Recall that $W_x = \text{dom}(\varphi_x^{(1)})$. Note that $W_x$, $x = 0, 1, 2, \ldots$, is the standard recursive enumeration of the recursively enumerable subsets of $\mathbb{N}$.

1. Which many-one reducibility relations hold or do not hold among the following sets and their complements?

   - $K = \{x \mid x \in W_x\}$
   - $H = \{x \mid 0 \in W_x\}$
   - $T = \{x \mid W_x = \mathbb{N}\}$
   - $E = \{x \mid W_x = \emptyset\}$
   - $S = \{x \mid W_x \text{ is infinite}\}$

   Prove your answers.

   Hint: Each of these sets is many-one complete within an appropriate level of the arithmetical hierarchy.

2. A set $P \subseteq \mathbb{N}$ is said to be productive if there exists a total recursive function $h(x)$ such that for all $x$, if $W_x \subseteq P$ then $h(x) \notin W_x$ and $h(x) \in P$. Such a function is called a productive function for $P$.

   A creative set is a recursively enumerable set whose complement is productive.

   Prove the following.

   (a) $K$ is creative.

   (b) If $A$ and $B$ are recursively enumerable sets and $A \leq_m B$ and $A$ is creative, then $B$ is creative.

   (c) If $B$ is recursively enumerable and many-one complete, then $B$ is creative.
(d) (Extra Credit) If $B$ is creative, then $B$ is many-one complete.

(e) (Extra Credit) If $A$ and $B$ are creative, then $A$ and $B$ are recursively isomorphic. This means that there exists a recursive permutation of $\mathbb{N}$, call it $g$, such that $x \in A$ if and only if $g(x) \in B$, for all $x$.

3. A set $I \subseteq \mathbb{N}$ is said to be immune if $I$ is infinite yet includes no infinite recursively enumerable set.

A simple set is a recursively enumerable set whose complement is immune.

Prove the following.

(a) If $A$ is simple, then $A$ is not recursive.

(b) If $A$ is simple, then $A$ is not creative.

4. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one total recursive function such that the range of $f$ is nonrecursive. The deficiency set of $f$ is defined as

$$D_f = \{ x \mid \exists y \; (x < y \land f(x) > f(y)) \}.$$ 

Prove that $D_f$ is a simple set.

Conclude that there exist recursively enumerable sets which are neither recursive nor many-one complete.

5. (Extra Credit) Generalize Exercises 2, 3, 4 to higher levels of the arithmetical hierarchy. Conclude that for each $n \geq 1$ there exist $\Sigma^0_n$ sets which are neither $\Delta^0_n$ nor many-one complete within the class of $\Sigma^0_n$ sets.