1. Let \( L \) be a language consisting of a binary predicate \( R \) and some additional predicates. Let \( M = (U_M, R_M, \ldots) \) be an \( L \)-structure such that \((U_M, R_M)\) is isomorphic to \((\mathbb{N}, <_\mathbb{N})\). Note that \( M \) contains no infinite \( R \)-descending sequence. Show that there exists an \( L \)-structure \( M' \) such that:

(a) \( M \) and \( M' \) satisfy the same \( L \)-sentences.

(b) \( M' \) contains an infinite \( R \)-descending sequence. In other words, there exist elements \( a'_1, a'_2, \ldots, a'_n, \ldots \in U_M' \) such that \( \langle a'_{n+1}, a'_n \rangle \in R_M' \) for all \( n = 1, 2, \ldots \).

Hint: Use the Compactness Theorem.

2. Let \( A \) be a sentence of the predicate calculus with identity. The spectrum of \( A \) is defined to be the set of positive integers \( n \) such that \( A \) is normally satisfiable in a domain of cardinality \( n \). A spectrum is a set \( X \) of positive integers, such that \( X = \text{spectrum}(A) \) for some \( A \).

The spectrum problem is the problem of characterizing the spectra, among all sets of positive integers. This is a famous and apparently difficult open problem. In particular, it is unknown whether the complement of a spectrum is necessarily a spectrum.

Some easy exercises:

(a) Show that if \( X \) is a finite set of positive integers, then \( X \) and the complement of \( X \) are spectra.

(b) Show that the set of even numbers is a spectrum.

(c) Show that the set of odd numbers is a spectrum.

(d) Show that, if \( r \) and \( m \) are positive integers, then

\[ \{ n \geq 1 : n \equiv r \mod m \} \]

is a spectrum.

(e) Show that if \( X \) and \( Y \) are spectra, then \( X \cup Y \) and \( X \cap Y \) are spectra.

3. Let \( A \) be a sentence of the predicate calculus with identity. Assume that \( A \) is normally satisfiable in arbitrarily large finite domains. (In other words, assume that the spectrum of \( A \) is infinite.) Show that \( A \) is normally satisfiable in some infinite domain.

Hint: Use the Compactness Theorem.
4. Let $A$ be a sentence of the predicate calculus with identity. Show that either spectrum$(A)$ or spectrum$(\neg A)$ is cofinite.

5. Let $L$ be the following language:

- $Pxyz$: $x + y = z \mod n$
- $Qxyz$: $x \times y = z \mod n$
- $Rxy$: $x < y$
- $Bx$: $x = 1$ (bottom)
- $Tx$: $x = n$ (top)
- $Nxy$: $x + 1 = y$
- $Ixy$: $x = y$ (identity predicate)

Exhibit a sentence $Z$ such that the finite normal $L$-structures $M$ satisfying $Z$ consist of the integers modulo $n$ for some positive integer $n$, with their usual ordering. In other words, for all finite normal $L$-structures $M$, $M$ satisfies $Z$ if and only if $M \cong Z_n$ for some $n$, where

$$Z_n = (U_n, P_n, Q_n, L_n, B_n, T_n, N_n, I_n)$$

and

- $U_n = \{1, \ldots, n\}$
- $P_n = \{(i, j, k) \in (U_n)^3 : i + j = k \mod n\}$
- $Q_n = \{(i, j, k) \in (U_n)^3 : i \times j = k \mod n\}$
- $R_n = \{(i, j) \in (U_n)^2 : i < j\}$
- $B_n = \{1\}$
- $T_n = \{n\}$
- $N_n = \{(i, j) \in (U_n)^2 : i + 1 = j\}$
- $I_n = \{(i, j) \in (U_M)^2 : i = j\}$

6. (a) Show that the set of squares $\{1, 4, 9, \ldots\}$ and its complement are spectra.
   (b) Show that the set of prime numbers and its complement are spectra.
   Hint: Use the result of Exercise 5 above.

7. Show that $\{2^n : n = 1, 2, 3, \ldots\}$ and its complement are spectra.

8. Let $L$ and $Z_n$ be as in Exercise 5 above. Show that there exists an infinite normal $L$-structure $M = Z_\infty$ with the following property: for all $L$-sentences $A$, if $Z_p$ satisfies $A$ for all sufficiently large primes $p$, then $Z_\infty$ satisfies $A$.
   Hint: Use the Compactness Theorem.