Strong Forcing

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This is a streamlined exposition of the basic facts about forcing. It replaces Chapter VII, Section 3, pages 192–204, in Kunen’s book. We follow the exposition in Shoenfield’s paper “Unramified Forcing.”

As usual, $M$ is a countable transitive model of $\text{ZFC}$, $P$ is a partial ordering in $M$, and $M^P$ is the set of $P$-names. For any $M$-generic filter $G \subseteq P$ we have $M[G] = \{ \tau_G : \tau \in M^P \}$ where $\tau_G = \{ \sigma_G : \langle \sigma, p \rangle \in \tau, p \in G \}$.

**Lemma 1.** For any $p$ there exists an $M$-generic filter $G \subseteq P$ such that $p \in G$.

*Proof.* This is easily proved, using countability of $M$. $\Box$

**Definition 2.** The *forcing language* consists of the language of $\text{ZFC}$ plus constant symbols $\tau$ for all $\tau \in M^P$. If $\varphi$ is a sentence of the forcing language, $M[G] \models \varphi$ means that $\varphi$ is true in $M[G]$ interpreting $\tau$ as $\tau_G$.

**Definition 3.** Let $p \in P$, and let $\varphi$ be a sentence of the forcing language. We define $p \forces \varphi$ ($p$ forces $\varphi$) to mean that $M[G] \models \varphi$ for all $M$-generic filters $G \subseteq P$ such that $p \in G$.

**Theorem 4 (definability of forcing).** For any formula $\varphi(x_1, \ldots, x_n)$ we have that $\{ \langle p, \tau_1, \ldots, \tau_n \rangle : p \forces \varphi(\tau_1, \ldots, \tau_n) \}$ is definable over $M$.

**Theorem 5 (forcing equals truth).** For all $M$-generic filters $G \subseteq P$, $M[G] \models \varphi$ if and only if $(\exists p \in G)(p \forces \varphi)$.

In order to prove Theorems 4 and 5, we introduce the notion of *strong forcing*. We assume that the forcing language has been set up with $\in$, $\not\in$, $\neg$, $\lor$, $\exists$ as primitives. We define $x \not\in y$ as $\neg(x \in y)$, and $x = y$ as $\neg(x \neq y)$.

**Definition 6.** We define $p \forces_s \varphi$ ($p$ strongly forces $\varphi$) as follows.

1. $p \forces_s \sigma \in \tau$ if and only if, for some $q \geq p$, $\langle \sigma', q \rangle \in \tau$ for some $\sigma'$ such that $p \forces_s \sigma = \sigma'$.

2. $p \forces_s \tau_1 \neq \tau_2$ if and only if, for some $q \geq p$ and some $\sigma$, either $\langle \sigma, q \rangle \in \tau_1$ and $p \forces_s \sigma \not\in \tau_2$, or $\langle \sigma, q \rangle \in \tau_2$ and $p \forces_s \sigma \not\in \tau_1$.

3. $p \forces_s \neg \varphi$ if and only if there does not exist $q \leq p$ such that $q \forces_s \varphi$. 

$\Box$
4. \( p \models_s \varphi \lor \psi \) if and only if \( p \models_s \varphi \) or \( p \models_s \psi \).

5. \( p \models_s \exists x \varphi(x) \) if and only if \( p \models_s \varphi(r) \) for some \( r \).

Note that, for clauses 1 and 2, the definition is by transfinite induction on the ranks of \( \sigma, \tau_1, \) and \( \tau_2 \) as \( P \)-names. For clauses 3, 4, and 5, the definition is by induction on the rank of \( \varphi \) as a sentence of the forcing language.

**Lemma 7.** If \( p \models_s \varphi \) and \( q \leq p \) then \( q \models_s \varphi \).

**Lemma 8 (definability of strong forcing).** For any formula \( \varphi(x_1, \ldots, x_n) \) we have that \( \{ (p, \tau_1, \ldots, \tau_n) : p \models_s \varphi(\tau_1, \ldots, \tau_n) \} \) is definable over \( M \).

Lemmas 7 and 8 are easily proved by induction, following the definition of \( \models_s \).

**Lemma 9 (strong forcing equals truth).** For all \( M \)-generic filters \( G \subseteq P, M[G] \models \varphi \) if and only if \( (\exists p \in G) (p \models_s \varphi) \).

**Proof.** The proof is by induction, following the definition of \( \models_s \).

1. “if”. Suppose \( p \in G \) and \( p \models_s \sigma \in \tau \). By definition there exist \( q \geq p \) and \( \langle \sigma', q \rangle \in \tau \) such that \( p \models_s \sigma = \sigma' \). Then \( q \in G \), hence \( \sigma_G' = \sigma_G \). Also, by inductive hypothesis, \( \sigma_G \in \tau_G \).

   “only if”. Suppose \( \sigma_G \in \tau_G \). By definition there exists \( \langle \sigma', q \rangle \in \tau \) such that \( \sigma_G = \sigma'_G \) and \( q \in G \). By inductive hypothesis, there exists \( r \in G \) such that \( r \models_s \sigma = \sigma' \). Since \( q, r \in G \) there exists \( p \in G \) such that \( p \leq q, r \).

   By Lemma 7 we have that \( p \models_s \sigma = \sigma' \). Thus \( p \models_s \sigma \in \tau \).

2. “if”. Suppose \( p \in G \) and \( p \models_{s} \tau_1 \neq \tau_2 \). Say \( q \geq p \), \( \langle \sigma, q \rangle \in \tau_1 \), \( p \models_s \sigma \notin \tau_2 \). Then \( q \in G \), hence \( \sigma_G \in \tau_{G} \). Also, by inductive hypothesis, \( \sigma_G \notin \tau_{G} \).

   Thus \( \tau_{1G} \neq \tau_{2G} \).

   “only if”. Suppose \( \tau_{1G} \neq \tau_{2G} \). Say \( \langle \sigma, q \rangle \in \tau_1 \), \( q \in G \), \( \sigma_G \notin \tau_{2G} \). By inductive hypothesis, there exists \( r \in G \) such that \( r \models_s \sigma \notin \tau_2 \).

   Since \( q, r \in G \) there exists \( p \in G \) such that \( p \leq q, r \). By Lemma 7 we have that \( p \models_s \sigma \notin \tau_2 \).

   Thus \( p \models_s \tau_1 \neq \tau_2 \).

3. “if”. Suppose \( p \in G \) and \( p \models_s \neg \varphi \). To show \( M[G] \models \neg \varphi \). Suppose \( M[G] \models \varphi \).

   By inductive hypothesis, there exists \( q \in G \) such that \( q \models_s \varphi \).

   Since \( p, q \in G \), they are compatible, so let \( r \leq p, q \). Then, by Lemma 7, \( r \models_s \varphi \), and \( r \leq p \), contradicting \( p \models_s \neg \varphi \).

   “only if”. Suppose \( M[G] \models \neg \varphi \). Put \( D = \{ p : p \models_s \varphi \text{ or } p \models_s \neg \varphi \} \). Using the definition of \( p \models_s \neg \varphi \), it is easy to see that \( D \) is dense. Let \( p \in D \cap G \). If \( p \models_s \varphi \), then by inductive hypothesis, \( M[G] \models \varphi \), a contradiction. Hence \( p \models_s \neg \varphi \).

4. “if”. Suppose \( p \in G \) and \( p \models_s \varphi \lor \psi \). Say \( p \models_s \varphi \). By inductive hypothesis, \( M[G] \models \varphi \).

   Hence \( M[G] \models \varphi \lor \psi \).

   “only if”. Suppose \( M[G] \models \varphi \lor \psi \). Say \( M[G] \models \varphi \). By inductive hypothesis, there exists \( p \in G \) such that \( p \models_s \varphi \).

   Hence \( p \models_s \varphi \lor \psi \).

2
5. “if”. Suppose \( p \in G \) and \( p \vDash_s \exists x \varphi(x) \). Then \( p \vDash_s \varphi(\tau) \) for some \( \tau \). By inductive hypothesis, \( M[G] \models \varphi(\tau) \). Hence \( M[G] \models \exists x \varphi(x) \).

“only if”. Suppose \( M[G] \models \exists x \varphi(x) \). Then \( M[G] \models \varphi(\tau) \) for some \( \tau \). By inductive hypothesis, there exists \( p \in G \) such that \( p \vDash_s \varphi(\tau) \). Then \( p \vDash_s \exists x \varphi(x) \).

This completes the proof.

Lemma 10. \( p \vDash \varphi \) if and only if \( \{ r \leq p : r \vDash_s \varphi \} \) is dense below \( p \).

Proof. “if”. Assume that \( \{ r \leq p : r \vDash_s \varphi \} \) is dense below \( p \). To show \( p \vDash \varphi \). Let \( G \) be generic with \( p \in G \). Then there exists \( r \in G \) such that \( r \vDash_s \varphi \). Hence, by Lemma 9, \( M[G] \models \varphi \).

“only if”. Assume \( p \vDash \varphi \). To show that \( \{ r \leq p : r \vDash_s \varphi \} \) is dense below \( p \). Given \( q \leq p \), by Lemma 1 let \( G \) be generic with \( q \in G \). Then \( p \in G \), hence \( M[G] \models \varphi \). By Lemma 9 there exists \( p' \in G \) such that \( p' \vDash_s \varphi \). Since \( p', q \in G \), they are compatible, so let \( r \leq p', q \). Then, by Lemma 7, \( r \vDash_s \varphi \).

This completes the proof.

Theorems 4 and 5 follow easily from Lemmas 8, 9, and 10.

Corollary 11. 1. If \( p \vDash \varphi \) and \( q \leq p \) then \( q \vDash \varphi \).

2. \( p \vDash \neg \varphi \) if and only if there does not exist \( q \leq p \) such that \( q \vDash \varphi \).

3. If \( p \vDash \varphi \lor \psi \) then there exists \( q \leq p \) such that \( q \vDash \varphi \) or \( q \vDash \psi \).

4. If \( p \vDash \exists x \varphi(x) \) then there exists \( q \leq p \) such that \( q \vDash \varphi(\tau) \) for some \( \tau \).