1. A real number $\alpha$ is said to be \textit{primitive recursive} if the function $f(n) = \text{the } n\text{th digit of } \alpha$ is primitive recursive. A real number $\alpha$ is said to be \textit{algebraic} if it is a root of a nonzero polynomial with integer coefficients. For example, $\sqrt{2}$ is a real algebraic number, because it is a root of the polynomial $x^2 - 2$.

Prove that all real algebraic numbers are primitive recursive.

\textit{Solution.} Let $p(x)$ be a nonzero polynomial of minimal degree with integer coefficients such that $p(\alpha) = 0$. Then $\alpha$ is a simple root of $p(x)$, i.e., $p'(\alpha) \neq 0$ where $p'(x)$ is the derivative of $p(x)$. Without loss of generality, assume that $\alpha > 0$ and $p'(\alpha) > 0$ and $\alpha$ is irrational. Let $a$ and $b$ be rational numbers such that $0 < a < \alpha < b$ and $p(x)$ is negative for $a < x < \alpha$ and positive for $\alpha < x < b$. Let $m$ be such that, for all sufficiently large $n$, the $(m+n)$th digit of $\alpha$ is $\text{Remainder}(g(n), 10)$ where $g(n)$ is the least $k$ such that

$$a < \frac{k}{10^n} < b \text{ and } p \left( \frac{k + 1}{10^n} \right) > 0.$$ 

Here we are applying the bounded least number operator, so $g$ is primitive recursive. From this it follows easily that $f$ is primitive recursive, where $f(n) = \text{the } n\text{th digit of } \alpha$.

2. We know that the Ackermann function is an example of a 1-place function which is recursive but not primitive recursive. Find an example of a 1-place predicate which is recursive but not primitive recursive.

\textit{Solution.} Our example will be obtained by diagonalizing over all 1-place primitive recursive functions. We first prove the following lemma and theorem.
Lemma. We can find a 2-place total recursive function \( r(e, n) \) with the following property. For all \( k \geq 1 \) and all \( k \)-place primitive recursive functions \( f \), there exists \( e \) such that \( r(e, \prod_{i=1}^{k} p_i^{n_i}) = f(n_1, \ldots, n_k) \) for all \( n_1, \ldots, n_k \).

Proof. Consider a system of indices for the primitive recursive functions, defined inductively as follows.

(a) Let 2 be an index for the constant zero function, \( Z(m) = 0 \).

(b) Let \( 2^2 \) be an index for the successor function, \( S(m) = m + 1 \).

(c) For \( 1 \leq i \leq k \) let \( 2^3 \cdot 3^k \cdot 5^i \) be an index of the \( k \)-place projection function \( P_{ki}(n_1, \ldots, n_k) = n_i \).

(d) If \( u_1, \ldots, u_l, v \) are indices of \( g_1(n_1, \ldots, n_k), \ldots, g_l(n_1, \ldots, n_k), h(t_1, \ldots, t_l) \) respectively, let \( 2^4 \cdot 3^k \cdot 5^v \cdot \prod_{j=1}^{l} p_j^{u_j+2} \) be an index of \( f(n_1, \ldots, n_k) \) given by generalized composition as

\[
  f(n_1, \ldots, n_k) = h(g_1(n_1, \ldots, n_k), \ldots, g_l(n_1, \ldots, n_k)).
\]

(e) If \( u \) and \( v \) are indices of \( g(n_1, \ldots, n_k) \) and \( h(m, t, n_1, \ldots, n_k) \) respectively, let \( 2^5 \cdot 3^{k+1} \cdot 5^v \cdot 7^u \) be an index of \( f(m, n_1, \ldots, n_k) \) given by primitive recursion as

\[
  f(0, n_1, \ldots, n_k) = g(n_1, \ldots, n_k), \quad f(m + 1, n_1, \ldots, n_k) = h(m, f(m, n_1, \ldots, n_k), n_1, \ldots, n_k).
\]

It is routine to show that the set of indices is primitive recursive. Moreover, if \( e \) is an index of a \( k \)-place primitive recursive function, then \( (e)_1 = k \). By the Recursion Theorem, let \( \psi(e, m) \) be a 2-place partial recursive function with the following properties. Writing \( n = \prod_{i=1}^{k} p_i^{n_i} \) and \( n' = \prod_{i=1}^{k} p_i^{n_i+1} \) and \( n'' = \prod_{i=1}^{k} p_i^{n_i+2} \) we have:

(a) \( \psi(2, p_1^{n}) \simeq 0 \).

(b) \( \psi(2^2, p_1^{n}) \simeq m + 1 \).

(c) \( \psi(e, n) \simeq n_i \) whenever \( e \) is an index of the form \( 2^3 \cdot 3^k \cdot 5^i \).

(d) \( \psi(e, n) \simeq \psi(v, \prod_{j=1}^{l} p_j^{u_j}) \) whenever \( e \) is an index of the form \( 2^4 \cdot 3^k \cdot 5^v \cdot \prod_{j=1}^{l} p_j^{u_j+2} \).
(e) $\psi(e, p^0_1 \cdot n') \simeq \psi(u, n)$ and $\psi(e, p^m_1 \cdot n') \simeq \psi(v, p^m_1 \cdot p_2 \psi(e, p^m_1 \cdot n') \cdot n'')$ whenever $e$ is an index of the form $2^5 \cdot 3^{k+1} \cdot 5^v \cdot 7^u$.

(f) $\psi(e, m) \simeq 0$ otherwise.

By induction on $e$ with a subsidiary induction on $m$, it is straightforward to prove that $\psi(e, m) \downarrow$ for all $e$ and $m$, and that $\psi(e, \prod_{i=1}^k p_i^{n_i}) = f(n_1, \ldots, n_k)$ whenever $e$ is an index of a $k$-ary primitive recursive function $f$. Letting $r(e, m) = \psi(e, m)$ we have our lemma.

**Theorem.** For each $k \geq 1$ we can find a $k+1$-place total recursive function $r_k$ with the following property. For any $k$-place primitive recursive function $f$ there exists $e$ such that $r_k(e, n_1, \ldots, n_k) = f(n_1, \ldots, n_k)$ for all $n_1, \ldots, n_k$.

**Proof.** Let $r_k(e, n_1, \ldots, n_k) = r(e, \prod_{i=1}^k p_i^{n_i})$ where $r$ is as in the lemma. Clearly $r_k$ has the desired property.

Now, define $d : \mathbb{N} \to \{0, 1\}$ by $d(n) = 1 - r_1(n, n)$. Clearly $d$ is the characteristic function of a 1-place predicate which is recursive but not primitive recursive.

3. Exhibit a register machine program showing that the function $f(m, n) = m^n$ is computable. (Note that $m^0 = 1$ for all $m \in \mathbb{N}$ including $m = 0$. This convention makes the recursion easier.)

**Solution.**