Muchnik Degrees: Results and Techniques

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Abstract

Let P and Q be sets of reals. P is said to be Muchnik reducible to Q if every member of Q Turing-computes a member of P. A Muchnik degree is an equivalence class of sets of reals under mutual Muchnik reducibility. It is easy to see that the Muchnik degrees form a distributive lattice under the partial ordering induced by Muchnik reducibility. Call this lattice L. We study not only L but also its countable distributive sublattice L_0 consisting of the Muchnik degrees of nonempty Π^0_1 subsets of the closed unit interval $[0,1]$. We present a variety of results and techniques which have been useful in recent investigations.

1 Introduction

Remark 1.1. Insofar as we are interested in the lattice of all Muchnik degrees, it doesn't matter whether we deal with subsets of the real line $\mathbb{R}$, or of the Baire space ω^ω, or the Cantor space 2^ω. However, when we turn to Π^0_1 sets, there is a difference, because 2^ω is compact while ω^ω is not.

Our main focus here is the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω. For technical reasons, we shall also discuss Π^0_1 subsets of ω^ω.

Definition 1.2. For $P, Q \subseteq \omega^\omega$, P is said to be Muchnik reducible to Q if for every $g \in Q$ there exists $f \in P$ such that $f \leq_T g$. (Note that $\leq_T$ stands for Turing reducibility.) This is abbreviated $P \leq_w Q$, where the w stands for "weak reducibility". In this terminology, "strong reducibility" is Medvedev reducibility, given by $P \leq_s Q$ if there exists a partial recursive functional $\Phi : Q \to P$, i.e., the domain of Φ includes Q, and for all $g \in Q$ we have $\Phi(g) \in P$.

Remark 1.3. Thus Medvedev ("strong") reducibility is the uniform variant of Muchnik ("weak") reducibility. Later we shall see an analogy

Muchnik reducibility / Medvedev reducibility =

Turing reducibility / truth-table reducibility.

Remark 1.4. The least upper bound operation for Muchnik or Medvedev reducibility is given by

$$P \times Q = \{f \oplus g \mid f \in P, g \in Q\}.$$

The greatest lower bound operation for Muchnik reducibility is given by $P \cup Q$. The greatest lower bound operation for Muchnik or Medvedev reducibility is given by

$$P + Q = \{\langle 0 \rangle^\frown f \mid f \in P\} \cup \{\langle 1 \rangle^\frown g \mid g \in Q\}.$$

It is easy to see that the Muchnik degrees form a distributive lattice under these operations. Similarly for the Medvedev degrees.

Remark 1.5. The smallest Muchnik degree is $\mathbf{0}$, the Muchnik degree of ω^ω, or equivalently the Muchnik degree of any $P \subseteq \omega^\omega$ containing a recursive element. Furthermore, 0 is meet irreducible, i.e., $\mathbf{0}$ is not the greatest lower bound of two nonzero Muchnik degrees. Similarly for Medvedev degrees.

Definition 1.6. A set $P \subseteq \omega^\omega$ is said to be Π^0_1 if there exists a recursive predicate $R \subseteq \omega \times \omega^\omega$ such that $P = \{f \in \omega^\omega \mid \forall n \, R(n, f)\}$.

Equivalently, $P \subseteq \omega^\omega$ is Π^0_1 if and only if

$$P = \{f \in \omega^\omega \mid f \text{ is a path through } T\}$$

for some recursive tree $T \subseteq \omega^{<\omega}$.

Example 1.7. An interesting example of a Π^0_1 subset of ω^ω of nonzero Muchnik degree is

$$\mathrm{DNR} = \{f \in \omega^\omega \mid \forall n \, f(n) \neq \varphi^{(1)}_n(n)\},$$

i.e., the set of $f : \omega \to \omega$ which are diagonally nonrecursive.

Definition 1.8. A set $P \subseteq \omega^\omega$ is said to be recursively bounded if there exists a recursive function $g \in \omega^\omega$ such that for all $f \in P$, $f(n) < g(n)$ for all n. Note that any $P \subseteq 2^\omega$ is recursively bounded.

The next lemma shows that the study of Medvedev and Muchnik degrees of recursively bounded Π^0_1 subsets of ω^ω is equivalent to the study of Π^0_1 subsets of 2^ω.

Definition 1.9. $P, Q \subseteq \omega^\omega$ are recursively homeomorphic if there exist partial recursive functionals $\Phi : P \to Q$ and $\Phi^{-1} : Q \to P$.

Lemma 1.10. *For any recursively bounded Π^0_1 set $P \subseteq \omega^\omega$, there exists a Π^0_1 set $P^* \subseteq 2^\omega$ such that P is recursively homeomorphic to P^*.*

Remark 1.11. If P, Q are Π^0_1 then $P \times Q$, $P \cup Q$, $P + Q$ are Π^0_1. Also, if $P, Q \subseteq 2^\omega$, then $P \times Q, P \cup Q, P + Q \subseteq 2^\omega$. Thus the Muchnik degrees of nonempty Π^0_1 subsets of 2^ω form a countable distributive sublattice of the lattice of all Muchnik degrees. Similarly for Medvedev degrees.

Definition 1.12. A nonempty Π^0_1 set $P \subseteq 2^\omega$ is said to be Muchnik complete if every nonempty Π^0_1 subset of 2^ω is Muchnik reducible to P. Medvedev completeness is defined similarly.

Remark 1.13. We use **1** to denote the degree of a Muchnik degree of a nonempty Π^0_1 subset of 2^ω which is Muchnik complete. Similarly for Medvedev degrees.

Example 1.14. The following Π^0_1 subsets of 2^ω are known to be Medvedev complete, hence Muchnik complete.

1. $P = \{\text{completions of } \mathsf{PA}\}$. Instead of PA we could use any effectively axiomatizable, effectively essentially undecidable theory. This is related to the Gödel/Rosser Theorem.

2. $P = \{f \in 2^\omega \mid f \text{ separates } A \text{ and } B\}$, where $A = \{n \mid \varphi_n^{(1)}(n) \simeq 0\}$ and $B = \{n \mid \varphi_n^{(1)}(n) \simeq 1\}$.

3. We can also give an explicit, recursion-theoretic construction of a Π^0_1 set P with the desired property. Roughly, $P = \prod_{n=0}^{\infty} P_n$ where P_n is the nth nonempty Π^0_1 subset of 2^ω.

Theorem 1.15 (Simpson 2000). *Any two Medvedev complete Π^0_1 subsets of 2^ω are recursively homeomorphic.*

Proof. The proof is by an effective back-and-forth argument, using the Recursion Theorem. It it similar to the proof of Myhill's result that any two creative, recursively enumerable subsets of ω are recursively isomorphic. □

The following example shows that Medvedev completeness is not the same as Muchnik completeness.

Example 1.16 (Jockusch 1989). For $k \geq 2$ let DNR_k be the set of functions $f : \omega \to \{1, \ldots, k\}$ which are DNR. It is easy to see that the sets DNR_k, $k = 2, 3, \ldots$, are Π^0_1 and recursively bounded, and that DNR_2 is Medvedev complete. Jockusch has shown that the the sets DNR_k, $k = 2, 3, \ldots$ are Muchnik complete but of different Medvedev degrees. Thus we have $\text{DNR}_2 \equiv_w \text{DNR}_3 \equiv_w \ldots$ yet $\text{DNR}_2 >_s \text{DNR}_3 >_s \ldots$.

An interesting relationship between Muchnik ("strong") and Medvedev ("weak") reducibility is given by the following theorem.

Theorem 1.17 (Simpson 2001). *Let $P, Q \subseteq 2^\omega$ be nonempty Π^0_1 sets. If $P \leq_w Q$, then there exists a nonempty Π^0_1 set $Q' \subseteq Q$ such that $P \leq_s Q'$.*

Proof. We shall prove this later, as a consequence of the Almost Recursive Basis Theorem. □

Corollary 1.18. *If $Q \subseteq 2^\omega$ is Π^0_1 and Muchnik complete, then there is a Π^0_1 set $Q' \subseteq Q$ such that Q' is Medvedev complete.*

Definition 1.19. $P, Q \subseteq \omega^\omega$ are said to be Turing degree isomorphic if there exists a Turing-degree-preserving one-to-one correspondence between P and Q. Clearly recursive homeomorphism implies Turing degree isomorphism.

Theorem 1.20 (Simpson 2001). *Any two Muchnik complete Π^0_1 subsets of 2^ω are Turing degree isomorphic.*

Proof. This follows easily from Theorem 1.15 and Corollary 1.18. □

Corollary 1.21. *A nonempty Π^0_1 subset of 2^ω is Muchnik complete if and only if it is Turing degree isomorphic to the set of completions of* PA*.*

Corollary 1.22. *Any two nonempty Π^0_1 subsets of $\bigcup_{k=0}^\infty \mathrm{DNR}_k$ are Turing degree isomorphic.*

Corollary 1.23. *If P is Muchnik complete, then the set of Turing degrees of members of P is upward closed.*

Proof. Let P be Muchnik complete. Put $Q = P \times 2^\omega$. Clearly Q is Muchnik complete, and the set of Turing degrees of members of Q is upward closed. By Theorem 1.20, P and Q are Turing degree isomorphic. □

Corollary 1.24 (Solovay). *The set of Turing degrees of completions of* PA *is upward closed.*

2 1-Random Reals

We use the "fair coin" measure on 2^ω. Thus $\mu(\{X \in 2^\omega : X(n) = 1\}) = 1/2$ for all $n \in \omega$.

Definition 2.1. An effective null G_δ is a set $S \subseteq 2^\omega$ of the form $S = \bigcap_{n=0}^\infty U_n$ where U_n, $n \in \omega$, is a recursive sequence of Σ^0_1 sets such that $\mu(U_n) < 1/2^n$ for all n.

Definition 2.2. $X \in 2^\omega$ is 1-random if $X \notin S$ for all effective null G_δ sets S. The set of 1-random reals is denoted R_1. Clearly $\mu(R_1) = 1$.

Theorem 2.3 (Martin-Löf 1966). *The union of all effective null G_δ sets is an effective null G_δ set.*

Proof. The proof is by a diagonal argument. □

Corollary 2.4. *$2^\omega \setminus R_1$ is an effective null G_δ set. Hence R_1 is Σ^0_2.*

Corollary 2.5. *$R_1 = \bigcup_{n=0}^\infty P_n$ where P_n, $n \in \omega$, is a sequence of Π^0_1 sets.*

Theorem 2.6. *Let $Q \subseteq 2^\omega$ be Π^0_1 of measure 0. Then Q is an effective null G_δ set.*

Proof. Straightforward. □

Corollary 2.7. *Let $Q \subseteq 2^\omega$ be Π^0_1. We have $\mu(Q) > 0$ if and only if $Q \cap R_1 \neq \emptyset$. In this case we actually have $Q \cap R_1 \supseteq P \neq \emptyset$, where P is Π^0_1 and $\mu(P) > 0$.*

Theorem 2.8 (Kučera 1985). *Let $Q \subseteq 2^\omega$ be Π^0_1 with $\mu(Q) > 0$. Then for all 1-random $X \in 2^\omega$ we have that $X^{(k)} \in Q$ for some k. Here $X^{(k)}(n) = X(k+n)$ for all n.*

Proof. Define Q^n for $n = 1, 2, 3, \ldots$. Then $2^\omega \setminus \bigcup_{n=1}^\infty Q^n$ is an effective null G_δ set. Hence $X \in Q^n$ for some n. It follows that $X^{(k)} \in Q$ for some k. □

Corollary 2.9. *Let $Q \subseteq 2^\omega$ be Π^0_1 with $\mu(Q) > 0$. Then $Q \leq_w R_1$.*

Corollary 2.10. *Let Q be a nonempty Π^0_1 subset of R_1. Then $Q \equiv_w R_1$.*

Corollary 2.11. *Among all Muchnik degrees of Π^0_1 sets $Q \subseteq 2^\omega$ with $\mu(Q) > 0$, there is a largest one, namely the Muchnik degree of R_1. Call this Muchnik degree $\mathbf{r}_1$.*

Theorem 2.12 (Jockusch/Soare 1972). *Let $A, B \subseteq \omega$ be recursively inseparable. Then $\mu(\{X \in 2^\omega : \exists Y \leq_T X \,(Y \text{ separates } A, B)\}) = 0$.*

Proof. Not difficult. □

Corollary 2.13. *The Muchnik degree $\mathbf{r}_1 = \deg_w(R_1)$ of Corollary 2.11 is not Muchnik complete. We have $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$.*

3 The Almost Recursive Basis Theorem

Definition 3.1. X is almost recursive (a.k.a., hyperimmune-free) if for all functions $f : \omega \to \omega$ recursive in X, there exists a recursive function $g : \omega \to \omega$ such that $f(n) < g(n)$ for all n.

The following is the Almost Recursive Basis Theorem.

Theorem 3.2 (Jockusch/Soare 1972). *Let $P \subseteq 2^\omega$ be Π^0_1 and nonempty. Then there exists $X \in P$ such that X is almost recursive.*

Proof. Let $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n \supseteq \cdots$ be a generic sequence of nonempty Π^0_1 sets. It can be shown that the unique $X \in \bigcap_{n=0}^\infty P_n$ is almost recursive. □

Corollary 3.3. *There exists a completion of* PA *which is almost recursive.*

Corollary 3.4. *There exists a 1-random $X \in 2^\omega$ which is almost recursive.*

Lemma 3.5. *Suppose X is almost recursive and $X \geq_T Y$. Then Y is truth-table reducible to X. In particular, there exists a total recursive functional $\Phi : 2^\omega \to 2^\omega$ such that $\Phi(X) = Y$.*

Proof. Let e be such that $Y = \{e\}^X$. Define $f : \omega \to \omega$ by $f(n) =$ the least s such that $\{e\}_s^{X[s]}(n)$ is defined. Clearly $f \leq_T X$. Let $g : \omega \to \omega$ be recursive such that $f(n) \leq g(n)$ for all n. Define a truth-table functional $\Phi : 2^\omega \to 2^\omega$ by putting $\Phi(Z)(n) = \{e\}_{g(n)}^{Z[g(n)]}(n)$ if this is defined, and $\Phi(Z)(n) = 0$ otherwise. Clearly $\Phi(X) = Y$. □

Theorem 3.6 (Simpson 2001). *Let $P, Q \subseteq 2^\omega$ be nonempty Π^0_1 sets. If $P \leq_w Q$, then there is a nonempty Π^0_1 set $Q' \subseteq Q$ such that $P \leq_s Q'$.*

Proof. Assume $P \leq_w Q$. By Theorem 3.2 let $Y \in Q$ be almost recursive. Let $X \in P$ be such that $X \leq_T Y$. By Lemma 3.5 let $\Phi : 2^\omega \to 2^\omega$ be a truth-table functional such that $\Phi(Y) = X$. Put $Q' = Q \cap \Phi^{-1}(P)$. Then Q' is a nonempty Π^0_1 subset of Q, and $P \leq_s Q'$ via Φ. □

Corollary 3.7. *Let X be 1-random and almost recursive. Then there is no completion of* PA *which is $\leq_T X$.*

Proof. Otherwise there would be a Medvedev complete Π^0_1 set $P \subseteq 2^\omega$ with $\mu(P) > 0$. □

4 The $\Sigma^0_3 \to \Pi^0_1$ Theorem

Theorem 4.1 (Simpson 2002). *If $S \subseteq \omega^\omega$ is Σ^0_3, then for all Π^0_1 sets $P \subseteq 2^\omega$ there is a Π^0_1 set $Q \subseteq 2^\omega$ such that $Q \equiv_w S \cup P$.*

Proof. First use a Skolem function technique to reduce to the case when S is a Π^0_1 subset of ω^ω. After that, let T_S be a recursive subtree of $\omega^{<\omega}$ such that S is the set of paths through T_S. Let T_P be a recursive subtree of $2^{<\omega}$ such that P is the set of paths through T_P. We may assume that, for all $\tau \in T_S$ and $n < \mathrm{lh}(\tau)$, $\tau(n) \geq 2$. Define T_Q to be the set of sequences $\rho \in \omega^{<\omega}$ of the form

$$\sigma_0{}^\frown\langle n_0\rangle^\frown\sigma_1{}^\frown\langle n_1\rangle^\frown \cdots {}^\frown\langle n_{k-1}\rangle^\frown\sigma_k$$

where $\langle n_0, n_1, \ldots, n_{k-1}\rangle \in T_S$, $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$, and $\rho(m) \leq m + 2$ for all $m < \mathrm{lh}(\rho)$. Thus T_Q is a recursive subtree of $\omega^{<\omega}$. Let $Q \subseteq \omega^\omega$ be the set of paths through T_Q. It is not hard to see that $Q \equiv_w S \cup P$. Note that Q is Π^0_1 and recursively bounded. Hence by Lemma 1.10 there is a Π^0_1 set $Q^* \subseteq 2^\omega$ which is recursively homeomorphic to Q. □

Corollary 4.2. *There is a Π^0_1 set $D \subseteq 2^\omega$ such that $D \equiv_w$ DNR. Put $\mathbf{d} = \deg_w(D) = \deg_w(\mathrm{DNR})$.*

Remark 4.3. It can be shown that $\mathbf{0} < \mathbf{d} < \mathbf{r}_1 < \mathbf{1}$.

Definition 4.4. X is 2-random if and only if it is 1-random relative to $0'$, the Turing degree of the Halting Problem. The set of 2-random reals is denoted R_2. We write $\mathbf{r}_2 = \deg_w(R_2)$.

Corollary 4.5. *There is a Π^0_1 set $R'_2 \subseteq 2^\omega$ such that $R'_2 \equiv_w R_2 \cup P$, where $P = \{$completions of $\mathsf{PA}\}$. Put $\mathbf{r}'_2 = \inf(\mathbf{r}_2, \mathbf{1}) = \deg_w(R'_2)$.*

Proof. Relativizing Corollary 2.4 we see that R_2 is a Σ^0_3 subset of 2^ω. Our theorem then follows by Theorem 4.1. □

Theorem 4.6. *If X is 2-random then X is not almost recursive.*

Proof. Martin 1967, unpublished, has shown that $\mu(\{X \in 2^\omega : X \text{ is not almost recursive}\}) = 1$. Our theorem follows from an analysis of Martin's proof. □

Theorem 4.7. *We have $\mathbf{0} < \mathbf{r}_1 < \mathbf{r}'_2 < \mathbf{1}$.*

Proof. Obviously $\mathbf{0} < \mathbf{r}_1 \leq \mathbf{r}'_2 \leq \mathbf{1}$. Theorem 2.12 implies that $\mathbf{r}'_2 < \mathbf{1}$. The fact that $\mathbf{r}_1 < \mathbf{r}'_2$ follows from Corollaries 3.4 and 3.7 and Theorem 4.6. □

5 Embedding the R. E. Degrees

We now use the $\Sigma^0_3 \to \Pi^0_1$ Theorem to embed the upper semilattice of Turing degrees of recursively enumerable subsets of ω into the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω.

Theorem 5.1. *Let $A \in 2^\omega$ be Δ^0_2, i.e., $A \leq_T 0'$. Then there is a Π^0_1 set $P_A \subseteq 2^\omega$ such that $P_A \equiv_w P \cup \{A\}$, where $P = \{$completions of $\mathsf{PA}\}$. We have $P_{A \oplus B} \equiv_w P_A \times P_B$.*

Proof. The first statement follows from Theorem 4.1 since $\{A\}$ is Π^0_2. The second statement is straightforward. □

Theorem 5.2 (Arslanov Completeness Criterion). *Let $A \subseteq \omega$ be recursively enumerable. If $f \in \text{DNR}$ and $f \leq_T A$, then A is Turing complete, i.e., $\deg_T(A) = 0'$.*

Proof. See Soare's book, Section V.5. □

Theorem 5.3. *Let $A, B \subseteq \omega$ be recursively enumerable. Then $A \leq_T B$ if and only if $P_A \leq_w P_B$.*

Proof. We identify $A, B \subseteq \omega$ with their characteristic functions $\chi_A, \chi_B \in 2^\omega$. Obviously $A \leq_T B$ implies $P_A \leq_w P_B$. For the converse, recall that P is Medvedev complete, hence $\text{DNR}_2 \leq_s P$. Thus for all $X \in P$ there is a DNR function $f \leq_T X$. Assume now that $P_A \leq_w P_B$. In particular we can find $X \in P \cup \{A\}$ such that $X \leq_T B$. If $X \in P$, then by the Arslanov Completeness Criterion, B is Turing complete, hence $A \leq_T B$. If $X \notin P$, then $X = A$, hence again $A \leq_T B$. □

Remark 5.4. Thus our embedding of the r. e. Turing degrees into the Muchnik lattice is given by $\deg_T(A) \mapsto \deg_w(P \cup \{A\})$, where $P = \{\text{completions of } \mathsf{PA}\}$. This embedding is order preserving and least upper bound preserving, carries 0 to $\mathbf{0}$, and carries $0'$ to $\mathbf{1}$.

6 Priority Arguments