1. Find a pair of numbers \( r, a \) such that \( \beta(r, a, 0) = 11, \beta(r, a, 1) = 19, \beta(r, a, 2) = 30, \beta(r, a, 3) = 37, \beta(r, a, 4) = 51. \)

Hint: First find an appropriate \( a \) by hand. Then write a small computer program to find \( r \) by brute force.

2. Recall that \( \mathbb{N} = \{0, 1, 2, \ldots \} \) = the natural numbers,
\( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \} \) = the integers, and
\( \mathbb{R} = (-\infty, \infty) \) = the real numbers.
According to Matiyasevich’s Theorem, we can find a polynomial
\[
f(w, x_1, \ldots, x_k)
\]
with integer coefficients, such that the set of \( a \in \mathbb{N} \) for which the equation \( f(a, x_1, \ldots, x_k) = 0 \) has a solution in \( \mathbb{N} \) is noncomputable.

(a) Discuss the analogous question in which “solution in \( \mathbb{N} \)” is replaced by “solution in \( \mathbb{Z} \)”.

(b) Discuss analogous questions in which “solution in \( \mathbb{N} \)” is replaced by “solution in \( \mathbb{R} \)”.

3. Prove König’s Theorem:
Let \( \langle \kappa_i \rangle_{i \in I} \) and \( \langle \lambda_i \rangle_{i \in I} \) be indexed sets of cardinal numbers with the same index set \( I \). If \( \kappa_i < \lambda_i \) for all \( i \in I \), then \( \sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i \).