This exam has 12 questions for a total of 100 points. **In order to obtain full credit for partial credit problems, all work must be shown. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work.** The point value for each question is in parentheses to the right of the question number. A table of Laplace transforms is attached as the last page of the exam.

**Please turn off and put away your cell phone.**

**You may not use a calculator on this exam.**

Do not write in this box.

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<td>1 through 8:</td>
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1. (5 points) A certain sixth order constant coefficient homogeneous linear equation has a characteristic equation \((r - 1)(r + 1)^2(r - 2)^3 = 0\). Which of the functions below is NOT a solution of this equation?

(a) \(y = e^t + 2te^{-t} + 3e^{-t} + 4t^2e^{2t}\)

(b) \(y = 2e^t + 3te^{-t} + 4e^{2t} + 5t^2e^{2t}\)

(c) \(y = 3e^{-t} + 4te^{-t} + 5te^{2t}\)

(d) \(y = 4e^t + 5te^{-t} + 6t^2e^{-t} - 7te^{2t}\)

2. (5 points) Let \(y(t)\) be the solution of the initial value problem

\[y''' + 4y'' + 4y' = 6e^{-2t}, \quad y(0) = -1, \quad y'(0) = 0, \quad y''(0) = 2.\]

Find its Laplace transform \(Y(s) = \mathcal{L}\{y(t)\}\).

(a) \(Y(s) = \frac{6}{s(s + 2)^3} - \frac{s^2 + 4s + 2}{s(s + 2)^2}\)

(b) \(Y(s) = \frac{6}{s(s + 2)^3} + \frac{s^2 + 4s + 6}{s(s + 2)^2}\)

(c) \(Y(s) = \frac{6}{(s + 2)^3} - \frac{s + 4}{(s + 2)^2}\)

(d) \(Y(s) = \frac{6}{(s + 2)} + \frac{2s^2 + 8s - 1}{s(s + 2)^2}\)
3. (5 points) Find the Laplace transform $\mathcal{L}\{u_2(t)t^2 e^{t-1}\}$.

(a) $F(s) = e^{-2s-3} \frac{2 - 4(s - 1) + 4(s - 1)^2}{(s - 1)^3}$

(b) $F(s) = e^{-2s} \frac{2 - 2(s - 1)^2}{(s - 1)^3}$

(c) $F(s) = e^{-2s-1} \frac{2}{s^3(s - 1)}$

(d) $F(s) = e^{-2s+1} \frac{2 + 4(s - 1) + 4(s - 1)^2}{(s - 1)^3}$

4. (5 points) Find the inverse Laplace transform of $F(s) = \frac{3s - 4}{s^2 - 2s + 5}$.

(a) $f(t) = 3e^t \cos(2t) - 2e^t \sin(2t)$

(b) $f(t) = 3e^t \cos(2t) - \frac{1}{2}e^t \sin(2t)$

(c) $f(t) = 3e^t \cos(2t) - \frac{7}{2}e^t \sin(2t)$

(d) $f(t) = 3e^t \cos(2t) - 4e^t \sin(2t)$
5. (5 points) A mass-spring system is initially resting at its equilibrium position. Starting at \( t = 0 \) an upward force of magnitude \( 5 \sin(2t) \) is applied to the system, until \( t = 15 \) when it is stopped. Between times \( t = 3 \) and \( t = 6 \) an additional constant external downward force of 20 Newtons is also applied. At \( t = 8 \) seconds the mass is struck with a hammer in such a fashion that an upward momentum of magnitude 50 kg-m/s is introduced to the system at that time. Given that the downward direction is positive, which of the following \( g(t) \) represents the combination of these forces?

- (a) \( g(t) = -5 \sin(2t) - 20 u(t) + 20 u(t) - 50 \delta(t - 8) + 5 u(t) \sin(2t) \)
- (b) \( g(t) = -5 \sin(2t) - 20 u(t) + 20 u(t) - 50 \delta(t - 8) + 5 u(t) \sin(2t) \)
- (c) \( g(t) = 5 \sin(2t) + 20 u(t) - 20 u(t) - 50 u(t) - 5 u(t) \sin(2t) \)
- (d) \( g(t) = 5 \sin(2t) - 20 u(t) + 20 u(t) + 50 \delta(t - 8) - 5 u(t) \sin(2t) \)

6. (5 points) Which system of first order equations below is equivalent to the following linear equation?

\[ y'' + 4y' - 4y = 0 \]

- (a) \[ \begin{aligned} x' &= -x_2 \\ x_2 &= 4x_1 + 4x_2 \end{aligned} \]
- (b) \[ \begin{aligned} x' &= -x_2 \\ x_2 &= 4x_1 - 4x_2 \end{aligned} \]
- (c) \[ \begin{aligned} x' &= -x_2 \\ x_2 &= -4x_1 + 4x_2 \end{aligned} \]
- (d) \[ \begin{aligned} x' &= -x_2 \\ x_2 &= -4x_1 - 4x_2 \end{aligned} \]
7. (5 points) Consider a certain system of two first order linear differential equations in two unknowns, $x' = Ax$, where $A$ is a matrix of real numbers. Suppose one of the eigenvalues of the coefficient matrix $A$ is $r = 3 - 2i$, which has a corresponding eigenvector $\begin{bmatrix} 1 \\ 4 + i \end{bmatrix}$. What is the system’s real-valued general solution?

(a) $C_1 e^{3t} \begin{bmatrix} \cos 2t \\ 4 \cos 2t - \sin 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t \\ 4 \sin 2t - \cos 2t \end{bmatrix}$

(b) $C_1 e^{3t} \begin{bmatrix} \cos 2t \\ 4 \cos 2t + \sin 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t \\ 4 \sin 2t + \cos 2t \end{bmatrix}$

(c) $C_1 e^{3t} \begin{bmatrix} \cos 2t \\ 4 \cos 2t + \sin 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t \\ 4 \sin 2t - \cos 2t \end{bmatrix}$

(d) $C_1 e^{3t} \begin{bmatrix} \cos 2t \\ 4 \cos 2t - \sin 2t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \sin 2t \\ 4 \sin 2t + \cos 2t \end{bmatrix}$

8. (5 points) Consider a certain $2 \times 2$ linear system $x' = Ax$, where $A$ is a matrix of real numbers. Suppose at least one of its nonzero solutions remain bounded as $t \to +\infty$. Which of the following is NOT a possible pair of eigenvalues of the coefficient matrix $A$?

(a) $\pm \sqrt{7}i$

(b) $1 \pm 2i$

(c) $-\sqrt{3}, -\sqrt{3}$

(d) $4, -4$
9. (18 points) Suppose $\mathcal{L}\{f(t)\} = \frac{s^3}{s^5 - 10}$, and that $f(0) = 2$, $f(1) = -4$, $f(2) = -1$, $f'(0) = 3$.

Answer each question below.

(a) Determine $\mathcal{L}\{t f(t)\}$.

(b) Determine $\mathcal{L}\{f''(t)\}$.

(c) Determine $\mathcal{L}\{-tf'(t)\}$.

(d) Determine $\mathcal{L}\{e^{6t} f(t)\}$.

(e) Determine $\mathcal{L}\{\delta(t - 1)(te^{2(t-1)} - 2f(t))\}$.

(f) Let $g(t) = \mathcal{L}^{-1}\left\{e^{-4s} \frac{2s^3}{s^5 - 10}\right\}$. Evaluate $g(2)$. 
10. (14 points) In parts (a) through (e), determine the type and stability of the critical point at \((0, 0)\) for each of the \(2 \times 2\) linear systems \(\mathbf{x}' = \mathbf{A}\mathbf{x}\) whose general solutions are given below. For the type, give the actual name. For the stability, use the letter \(A\) if the point is asymptotically stable, \(U\) if it is unstable, \(S\) if it is (neutrally) stable.

<table>
<thead>
<tr>
<th>Type</th>
<th>Stability</th>
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<tbody>
<tr>
<td>(a) (C_1 e^{-5t} \begin{bmatrix} 3 \ -1 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} 1 \ 4 \end{bmatrix})</td>
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<td>(b) (C_1 \begin{bmatrix} -3 \cos \pi t \ 4 \sin \pi t \end{bmatrix} + C_2 \begin{bmatrix} 3 \sin \pi t \ 4 \cos \pi t \end{bmatrix})</td>
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<tr>
<td>(c) (C_1 e^{-4t} \begin{bmatrix} 6 \cos 2t \ 5 \sin 2t \end{bmatrix} + C_2 e^{-4t} \begin{bmatrix} -5 \sin 2t \ 6 \cos 2t \end{bmatrix})</td>
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<tr>
<td>(d) (C_1 e^{(5+\sqrt{3})t} \begin{bmatrix} 1 \ 1 \end{bmatrix} + C_2 e^{(5-\sqrt{3})t} \begin{bmatrix} 2 \ -1 \end{bmatrix})</td>
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<tr>
<td>(e) (C_1 e^{\sqrt{10}t} \begin{bmatrix} 1 \ 2 \end{bmatrix} + C_2 e^{\sqrt{10}t} \begin{bmatrix} 1 \ -2 \end{bmatrix})</td>
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(f) One of the solutions above is quasi-periodic, what is its quasi-frequency?

(g) Which solution above represents a system that is equivalent to the second order equation \(y'' - 10y' + 22y = 0\)?
11. (16 points) Use the Laplace transform to solve the following initial value problem.

\[ y'' + 5y' + 6y = 3\delta(t - 6) - u_{2\pi}(t)e^{t - 2\pi}, \quad y(0) = 2, \quad y'(0) = 1. \]
12. (12 points)
   (a) (7 points) Find the general solution of the system of linear equations

   \[ x' = \begin{bmatrix} -6 & 3 \\ -3 & 0 \end{bmatrix} x. \]

   (b) (3 points) Find the solution satisfying \( x(-201) = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \). Completely simplify your answer into a single vector.

   (c) (2 points) Classify the type and stability of the critical point at \((0, 0)\).
<table>
<thead>
<tr>
<th></th>
<th>$f(t) = \mathcal{L}^{-1}{F(s)}$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
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<tbody>
<tr>
<td>1</td>
<td>$1$</td>
<td>$\frac{1}{s}$</td>
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<tr>
<td>2</td>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$</td>
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<td>3</td>
<td>$t^n, \quad n = \text{positive integer}$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
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<td>4</td>
<td>$t^p, \quad p &gt; -1$</td>
<td>$\frac{\Gamma(p+1)}{s^{p+1}}$</td>
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<td>5</td>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
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<tr>
<td>6</td>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
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<td>7</td>
<td>$\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$</td>
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<td>8</td>
<td>$\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$</td>
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<td>9</td>
<td>$e^{at} \sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$</td>
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<td>10</td>
<td>$e^{at} \cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$</td>
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<td>11</td>
<td>$t^n e^{at}, \quad n = \text{positive integer}$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$</td>
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<tr>
<td>12</td>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$</td>
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<td>13</td>
<td>$u_c(t)f(t-c)$</td>
<td>$e^{-cs} F(s)$</td>
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<td>14</td>
<td>$e^{ct} f(t)$</td>
<td>$F(s-c)$</td>
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<td>15</td>
<td>$f(ct)$</td>
<td>$\frac{1}{c} F\left(\frac{s}{c}\right)$</td>
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<tr>
<td>16</td>
<td>$(f * g)(t) = \int_0^t f(t-\tau)g(\tau) , d\tau$</td>
<td>$F(s)G(s)$</td>
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<td>17</td>
<td>$\delta(t-c)$</td>
<td>$e^{-cs}$</td>
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<td>18</td>
<td>$f^{(n)}(t)$</td>
<td>$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$</td>
</tr>
<tr>
<td>19</td>
<td>$(-t)^n f(t)$</td>
<td>$F^{(n)}(s)$</td>
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