $\Sigma_1^{0,0^{(n)}}$.

Lemma 24.2. $0^{(\alpha)}$ is a Π^0_2 singleton.

Proof. We omit the proof, but see Theorem 10.6. □

Remark 24.3. It follows that $\{0^{(\alpha)}\}$ is Σ^0_3. Let $\mathbf{h}_\alpha = \deg_w(\{0^{(\alpha)}\})$. Then $\mathbf{h}^*_\alpha = \deg_w(\{X \mid 0^{(\alpha)} \leq_{\mathrm{BLR}} X\})$. By Theorem 23.15 plus the Embedding Lemma we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) \in \mathcal{E}_w$. We shall prove that $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) < \inf(\mathbf{h}^*_\beta, \mathbf{1})$ for all $\alpha < \beta < \omega_1^{\mathrm{CK}}$. Thus we shall have an embedding of the hyperarithmetical hierarchy into $\mathcal{E}_w$.

Lemma 24.4. Given a Turing oracle X, we can find a 1-generic $g \in \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{BLR}(X) = \mathrm{BLR}(g)$, i.e., $X \equiv_{\mathrm{BLR}} g$.

Proof. This is similar to the proof of the Friedberg Jump Theorem. Recall that g is 1-generic if for each n either $\varphi_n^{(1),g}(0)\downarrow$ or $(\exists \sigma \subset g)\,(\forall \tau \supseteq \sigma)\,\varphi_n^{(1),\tau}(0)\uparrow$.

Given X, define $h(n) = \varphi_n^{(1),X}(n) + 1$ if $\varphi_n^{(1),X}(n)\downarrow$, 0 if $\varphi_n^{(1),X}(n)\uparrow$. Recall that $X \leq_{\mathrm{BLR}} Y \Leftrightarrow h \in \mathrm{BLR}(Y)$. We shall obtain g as $g = \bigcup_{n=0}^\infty \tau_n$ where $\tau_0 \subset \tau_1 \subset \tau_2 \subset \cdots \subset \tau_n \subset \tau_{n+1} \subset \cdots$.

Stage 0: Let $\tau_0 = \langle\rangle$.

Stage $n+1$: Let $\tau'_n = \tau_n{}^\frown\langle h(n)\rangle$. Ask the oracle $0'$ whether $\exists \tau \supseteq \tau'_n$ such that $\varphi^{(1),\tau}_{n,|\tau|}(n)\downarrow$. If so, let τ_{n+1} = the least such τ. Otherwise let $\tau_{n+1} = \tau'_n$.

By construction $g = \bigcup_{n=1}^\infty \tau_n$ is 1-generic. Moreover, the entire construction (i.e., the sequence of strings τ_n, $n = 0, 1, 2, \ldots$) is boundedly limit-recursive in each of the oracles X and g, with bounding function 4^n. Because $h(n) = \tau_{n+1}(|\tau_n|) = g(|\tau_n|)$ and $\varphi_n^{(1),g}(n) \simeq \varphi^{(1),\tau_{n+1}}_{n,|\tau_{n+1}|}(n)$, it follows that $\mathrm{BLR}(X) = \mathrm{BLR}(g)$, Q.E.D. □

Theorem 24.5. For each $\alpha < \omega_1^{\mathrm{CK}}$ we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) \in \mathcal{E}_w$. Moreover, for all $\alpha < \beta < \omega_1^{\mathrm{CK}}$ we have $\inf(\mathbf{h}^*_\alpha, \mathbf{1}) < \inf(\mathbf{h}^*_\beta, \mathbf{1})$, in fact $\inf(\mathbf{h}^*_\alpha, \mathbf{d}) < \inf(\mathbf{h}^*_\beta, \mathbf{d})$.

Proof. Apply the previous lemma to get a 1-generic g such that $0^{(\alpha)} \equiv_{\mathrm{BLR}} g$. Since $0^{(\beta)} \not\leq_{\mathrm{BLR}} 0^{(\alpha)}$ we have $0^{(\beta)} \not\leq_{\mathrm{BLR}} g$, i.e., $\{X \mid 0^{(\beta)} \leq_{\mathrm{BLR}} X\} \not\leq_w \{g\}$. By Lemma 19.9 we also have $\mathrm{DNR} \not\leq_w \{g\}$. Thus $\inf(\mathbf{h}^*_\beta, \mathbf{d}) \not\leq \mathbf{h}^*_\alpha$. It follows that $\inf(\mathbf{h}^*_\alpha, \mathbf{d}) < \inf(\mathbf{h}^*_\beta, \mathbf{d})$, Q.E.D. □