1. Hoeffding’s Inequality says that the probability space $2^\mathbb{N}$ with the fair coin probability measure satisfies

$$\operatorname{Prob}\left(\left|\frac{\sum_{i=0}^{n-1} X(i)}{n} - \frac{1}{2}\right| > \epsilon\right) < \frac{2}{\exp(2n\epsilon^2)}.$$ 

Use Hoeffding’s Inequality to prove that if a point $X \in 2^\mathbb{N}$ is random (i.e., random in the sense of Martin-Löf), then $X$ obeys the Strong Law of Large Numbers:

$$\frac{\sum_{i=0}^{n-1} X(i)}{n} \to \frac{1}{2} \quad \text{as} \quad n \to \infty.$$ 

2. Prove that there exist weakly 1-random points in $2^\mathbb{N}$ which do not obey the Strong Law of Large Numbers.

Hint: Use finite approximation.

3. In problem 1, can you say anything about the rate of convergence to $1/2$?

4. Prove that if $X \oplus Y \in 2^\mathbb{N}$ is random (i.e., random in the sense of Martin-Löf), then $X \not\leq_T Y$ and $Y \not\leq_T X$.

5. Prove that there exist points $X, Y \in 2^\mathbb{N}$ such that $X \oplus Y$ is weakly 1-random yet $X \equiv_T Y$.

6. A set $B \subseteq \mathbb{N}$ is said to be biimmune if both $B$ and its complement $\mathbb{N} \setminus B$ are immune. Prove that if $X \in 2^\mathbb{N}$ is weakly 1-random then $X$ is the characteristic function of a biimmune set.
7. Let $f$ be a Turing oracle.

For each $i \in \mathbb{N}$ define

$$U^f_i = \{ X \in 2^\mathbb{N} \mid \varphi^{(1)}_i, f \oplus X(0) \downarrow \}.$$ 

Thus $U^f_i$, $i = 0, 1, 2, \ldots$ is the standard recursive enumeration of all $\Sigma^0_{1,f}$ subsets of $2^\mathbb{N}$.

Given a sequence of sets $V_n \subseteq 2^\mathbb{N}$, $n = 0, 1, 2, \ldots$, prove that the following are pairwise equivalent.

(a) There exists a total recursive function $g$ such that $V_n = U^f_{g(n)}$ for all $n$.

(b) There exists a total $f$-recursive function $h$ such that $V_n = U^f_{h(n)}$ for all $n$.

(c) The predicate $P \subseteq 2^n \times \mathbb{N}$ given by

$$P(X, n) \equiv X \in V_n$$

is $\Sigma^0_{1,f}$.

In this case we say that the sequence of sets $V_n$, $n = 0, 1, 2, \ldots$ is uniformly $\Sigma^0_{1,f}$ or uniformly $\Sigma^0_1$ relative to $f$.

Note: This concept will be part of the definition of what it means for a point $X \in 2^\mathbb{N}$ to be random relative to the oracle $f$. 