1. Let $L$ be the language consisting of the identity predicate, $I$, and one additional binary predicate, $R$. A graph $G$ is a normal $L$-structure satisfying the $L$-sentences $\forall x \forall y (Rxy \leftrightarrow Ryx)$ and $\forall x \neg Rx x$. Thus

$$G = (U_G, R_G, I_G)$$

where $U_G = \{\text{vertices of } G\}$, $R_G = \{(u, v) \in (U_G)^2 : u \text{ is adjacent to } v\}$, and $I_G = \{(v, v) : v \in U_G\}$.

A graph $G$ is said to be connected if for any pair of vertices $u, v$ in $G$ there exists a path from $u$ to $v$, i.e., a finite sequence of vertices $u = v_0, v_1, \ldots, v_n = v$ such that for each $i < n$, $v_i$ is adjacent to $v_{i+1}$. Here $n$ is the length of the path. The distance between $u$ and $v$ is the minimum length of a path from $u$ to $v$, if such a path exists, and $\infty$ otherwise. The diameter of $G$ is the supremum of the distances between vertices of $G$.

(a) For each positive integer $n$, construct an $L$-sentence $C_n$ such that for all graphs $G$, $G$ satisfies $C_n$ if and only if $G$ is connected of diameter $\leq n$.

(b) Let $S$ be a set of $L$-sentences. Suppose there exist connected graphs of arbitrarily large finite diameter satisfying $S$. Show that there exists a non-connected graph satisfying $S$.

(c) In particular, show that there is no $L$-sentence $C$ such that for all graphs $G$, $G$ satisfies $C$ if and only if $G$ is connected.

Solution.

(a) As $C_n$ we may take $\forall x \forall y (A_0 \lor A_1 \lor \cdots \lor A_n)$ where $A_n$ is the formula

$$\exists x_1 \cdots \exists x_{n-1} (Rx x_1 \& Rx_1 x_2 \& \cdots \& Rx_{n-1} y).$$

Intuitively, $A_n$ says “there exists a path of length $n$ from $x$ to $y$”.

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(b) Let $L^*$ be the language consisting of $L$ plus two additional unary predicates, $P$ and $Q$. Let $S'$ consist of $S$ together with the sentences $\forall x \forall y (Rx \Leftrightarrow Ry)$ and $\forall x \neg Rxx$ and $\exists x Px$ and $\exists y Qy$ and $D_n$, $n = 0, 1, 2, \ldots$. Here $D_n$ is the sentence $\forall x \forall y ((Px \& Qy) \Rightarrow \neg A_n)$. Intuitively, $D_n$ says “there is no path of length $n$ from $P$ to $Q$”. Note that $S'$ is a set of $L^*$-sentences.

We claim that $S'$ is normally satisfiable. By the Compactness Theorem, it suffices to show that each finite subset of $S'$ is normally satisfiable. Let $S''$ be a finite subset of $S'$. Let $n''$ be so large that, for all $n \geq n''$, $D_n$ does not belong to $S''$. By hypothesis, there exists a connected graph of diameter $> n''$ satisfying $S'$. Let $G = (U_G, R_G, I_G)$ be such a graph. Pick two vertices $a$ and $b$ of $G$ whose distance is $> n''$. Define $G'' = (U_G, P_{G''}, Q_{G''}, R_G, I_G)$ where $P_{G''} = \{a\}$ and $Q_{G''} = \{b\}$. Thus $G''$ is a normal $L^*$-structure satisfying $S''$. This proves our claim.

By our claim, let $G^* = (U_{G^*}, P_{G^*}, Q_{G^*}, R_{G^*}, I_{G^*})$ be a normal $L^*$-structure satisfying $S'$. In particular, $(U_{G^*}, R_{G^*}, I_{G^*})$ is a graph and satisfies $S$. Also, since $G^*$ satisfies $\exists x Px$ and $\exists y Qy$, pick vertices $a^* \in P_{G^*}$ and $b^* \in Q_{G^*}$. For each $n$, since $G^*$ satisfies $D_n$, the distance from $a^*$ to $b^*$ is $> n$. Since this holds for all $n$, we see that there is no (finite) path from $a^*$ to $b^*$, i.e., the distance from $a^*$ to $b^*$ is $\infty$. Thus the graph $(U_{G^*}, R_{G^*}, I_{G^*})$ is not connected.

(c) Assume for a contradiction that $C$ is an $L$-sentence with the property that, for all graphs $G$, $G$ satisfies $C$ if and only if $G$ is connected. In particular, by the “if” part of the assumption, there exist finite connected graphs of arbitrarily large diameter satisfying $C$. (For example, consider $n$-cycles, where $n$ is large.) Hence, by part (b) above with $S$ consisting of $C$ alone, there exists a non-connected graph satisfying $C$. This violates the “only if” part of the assumption.

Note: Apropos the result in part (c) above. Using a more sophisticated technique known as Ehrenfeucht-Fraïssé games, one can prove the following stronger result. There is no $L$-sentence $C$ such that, for all finite graphs $G$, $G$ satisfies $C$ if and only if $G$ is connected.
2. Show that the following $L$-$V$-sentences are derivable in $LH$. You may freely use Lemma 3.3.5, which says that $LH$ is closed under quasitautological consequence.

(a) $(\forall x (A \Rightarrow B)) \Rightarrow ((\forall x A) \Rightarrow (\forall x B))$

(b) $(\exists x (A \lor B)) \Leftrightarrow ((\exists x A) \lor (\exists x B))$

(c) $(\exists x A) \Leftrightarrow (\neg \forall x \neg A)$

(d) $(\forall x (A \lor C)) \Leftrightarrow ((\forall x A) \lor C)$

Note: The variable $x$ does not occur freely in $C$. This follows from the fact that (d) is an $L$-$V$-sentence.

**Solutions.**

(a) Let $D$ be sentence $(\forall x (A \Rightarrow B)) \Rightarrow ((\forall x A) \Rightarrow (\forall x B))$. To obtain a derivation of $D$ in $LH$, we follow the method of our proof of completeness of $LH$, via the Companion Theorem.

From the unsigned tableau proof of $D$ (i.e., the obvious finite closed unsigned tableau starting with $\neg D$), we obtain a companion sequence $C_1, C_2, C_3$ for $D$, where

- $C_1 = B[x/a] \Rightarrow (\forall x B)$
- $C_2 = (\forall x A) \Rightarrow A[x/a]$
- $C_3 = (\forall x (A \Rightarrow B)) \Rightarrow (A[x/a] \Rightarrow B[x/a])$

and the parameter $a$ does not occur in $D$ or in $B$. Thus

$$(C_1 \& C_2 \& C_3) \Rightarrow D$$

is a quasitautology. Note also that $C_2$ and $C_3$ are instances of the universal instantiation rule of $LH$. Thus we have the following derivation of $D$ in $LH$.

i. $(C_1 \& C_2 \& C_3) \Rightarrow D$ (quasitautology)

ii. $C_2$ (universal instantiation)

iii. $C_3$ (universal instantiation)

iv. $C_1 \Rightarrow D$ (quasitautological consequence of i, ii, iii)

v. $(\neg D) \Rightarrow B[x/a]$ (quasitautological consequence of iv)

vi. $(\neg D) \Rightarrow (\neg \forall x B)$ (quasitautological consequence of iv)

vii. $(\neg D) \Rightarrow (\forall x B)$ (universal generalization applied to v, noting that $a$ does not occur in the conclusion)

viii. $D$ (quasitautological consequence of vi, vii)

(b) Left to the student.

(c) Left to the student.

(d) Left to the student.
3. Consider the following proof system \( LH' \). (\( LH' \) is a stripped down version of \( LH \).) The objects of \( LH' \) are \( L \)-\( V \)-sentences containing only \( \forall, \Rightarrow, F \) (i.e., not containing \( \exists, \Leftrightarrow, \&\), \( \lor, \neg, T \)). The rules of inference of \( LH' \) are:

(a) quasitautologies
(b) \((\forall x \ B) \Rightarrow B[x/a]\)
(c) \((\forall x (A \Rightarrow B)) \Rightarrow (A \Rightarrow \forall x B)\)
(d) \( \frac{A \ A \Rightarrow B}{B} \) (modus ponens)
(e) \( \frac{B[x/a]}{\forall x B} \) (generalization), where \( a \) does not occur in \( B \).

Show that \( LH' \) is sound and complete.

\textit{Solution.}

Soundness is proved just as for \( LH \).

Completeness.

Just as for the full tableau method, we can prove soundness and completeness of the restricted tableau method with \( \forall, \Rightarrow, F \), and from this we obtain the restricted Companion Theorem. In this context there are only two kinds of companions, the ones involving \( \forall \). It remains to prove the following lemma: If \( C \) is a companion of \( A \), and if \( C \Rightarrow A \) is derivable in \( LH' \), then \( A \) is derivable in \( LH' \).

We deal only with companions of the form \( B[x/a] \Rightarrow (\forall x B) \). Assume that \((B[x/a] \Rightarrow (\forall x B)) \Rightarrow A \) is derivable in \( LH' \), where \( a \) does not occur in \( A, B \). It follows quasitautologically that both (i) \((\neg A) \Rightarrow B[x/a]\) and (ii) \((\neg A) \Rightarrow \neg \forall x B \) are derivable in \( LH' \). From (i) and the generalization rule (e) of \( LH' \), we see that \( \forall x ((\neg A) \Rightarrow B) \) is derivable in \( LH' \). Also, by rule (c) of \( LH' \), \((\forall x ((\neg A) \Rightarrow B)) \Rightarrow ((\neg A) \Rightarrow \forall x B) \) is derivable in \( LH' \). Hence, by modus ponens, \((\neg A) \Rightarrow \forall x B \) is derivable in \( LH' \). It follows quasitautologically from this and (ii) that \( A \) is derivable in \( LH' \). This completes the proof.
4. (a) Let $S$ be a set of $L$-sentences. Consider the proof system $LH(S)$ consisting of $LH$ with additional rules of inference $(A)$, $A \in S$. Show that an $L$-V-sentence $B$ is derivable in $LH(S)$ if and only if $B$ is a logical consequence of $S$.

(b) Indicate the modifications needed when $S$ is a set of $L$-V-sentences.

Notation: We write $S \vdash B$ to indicate that $B$ is derivable in $LH(S)$.

Solution.

(a) By induction on the length of derivations, it is straightforward to prove that each sentence derivable in $LH(S)$ is a logical consequence of $S$. The assumption that $S$ is a set of $L$-sentences (not $L$-V-sentences) is used in the inductive steps corresponding to rules 4(a) and 4(b), universal and existential generalization, because we need to know that the parameter $a$ does not occur in $S$.

Conversely, assume $B$ is a logical consequence of $S$. By the Compactness Theorem, it follows that $B$ is a logical consequence of a finite subset of $S$, say $A_1, \ldots, A_n$. Hence $(A_1 \& \cdots \& A_n) \Rightarrow B$ is logically valid. Hence, by completeness of $LH$, $(A_1 \& \cdots \& A_n) \Rightarrow B$ is derivable in $LH$. Since $LH(S)$ includes $LH$, we have that

$$(A_1 \& \cdots \& A_n) \Rightarrow B$$

is derivable in $LH(S)$. But $A_1, \ldots, A_n$ are derivable in $LH(S)$. It follows quasitautologically that $B$ is derivable in $LH(S)$. This completes the proof.

(b) If $S$ is a set of $L$-V-sentences, we need to modify our system as follows.

Let $V'$ be a countably infinite set of new parameters, disjoint from $V$. Define $LH(S)$ as before, but allowing parameters from $V \cup V'$. The objects are $L$-V$\cup V'$-sentences. In rules 4(a) and 4(b), one must impose the restriction that $a$ does not occur in $A, B, S$. With this modification, everything goes through as before.