1. Construct $LH$-derivations of the following logically valid sentences. To make this easier, you may supplement the rules of $LH$ with

$$\frac{A_1 \ldots A_k}{B}$$

whenever $B$ is a quasitautological consequence of $A_1, \ldots, A_k$.

(a) $(\forall x (P_x \& Q_x)) \Leftrightarrow ((\forall x P_x) \& (\forall x Q_x))$

Solution.

1. $(\forall x (P_x \& Q_x)) \Rightarrow (P_a \& Q_a)$ UI
2. $(\forall x (P_x \& Q_x)) \Rightarrow P_a$ 1, QT
3. $(\forall x (P_x \& Q_x)) \Rightarrow \forall x P_x$ 2, UG
4. $(\forall x (P_x \& Q_x)) \Rightarrow Q_a$ 1, QT
5. $(\forall x (P_x \& Q_x)) \Rightarrow \forall x Q_x$ 4, UG
6. $(\forall x P_x) \Rightarrow P_a$ UI
7. $(\forall x Q_x) \Rightarrow Q_a$ UI
8. $((\forall x P_x) \& (\forall x Q_x)) \Rightarrow (P_a \& Q_a)$ 6, 7, QT
9. $((\forall x P_x) \& (\forall x Q_x)) \Rightarrow \forall x (P_x \& Q_x)$ 8, UG
10. $(\forall x (P_x \& Q_x)) \Rightarrow ((\forall x P_x) \& (\forall x Q_x))$ 3, 5, 9, QT

(b) $(\exists x \forall y R xy) \Rightarrow (\forall x \exists y R xy)$

Solution.

1. $(\forall y Ray) \Rightarrow Rab$ UI
2. $Rab \Rightarrow \exists y Ryb$ EI
3. $(\forall y Ray) \Rightarrow \exists y Ryb$ 1, 2, QT
4. $(\exists x \forall y R xy) \Rightarrow \exists y Ryb$ 3, EG
5. $(\exists x \forall y R xy) \Rightarrow \forall x \exists y Ryx$ 3, UG
(c) \( \neg \exists x \forall y (Eyx \iff \neg Ey) \)

Solution.

1. \( \forall y (Eya \iff \neg Ey) \Rightarrow (Eaa \iff \neg Eaa) \)  
   \( \text{UI} \)

2. \( \neg \forall y (Eya \iff \neg Ey) \)  
   \( 1, \text{QT} \)

3. \( (\forall y (Eya \iff \neg Ey)) \Rightarrow \exists x \forall y (Eyx \iff \neg Ey) \)  
   \( 2, \text{QT} \)

4. \( (\exists x \forall y (Eyx \iff \neg Ey)) \Rightarrow \exists x \forall y (Eyx \iff \neg Ey) \)  
   \( 3, \text{EG} \)

5. \( \neg \exists x \forall y (Eyx \iff \neg Ey) \)  
   \( 4, \text{QT} \)

2. The proof system \( LH' \).

Consider the following proof system \( LH' \), which is a “stripped down” version of \( LH \). The objects of \( LH' \) are \( L-V \)-sentences containing only \( \forall, \Rightarrow, \neg \) (i.e., not containing \( \exists, \iff, \& , \lor \) ). The rules of \( LH' \) are:

(a) quasitautologies
(b) \( (\forall x B) \Rightarrow B[x/a] \)
(c) \( (\forall x (A \Rightarrow B)) \Rightarrow (A \Rightarrow \forall x B) \)
(d) \( A \ A \Rightarrow B \quad (\text{modus ponens}) \)
(e) \( B[x/a] \forall x B \quad (\text{generalization}), \text{ where } a \text{ does not occur in } B. \)

Show that \( LH' \) is sound and complete.

Solution. In outline, the proof of soundness and completeness of \( LH' \) is the same as for \( LH \). Define a companion of \( A \) to be any \( L-V \)-sentence of the form \( (\forall x B) \Rightarrow B[x/a] \), or \( B[x/a] \Rightarrow \forall x B \) where \( a \) does not occur in \( A, B \). The Companion Theorem and its proof are as for \( LH \).

The only new point is contained in the proof of the following lemma.

Lemma. If \( C \) is a companion of \( A \) and if \( C \Rightarrow A \) is derivable in \( LH' \), then \( A \) is derivable in \( LH' \).

Proof. Assume \( C \Rightarrow A \) is derivable in \( LH' \).

If \( C \) is of the form \( (\forall x B) \Rightarrow B[x/a] \), then \( C \) is an axiom of \( LH' \), hence derivable, hence by Modus Ponens \( A \) is derivable.

Now suppose \( C \) is of the form \( B[x/a] \Rightarrow \forall x B \) where \( a \) does not occur in \( A, B \). Since \( C \Rightarrow A \) is derivable, it follows by quasitautology that \( (\neg A) \Rightarrow B[x/a] \) and \( (\neg A) \Rightarrow \neg \forall x B \) are derivable. From the former plus generalization we have that \( \forall x ((\neg A) \Rightarrow B) \) is derivable. But
\((\forall x ((\neg A) \Rightarrow B)) \Rightarrow ((\neg A) \Rightarrow \forall x B)\) is an axiom of \(LH'\), so by Modus Ponens we have that \((\neg A) \Rightarrow \forall x B\) is derivable. Now by quasitautology we see that \(A\) is derivable. This completes the proof.

3. The proof system \(LH(S)\).

(a) Let \(S\) be a set of \(L\)-sentences. Consider a proof system \(LH(S)\) consisting of \(LH\) with additional rules of inference \((A), A \in S\). Show that an \(L-V\)-sentence \(B\) is derivable in \(LH(S)\) if and only if \(B\) is a logical consequence of \(S\).

\textit{Solution.} If \(B\) is derivable in \(LH(S)\), then by a straightforward induction on the length of a derivation of \(B\) in \(LH(S)\), it follows that \(B\) is a logical consequence of \(S\).

For the converse, assume \(B\) is a logical consequence of \(S\). By the Compactness Theorem, there exist \(A_1, \ldots, A_n \in S\) such that \(B\) is a logical consequence of \(A_1, \ldots, A_n\). It follows that

\[ A_1 \Rightarrow (A_2 \Rightarrow \cdots (A_n \Rightarrow B)) \]

is logically valid, hence derivable in \(LH\) (using completeness of \(LH\)), hence derivable in \(LH(S)\), since \(LH(S)\) includes \(LH\). But \(A_1, \ldots, A_n\) are axioms of \(LH(S)\), so by \(n\) application of Modus Ponens we see that \(B\) is derivable in \(LH(S)\). This completes the proof.

(b) Indicate the modifications needed when \(S\) is a set of \(L-V\)-sentences.

\textit{Solution.} Let \(V'\) be a countably infinite set of new parameters, with \(V' \cap V = \emptyset\). The objects of \(LH(S)\) are \(L-V \cup V'\)-sentences. The rules of \(LH(S)\) are as before except that, in the generalization rules, we stipulate that \(a\) does not occur in \(A, B, S\).