1. Find a primitive recursive function \( f(x, y) \) such that, for all \( x \) and \( y \),
\[
\varphi_{f(x,y)}^{(1)} = \varphi_{x}^{(1)} \circ \varphi_{y}^{(1)}.
\]
Hint: Use the Parametrization Theorem.

2. A function \( f : \mathbb{N}^k \to \mathbb{N} \) is said to be limit recursive if there exists a recursive function \( g : \mathbb{N}^{k+1} \to \mathbb{N} \) such that, for all \( x_1, \ldots, x_k \),
\[
f(x_1, \ldots, x_k) = \lim_{s \to \infty} g(x_1, \ldots, x_k, s).
\]
Prove that \( f \) is \( \Delta^0_2 \) if and only if \( f \) is limit recursive.

3. Two sets \( A, B \subseteq \mathbb{N} \) are said to be recursively inseparable if there is no recursive set \( X \subseteq \mathbb{N} \) such that \( A \subseteq X \) and \( B \cap X = \emptyset \). Find a disjoint pair of \( \Sigma^0_1 \) sets which are recursively inseparable.
Hint: Let \( A = K_0 \) and \( B = K_1 \), where \( K_n = \{ x \mid \varphi_{n}^{(1)}(x) \simeq n \} \).

4. Show that the following number-theoretic predicates are arithmetically definable, by exhibiting formulas which define them over the structure \((\mathbb{N}, +, \cdot, 0, 1, =)\).
   (a) \( \text{GCD}(x, y) = z \).
   (b) \( \text{LCM}(x, y) = z \).
   (c) \( \text{Quotient}(x, y) = z \).
   (d) \( \text{Remainder}(x, y) = z \).
   (e) \( \beta(x, y, z) = w \).
   (f) \( x \) is the largest prime number less than \( y \).
   (g) \( x \) is the product of all the prime numbers less than \( y \).

5. Which of the following number-theoretic predicates are arithmetically definable? Prove your answers.
(a) $x$ is the sum of all the prime numbers less than $y$.

(b) $x^y = z$.

(c) $x! = y$.

6. Find a pair of numbers $r, a$ such that $\beta(r, a, 0) = 11$, $\beta(r, a, 1) = 19$, $\beta(r, a, 2) = 30$, $\beta(r, a, 3) = 37$, $\beta(r, a, 4) = 51$.

(Hint: First find an appropriate $a$ by hand. Then write a small computer program to find $r$ by brute force.)

7. Let $A$ and $B$ be subsets of $\mathbb{N}$. Prove that if $A$ is reducible to $B$ and $B$ is arithmetically definable, then $A$ is arithmetically definable.

8. For each $n \geq 1$ let $C_n$ be a subset of $\mathbb{N}$ which is $\Sigma^0_n$ complete. Consider the set $B = \{2^n3^x \mid x \in C_n\}$. Prove that $B$ is not arithmetically definable.

9. Prove that the set $\text{Fml}$ of all Gödel numbers of formulas is primitive recursive.

10. Prove that the set $\text{Snt}$ of all Gödel numbers of sentences is primitive recursive.