1. Let $K(\tau)$ denote the prefix-free complexity of a bitstring $\tau$. Prove that
   \[ K(\tau_1 \natural \tau_2) \leq K(\tau_1) + K(\tau_2) + O(1). \]

2. (a) Give an example of a subset of $\mathbb{N}^\mathbb{N}$ which is $\Sigma^0_2$ but not $\Sigma^0_1$.0'.
   (b) Can you replace $\mathbb{N}^\mathbb{N}$ by $2^{\mathbb{N}}$ here?

   Note: Recall Post’s Theorem, which says (among other things) that a subset of $\mathbb{N}$ is $\Sigma^0_2$ if and only if it is $\Sigma^0_1.0'. The point of (a) is to show that Post’s Theorem does not hold for subsets of $\mathbb{N}^\mathbb{N}$.

   Hint: Recall that a set is open if and only if it is $\Sigma^0_1$ relative to an oracle. Therefore, it suffices to find a set which is $\Sigma^0_2$ and not open.

3. A real number is said to be left recursively enumerable (respectively right recursively enumerable) if it is the limit of an increasing (respectively decreasing) recursive sequence of rational numbers.
   (a) If $A$ is a recursively enumerable subset of $\mathbb{N}$, show that the real number $\sum_{n \in A} 1/2^n$ is left recursively enumerable.
   (b) Show that there exist real numbers which are left recursively enumerable but not recursive.
   (c) Show that a real number is recursive if and only if it is both left recursively enumerable and right recursively enumerable.

4. Let $P$ be a $\Pi^0_1$ subset of $2^{\mathbb{N}}$. We have seen how to construct a recursive tree $T \subseteq 2^{\leq \mathbb{N}}$ such that $P = \{\text{paths through } T\}$. For each $n = 0, 1, 2, \ldots$ let $T_n$ be the set of strings in $T$ of length $n$.
   (a) Show that $T_n$ is prefix-free.
(b) Show that the set
\[ V_n = \bigcup_{\tau \in T_n} N_\tau \]
is $\Delta_1^0$. (Note that $V_n$ is a subset of $2^N$.)
(c) Show that $P$ is the intersection of the $V_n$’s. In other words,
\[ P = \bigcap_{n=0}^{\infty} V_n. \]
(d) Show that the measure of $P$ is given by
\[ \mu(P) = \lim_{n \to \infty} \frac{|T_n|}{2^n}. \]
Here $|T_n|$ denotes the number of strings in $T_n$.
(e) Show that the real number $\mu(P)$ is right recursively enumerable.
(f) Show that $\mu(P)$ is not necessarily a recursive real number.

5. Given a nonempty $\Pi_1^0$ set $P \subseteq 2^N$, can we always find a member of $P$ which is recursive?

Hint: Consider a recursively inseparable pair of r.e. sets.

6. Two sets $P, Q \subseteq \mathbb{N}$ are said to be Turing isomorphic if the members of $P$ and $Q$ have the same Turing degrees, i.e.,
\[ \{\deg_T(f) \mid f \in P\} = \{\deg_T(g) \mid g \in Q\}. \]
(a) Prove that every $\Pi_2^0$ subset of $\mathbb{N}$ is Turing isomorphic to a $\Pi_1^0$ subset of $\mathbb{N}$.
(b) Prove that every $\Pi_2^0$ subset of $\mathbb{N}$ is Turing isomorphic to a $\Pi_2^0$ subset of $2^\mathbb{N}$.
(c) Is every $\Pi_2^0$ subset of $2^\mathbb{N}$ Turing isomorphic to a $\Pi_1^0$ subset of $2^\mathbb{N}$?

Justify your answer.

Hints: (a) If $\forall x \exists y R(f, x, y)$ holds, map $f$ to $f \oplus g$ where $g(x) = \mu y R(f, x, y)$.
(b) Map $f$ to the characteristic function of the set $G_f = \{3^x5^y \mid f(x) = y\} = \text{the “graph” of } f.$