1. Exhibit construction sequences showing that $\lambda xy[xy]$ and $\lambda x[2^x]$ are primitive recursive.

2. Show that $\lambda m n k[k$th digit of $F(m)/F(n)]$ is primitive recursive, where $F(n) = n$th Fibonacci number. You may take for granted the material of Section 1 of the notes.

Solution. The following are primitive recursive:

\[
\ell(n) = \text{number of digits in } n = \text{least } \ell \text{ such that } 10^\ell > n
\]

assuming $n > 0$;

\[
d_0(n, k) = k^{th} \text{ digit of } n = \text{Rem}(\text{Quot}(n, 10^{\ell(n)-k}), 10)
\]

assuming $n > 0$ and has at least $k$ digits;

\[
d_1(m, n, k) = d_0(\text{Quot}(10^j m, n), k)
\]

where $j$ is sufficiently large so that $\text{Quot}(10^j m, n)$ has at least $k$ digits, say $j = \text{least } j \text{ such that } 10^j m > 10^k n$. Note that $d_1(m, n, k)$ is the $k$th digit of the rational number $m/n$ provided $m > n > 0$. Next define

\[
d_2(m, n, k) = \begin{cases} 
  d_1(m, n, k) & \text{if } m > n \\
  d_1(m + n, n, k + 1) & \text{if } m < n \\
  1 & \text{if } m = n \text{ and } k = 1 \\
  0 & \text{if } m = n \text{ and } k > 1 
\end{cases}
\]
Thus $d_2(m, n, k)$ is the $k^{th}$ digit of the rational number $m/n$ for all $m, n > 0$. Finally the desired function is $d_2(F(m), F(n), k)$.

3. Exhibit a register machine program to compute $\lambda xy[xy]$.

Solution. See Chapter 1.

4. Exercise on the Ackermann function.

1. Show that each $F_n$ is primitive recursive.

2. (a) $F_n(x) > 0$.
   (b) $F_n(x + 1) > F_n(x)$, i.e. $F_n$ is monotone.
   (c) $F_n(x) > x$.
   (d) $F_{n+1} \geq F_n(x + 1)$.

3. Every primitive recursive function is dominated by $F_n$ for some $n$.

4. $\lambda x[F_n(x)]$ is not primitive recursive.

Solution. Recall the definition

\[
F_0(x) = x + 1 \\
F_{n+1}(x) = \underbrace{F_n \cdots F_n}_x(1).
\]

1. By induction on $n$. Obviously $F_0$ is primitive recursive. The recursion equations

\[
F_{n+1}(0) = F_n(1) \\
F_{n+1}(x + 1) = F_n(F_{n+1}(x))
\]

show that if $F_n$ is primitive recursive then so is $F_{n+1}$.

2. $F_n(x) > 0$ is easily proved by induction on $n$. Prove $F_n$ monotone and $F_n(x) > x$ simultaneously by induction on $n$: $F_n(0) > 0$ and monotonicity imply $F_n(x) > x$ for all $x$, hence $F_{n+1}(x + 1) = F_n(F_{n+1}(x)) > F_{n+1}(x)$. Prove $F_{n+1}(x) \geq F_n(x + 1)$ by induction on $x$: For $x = 0$ we have $F_{n+1}(0) = F_n(1)$, and for $x + 1$ we have $F_{n+1}(x + 1) = F_n(F_{n+1}(x)) \geq F_n(F_n(x + 1))$ by inductive hypothesis, and $F_n(F_n(x + 1)) \geq F_n(x + 2)$ since $F_n(x + 1) \geq x + 2$ and $F_n$ is monotone.
3. Recall that \( f(x_1, \ldots, x_k) \) is said to be dominated by \( F_n \) if
\[
f(x_1, \ldots, x_k) \leq F_n(\text{max}(x_1, \ldots, x_k))
\]
for all \( x_1, \ldots, x_k \). Clearly the initial functions are dominated by \( F_0 \). For composition, if \( f = h(g_1, \ldots, g_m) \), let \( n \) be sufficiently large so that \( F_{n+1} \) dominates \( g_1, \ldots, g_m \) and \( F_n \) dominates \( h \). Then an easy computation shows that \( F_{n+1} \) dominates \( f \). For primitive recursion, if \( f \) is obtained by primitive recursion from \( g \) and \( h \), let \( n \) be sufficiently large so that \( F_{n+1} \) dominates \( g \) and \( F_n \) dominates \( h \). We claim that
\[
f(y, x_1, \ldots, x_k) \leq F_{n+1}(y + \text{max}(x_1, \ldots, x_k))
\]
for all \( x_1, \ldots, x_k, y \). This is easily proved by induction on \( y \). Note also that
\[
F_{n+2}(z) = F_{n+1}F_{n+1} \cdots F_{n+1}(1) \geq F_{n+1}(2z + 1)
\]
for all \( z \), since \( F_{n+1}(w) \geq F_1(w) = w + 2 \) for all \( w \). Thus we have
\[
f(y, x_1, \ldots, x_k) \leq F_{n+1}(y + \text{max}(x_1, \ldots, x_k)) \\
\leq F_{n+1}(2 \text{max}(y, x_1, \ldots, x_k)) \\
\leq F_{n+2}(\text{max}(y, x_1, \ldots, x_k))
\]
and this completes the proof.

4. If \( \lambda x[F_x(x)] \) were primitive recursive, then \( \lambda x[F_x(x) + 1] \) would be primitive recursive, hence dominated by \( F_n \) for some \( n \), in particular \( F_n(n) + 1 \leq F_n(n) \), a contradiction.

5. Show that
\[
T = \{ x \mid \varphi_x^{(1)} \text{ is total} \}
\]
and
\[
E = \{ x \mid \varphi_x^{(1)} \text{ is the empty function} \}
\]
are nonrecursive.

Solution. By the Enumeration and Parametrization theorems, we can find a primitive recursive function \( f \) such that
\[
\varphi_{\varphi_{f(x)}^{(1)}}(y) \simeq \varphi_x^{(1)}(x)
\]
for all $x$ and $y$. Then $x \in K$ implies $f(x) \in T$, while $x \notin K$ implies $f(x) \in E$. Thus $f$ reduces $K$ to $T$ and to the complement of $E$.

6. Prove Rice’s Theorem: If $C$ is any nontrivial class of 1-place partial recursive functions, then the index set $I_C = \{ x \mid \varphi_x^{(1)} \in C \}$ is nonrecursive.

Solution. Let $x_0$ be an index of the empty function, and let $x_1$ be an index such that $\varphi_{x_1}^{(1)} \in C$ if and only if $\varphi_{x_0}^{(1)} \notin C$. By the Enumeration and Parametrization theorems, we can find a primitive recursive function $f$ such that

$$\varphi_{f(x)}^{(1)}(y) \simeq \begin{cases} 
\varphi_{x_1}^{(1)}(y) & \text{if } \varphi_x^{(1)}(x) \text{ is defined} \\
\text{undefined} & \text{otherwise}
\end{cases}$$

for all $x$ and $y$. Thus $x \in K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_1}^{(1)}$, while $x \notin K$ implies $\varphi_{f(x)}^{(1)} = \varphi_{x_0}^{(1)}$. Thus $f$ reduces $K$ either to $I_C$ (if $\varphi_{x_1}^{(1)} \in C$) or to the complement of $I_C$ (if $\varphi_{x_1}^{(1)} \notin C$). In either case it follows that $I_C$ is not recursive.

7. Write down a sentence expressing Goldbach’s Conjecture: every even number $> 2$ is the sum of two primes. Write down a formula defining the function $\lambda xy$ (least common multiple of $x$ and $y$).

8. Show that a total recursive function $f(x_1, \ldots, x_k)$ is primitive recursive if and only if there exists an index $e$ of $f$ such that $\lambda x_1 \cdots x_k[\text{Stop}(e, x_1, \ldots, x_k)]$ is dominated by some primitive recursive function.

9. Show that the definitions of $\sum_{i \in I} \kappa_i$ and $\Pi_{i \in I} \kappa_i$ are valid. Specifically, you need to prove the following:

1. Given an indexed set of cardinal numbers $\langle \kappa_i \rangle_{i \in I}$, there exists an indexed set of sets $\langle X_i \rangle_{i \in I}$ such that $\text{card}(X_i) = \kappa_i$ for all $i \in I$ and $X_i \cap X_j = \emptyset$ for all $i, j \in I$ with $i \neq j$.

2. If $\langle X'_i \rangle_{i \in I}$ is another such indexed set of sets, then $\bigcup_{i \in I} X_i \approx \bigcup_{i \in I} X'_i$ and $\prod_{i \in I} X_i \approx \prod_{i \in I} X'_i$.

10. Prove König’s theorem: If $\langle \kappa_i \rangle_{i \in I}$ and $\langle \lambda_i \rangle_{i \in I}$ are indexed sets of cardinal numbers with $\kappa_i < \lambda_i$ for all $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.

11. Define $\aleph_0 = \text{card}(\mathbb{N})$. Show that $\text{card}(\mathbb{Q}) = \aleph_0$, $\text{card}(\mathbb{R}) = 2^{\aleph_0}$, $\text{card}(\mathbb{R}^\mathbb{N}) = 2^{\aleph_0}$, $\text{card}(\mathbb{R}^\mathbb{R}) = 2^{2^{\aleph_0}}$. 

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12. If $F$ is a continuous increasing function from ordinals to ordinals, show that $F$ has arbitrarily large fixed points. (*Increasing* means that $F(\alpha) \geq \alpha$ for all $\alpha$. *Continuous* means $F(\delta) = \sup\{F(\alpha) \mid \alpha < \delta\}$ for limit ordinals $\delta$. A *fixed point* of $F$ is an ordinal $\alpha$ such that $F(\alpha) = \alpha$.)

13. Show that $2^{\aleph_0} \neq \aleph_\omega$. (Hint: Use König’s theorem.)

14. Show that, if there exist at least $n + 1$ inaccessible cardinals, then there is a transitive model of ZFC + “there exist exactly $n$ inaccessible cardinals.”