1. Let $f$ and $g$ be Turing oracles. Define $f \leq_{LK} g$ to mean that

$$K^g(\tau) \leq K^f(\tau) + O(1)$$

for all bitstrings $\tau$. Define $f \leq_{LR} g$ to mean that

$$(\forall X \in 2^N) \text{ (if } X \text{ is } g\text{-random then } X \text{ is } f\text{-random).}$$

(a) Show that $f \leq_T g$ implies both $f \leq_{LK} g$ and $f \leq_{LR} g$.

(b) Let $X \in 2^N$ be such that $X \leq_{LK} 0$. Show that $X$ is $K$-trivial, i.e.,

$$K(X \upharpoonright n) \leq K(n) + O(1) \text{ for all } n.$$ 

Note: It can be shown that the properties $f \leq_{LK} g$ and $f \leq_{LR} g$ are equivalent to each other. However, they are not equivalent to $f \leq_T g$. In fact, we can find a nonrecursive $X \in 2^N$ such that $X \leq_{LK} 0$. It can be shown that $X \leq_{LK} 0$ if and only if $X$ is $K$-trivial.

2. For convenience in stating this problem, let us identify subsets of $\mathbb{N}$ with their characteristic functions. In other words, we identify $A \subseteq \mathbb{N}$ with $\chi_A \in 2^N$. Thus $2^N$ is the set of all subsets of $\mathbb{N}$.

Let $J : 2^N \rightarrow 2^N$ be the Turing jump operator:

$$J(X) = X' = H^X = \text{the Halting Problem relative to } X.$$ 

Recall that $0^{(1)} = 0' = J(0)$ and in general $0^{(n+1)} = (0^{(n)})' = J(0^{(n)})$ for all $n$. By Post’s Theorem we know that for each $n \geq 1$ the set $0^{(n)}$ is $\Sigma^0_n$ and not $\Delta^0_n$. Define

$$0^{(\omega)} = \bigoplus_{n=1}^{\infty} 0^{(n)} = \{3^m5^n | m \in 0^{(n)}\}.$$ 

Note that the set $0^{(\omega)}$ is not arithmetical, i.e., it is not $\Delta^0_n$ for any $n$. 


(a) Show that the 2-place predicate $P \subseteq 2^\mathbb{N} \times 2^\mathbb{N}$ given by

$$P(X,Y) \equiv J(X) = Y$$

is $\Pi^0_2$.

(b) Show that for each $n \geq 1$ the singleton set $\{0^{(n)}\}$ is $\Pi^0_2$.

(c) Show that the singleton set $\{0^{(\omega)}\}$ is $\Pi^0_2$.

Note: These singleton sets are subsets of $2^\mathbb{N}$.

3. (a) Show that every nonempty $\Pi^0_1$ subset of $2^\mathbb{N}$ contains a member which is $\Delta^0_n$ for some $n$.

(b) In part (a), what is the optimal value of $n$?

(c) In parts (a) and (b), what if we replace $\Pi^0_1$ sets by $\Pi^0_2$ sets?

(d) Is every $\Pi^0_2$ subset of $2^\mathbb{N}$ Turing isomorphic to a $\Pi^0_1$ subset of $2^\mathbb{N}$?

4. Let $X \in 2^\mathbb{N}$. We say that $X$ is 2-random if $X$ is random relative to $0'$. Recall also that $X$ is weakly 2-random if $X \notin \Pi^0_2$ set of measure 0. Let $a = \deg_T(X) = \text{the Turing degree of } X$.

(a) Show that if $X$ is 2-random then $X$ is weakly 2-random.

(b) Show that if $X$ is weakly 2-random then $\inf(a, 0') = 0$.

(c) In part (b) what if we assume only that $X$ is random?

(d) Show that if $X$ is 2-random then $\sup(a, 0') = a'$.

(e) In part (d) what if we assume only that $X$ is weakly 2-random?

5. Show that every $\Pi^0_2$ subset of $2^\mathbb{N}$ includes a $\Sigma^0_2,0'$ set of the same measure.