Math 312, Intro. to Real Analysis:
Midterm Exam #1 Solutions

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1. True or False (3 points each)

(a) Every ordered field has the Archimedean property.
    Answer: False.

(b) The ordered field axioms imply $|a - b| \leq |a| + |b|$ for all $a, b$.
    Answer: True.

(c) If $\lim a_n = -\infty$ then $\lim \sup a_n = -\infty$.
    Answer: True.

(d) For any sequence of real numbers, the lim inf and the lim sup always exist and furthermore the lim inf is always $\leq$ the lim sup.
    Answer: True.

(e) The equation $3x^3 + 2x^2 + 3x + 2 = 0$ has a rational solution.
    Answer: True. A rational solution is $x = -2/3$.

(f) $\sqrt[3]{216}$ is an irrational number.

(g) The limit of a convergent sequence of negative numbers is negative.
    Answer: False. For example, the sequence $-1/n$ for $n = 1, 2, 3, \ldots$ has 0 as its limit.

(h) The limit of a convergent sequence of rational numbers is rational.
    Answer: False.

(i) Every interval contains at least three rational numbers.
    Answer: True.

(j) Every bounded sequence of real numbers is convergent.
    Answer: False.

(k) Every convergent sequence of real numbers is bounded.
    Answer: True.

(l) Every monotone sequence of real numbers is convergent.
    Answer: False. For example, the sequence 1, 2, 3, \ldots is divergent to $\infty$. 


(m) If \((a_n)\) is a monotone sequence of real numbers, then \(\lim a_n\) exists and belongs to the interval \((-\infty, \infty)\).

**Answer:** False. Same example as above.

2. (7 points each)

(a) Give an example of a sequence of real numbers such that

\[-\infty < \inf a_n < \lim a_n < \sup a_n < \infty.\]

**Answer:** An example is \(a_n = \frac{(-1)^n}{n}\). The inf is \(-1\), the sup is \(1/2\), and the limit is 0.

(b) Give an example of a sequence of real numbers such that

\[\lim \sup a_n, \lim \inf a_n, \sup a_n, \inf a_n\]

are four distinct real numbers.

**Answer:** An example is \(a_n = (-1)^n \left(1 + \frac{1}{n}\right)\). The inf is \(-2\), the sup is \(1 + 1/2\), the lim inf and lim sup are \(-1\) and 1.

(c) Give an example of a sequence of real numbers such that

\[\lim \inf a_n = -\infty \quad \text{and} \quad \lim \sup a_n = \sqrt{2}.\]

**Answer:** An example is \(-1, \sqrt{2}, -2, \sqrt{2}, -3, \sqrt{2}, -4, \ldots\). In other words, \(a_{2n} = \sqrt{2}\) and \(a_{2n-1} = -n\) for all \(n\).

3. (8 points) It can be shown that \(\sqrt{1 + \sqrt{5}}\) is an algebraic number, i.e., it is a solution of some polynomial equation with integer coefficients. Find such an equation.

**Answer:** Let \(\alpha = \sqrt{1 + \sqrt{5}}\). Then \(\alpha^3 = 1 + \sqrt{5}\), i.e., \(\alpha^3 - 1 = \sqrt{5}\) hence \(\alpha^6 - 2\alpha^3 + 1 = 5\), i.e., \(\alpha^6 - 2\alpha^3 - 4 = 0\). The desired equation is \(x^6 - 2x^3 + 4 = 0\).

4. (8 points) Find all candidates for rational solutions of the equation

\[2x^2 - ax + 5 = 0\]

where \(a\) is an unspecified integer.

**Answer:** The candidates are \(\pm 5, \pm \frac{5}{2}, \pm 1, \pm \frac{1}{2}\). This is according to the Rational Zeros Theorem, page 9 in the Ross textbook.

5. (12 points) Use algebra plus limit laws to calculate

\[\lim \frac{\sqrt{2n^2 + 5n}}{n + 4}.\]
Answer:

\[
\lim \frac{\sqrt{2n^2 + 5n}}{n+4} = \lim \frac{\sqrt{\frac{2+5}{n} \frac{1}{4}}}{\lim \frac{1+\frac{4}{n}}{n}} = \frac{\sqrt{\frac{2+5}{n}}}{\frac{1+\frac{4}{n}}{n}} = \frac{\sqrt{2} + 0}{1 + 0} = \sqrt{2}.
\]

6. (12 points) It can be shown that

\[
\lim \frac{\sqrt{n} - 5001}{\sqrt{n} - 1001} = 1.
\]

Given \( \varepsilon > 0 \), find an \( N \) so large that

\[
\left| \frac{\sqrt{n} - 5001}{\sqrt{n} - 1001} - 1 \right| < \varepsilon
\]

holds for all \( n > N \).

Answer: Given \( \varepsilon > 0 \), we want

\[
\left| \frac{\sqrt{n} - 5001}{\sqrt{n} - 1001} \right| < \varepsilon
\]

and by algebra this is equivalent to saying that

\[
\left| \frac{-4001}{\sqrt{n} - 1001} \right| < \varepsilon.
\]

If \( n > 1001^3 \) then \( \sqrt{n} - 1001 > 0 \), and \( | -4001| = 4001 \), so the above is equivalent to saying that

\[
\frac{4001}{\sqrt{n} - 1001} < \varepsilon.
\]

By algebra, this is equivalent to

\[
\frac{4001}{\varepsilon} + 1001 < \sqrt{n}
\]

i.e.,

\[
n > \left( \frac{4001}{\varepsilon} + 1001 \right)^3.
\]

Thus we may take

\[
N = \left( \frac{4001}{\varepsilon} + 1001 \right)^3.
\]