Richard Elwes, Math in 100 Key Breakthroughs, Quercus Press, New York, 2013, 416 pages.

# math

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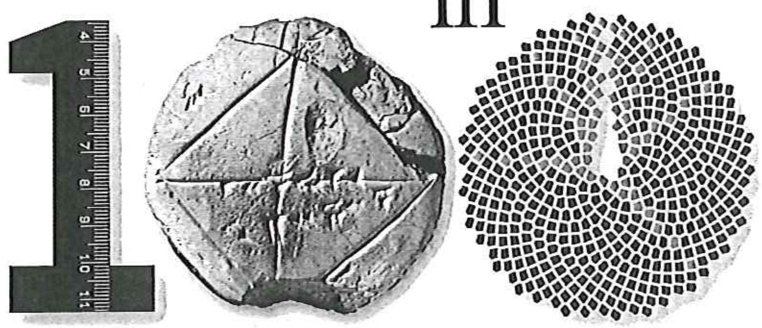

# key breakthroughs

richard elwes

Quercus

# Contents

# 98 Reverse mathematics

**BREAKTHROUGH** REVERSE MATHEMATICIANS INVESTIGATE HOW THE FACTS AND THEOREMS OF MAINSTREAM MATHEMATICS RELATE TO THE UNDERLYING LOGICAL SYSTEMS BEING USED.

**DISCOVERER** GERHARD GENTZEN (1909–45), HARVEY FRIEDMAN (1948–), STEPHEN SIMPSON (1948–).

**LEGACY** FRIEDMAN'S SEEMINGLY SIMPLE STATEMENTS ABOUT WHOLE NUMBERS WHICH REQUIRE THE STRONGEST LOGICAL AXIOMS THREATENS TO USHER UNPROVABILITY IN TO THE HEART OF MATHEMATICS.

Gödel's incompleteness theorems (see page 277) turned the world of mathematical logic upside-down. It meant that the long-held goal of a single set of laws underlying the arithmetic of the whole numbers had to be abandoned. In the years since, proof theorists have compared the strengths of different logical laws for arithmetic, while reverse mathematicians have asked which laws are required to prove specific theorems. One of the most spectacular discoveries involves statements that are not provable within any of the usual logical frameworks.

The standard laws for describing the whole numbers are those of Peano arithmetic, named after its inventor, the 19th-century logician Giuseppe Peano. This is a list of seven simple axioms, which largely seem to be statements of the obvious. For example, if $x$ and $y$ are unequal numbers, then so are $x + 1$ and $y + 1$. It had once been hoped that Peano's laws might provide a solid underpinning for the whole of mathematics. But in 1931, Gödel's incompleteness theorem killed that idea stone dead. Despite being incomplete, it seemed adequate for most practical purposes.

Even if it was technically incomplete, one might at least hope that Peano arithmetic should be consistent, meaning that it doesn't contain any hidden contradictions. If it is to be any use at all, then following the rules of Peano arithmetic should never produce a nonsensical outcome such as $1 + 2 = 4$.

**OPPOSITE** A reverse mathematician aims to reconstruct a logical framework by examining its mathematical consequences. Just as the patterns within a stained-glass window produce a complex interplay of colored light, so the relationship between the logical laws of numbers and the theorems that mathematicians prove about them is far from straightforward.