Math 558 – Homework #3

March 2, 2007

1. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\} = \text{the natural numbers}$,

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\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \text{the integers},
$$

and $\mathbb{R} = (-\infty, \infty) = \text{the real numbers}$.

According to Matiyasevich’s Theorem, we can find a polynomial

$$f(w, x_1, \ldots, x_k)$$

with integer coefficients, such that the set of $a \in \mathbb{N}$ for which the equation $f(a, x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}$ is noncomputable.

(a) Discuss the analogous question in which “solution in $\mathbb{N}$” is replaced by “solution in $\mathbb{Z}$”.

Hint: You may use the following well-known theorem of Lagrange:

For each $n \in \mathbb{N}$ there exist $a, b, c, d \in \mathbb{N}$ such that

$$n = a^2 + b^2 + c^2 + d^2.$$

(b) Discuss analogous questions in which “solution in $\mathbb{N}$” is replaced by “solution in $\mathbb{R}$”.

2. Explain in detail how you would translate the following statements of Euclidean plane geometry into sentences of the language

$$+, -, \cdot, 0, 1, <, =$$

over the real number system.

(a) For every two points, there is a unique line passing through them.

(b) For every three non-collinear points, there is a unique circle passing through them.

(c) For every line $L$ and circle $C$, the intersection of $L$ and $C$ consists of at most two points.

(d) For every circle $C$ and point $P$ lying on $C$, there exists one and only one line $L$ such that $L \cap C = P$. (I.e., a tangent line.)

(e) Every line segment has a unique midpoint.

(f) Every angle can be uniquely bisected.

(g) The three angle bisectors of any triangle meet in a single point.

(h) Every angle can be uniquely trisected.