This exam has 13 questions for a total of 100 points. Show all your work! **In order to obtain full credit for partial credit problems, all work must be shown.** Credit will not be given for an answer not supported by work. For other problems, points might be deducted, at the sole discretion of the instructor, for an answer not supported by a reasonable amount of work. The point value for each question is in parentheses to the right of the question number.

**You may not use a calculator on this exam. Please turn off and put away your cell phone.**

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1. (5 points) Consider the equation

\[(y'')^2 - (y''')^4 + \tan(t + 1)y = 0.\]

Which of the following statements are true?

I The equation is second order.
II The equation is nonlinear.
III The equation has a solution \(y = 0\).

(a) I and II only.
(b) II and III only.
(c) II only.
(d) III only.

2. (5 points) Consider the initial value problem

\[t(t^2 - 4)\sin(2t)y' + y = \frac{1}{t - 7}, \quad y(9) = \beta,\]

where \(\beta > 0\) is an arbitrary real number.

According to the Existence and Uniqueness Theorem, what is the largest interval in which a unique solution is guaranteed to exist?

(a) It depends on the value of \(\beta\).
(b) \((7, \infty)\)
(c) \((\frac{5\pi}{2}, 3\pi)\)
(d) \((7, 3\pi)\)
3. (5 points) What is a suitable integrating factor that can be used to solve the equation

\[(t^2 + 1) \frac{dy}{dt} + t^3 y = \frac{t}{e^{t^2}}.\]

DO NOT solve this differential equation.

(a) \( \mu(t) = e^{\left(\frac{t^2}{1}\right)} \)

(b) \( \mu(t) = e^{\left(\frac{t^2 - \ln(1 + t^2)}{2}\right)} \)

(c) \( \mu(t) = e^{\left(\frac{2}{t^2 - \ln(1 + t^2)}\right)} \)

(d) \( \mu(t) = e^{\left(\frac{t}{t^{2/3}}\right)} \)

4. (5 points) A 820-gallon mixing tank initially contains 220 gallons of water with concentration 0.1 lb/gal of salt. Fresh water enters the tank at a rate of 5 gal/min. In addition, salt powder is poured into the tank at rate 3 lb/min. The well-mixed solution leaves at a rate of 4 gal/min. Which of the problems below models the change of the amount of salt \( Q(t) \) in the tank during the time interval \( 0 \leq t \leq 600 \) min?

(a) \( Q'(t) + \frac{4Q(t)}{220 + t} = 3, \quad Q(0) = 22. \)

(b) \( Q'(t) = 15 - \frac{4Q(t)}{220 + t}, \quad Q(0) = 22. \)

(c) \( Q'(t) = 3 - \frac{5Q(t)}{220}, \quad Q(0) = 0.1. \)

(d) \( Q'(t) - \frac{4Q(t)}{220 + t} = 15, \quad Q(0) = 0.1. \)
5. (5 points) Consider the autonomous equation

\[ y' = y^2(y + 2)^3(4 - y^2). \]

Which of the statements below regarding its equilibrium solutions is true?

(a) There are exactly 4 equilibrium solutions.
(b) \( y = -2 \) is unstable.
(c) There are exactly 2 semistable equilibrium solutions.
(d) \( y = 0 \) is asymptotically stable.

6. (5 points) Consider the autonomous equation

\[ y' = y^2(y + 2)^3(4 - y^2). \]

Suppose \( y(-12) = 4 \). To what value will \( y(t) \) approach after a very long time?

(a) 4
(b) 2
(c) 0
(d) \( \infty \)
7. (5 points) A function of the form \( y = Cte^{-t} \), for one or more values of \( C \neq 0 \), satisfies three of the four equations below. Which equation does NOT have any solution in the form \( y = Cte^{-t} \)?

(a) \( y'' + 9y = -2te^{-t} \)
(b) \( y'' + 2y' + y = 0 \)
(c) \( y''' - 2y'' + y = 0 \)
(d) \( y'' - y = 4e^{-t} \)

8. (5 points) Which of the functions below is not a solution of the third order linear equation

\[ y''' + 2y'' - y' - 2y = 0 \]

(a) \( y(t) = \frac{1}{\pi} e^{t-5} + 5e^{1-t} \)
(b) \( y(t) = 3e^{t+2} - \sqrt{7}e^{-2t-\pi} \)
(c) \( y(t) = e^{-2+2t} + 4\pi e^{-2t} \)
(d) \( y(t) = 0 \)
9. (12 points) Consider the equation

\[ y' = \frac{9xe^{3x} - 3e^{3x} - y}{2y - 6 + x} . \]

(a) (4 points) Rewrite it into an exact equation, and verify its exactness.

(b) (6 points) Find its general solution. You may leave your answer in implicit form.

(c) (2 points) Find the solution satisfying the initial condition \( y(0) = -2 \).
10. (10 points) Consider the second order linear equation

\[ y'' + y' - 12y = 0. \]

(a) (3 points) Find the general solution of the equation.

(b) (3 points) Find the solution satisfying the initial conditions \( y(2019) = 6, \ y'(2019) = -10. \)

(c) (4 points) Let \( y_1 \) be the solution satisfying initial conditions \( y(0) = 8, \ y'(0) = \beta. \) Suppose \( \lim_{t \to \infty} y_1(t) = 0. \) Find all the possible value(s) of \( \beta. \)
11. (12 points) Given that \( y_1(t) = t^2 + 2 \) and \( y_2(t) = \sqrt{2} t \) are both solutions of the second order homogeneous linear equation

\[
y'' + p(t)y' + q(t)y = 0.
\]

Answer each question below. State a brief reason that justifies each answer.

(a) (2 points) Find their Wronskian \( W(y_1, y_2)(t) \).

(b) (2 points) True or false: \( y_1 \) and \( y_2 \) form a set of fundamental solutions of this equation.

(c) (2 points) True or false: \( y_3(t) = t \) is also a solution of the equation.

(d) (2 points) True or false: \( y_4(t) = -t^2 - 1 + \sqrt{2} t \) is also a solution of the equation.

(e) (2 points) True or false: \( y_5(t) = 2 \) is also a solution of the equation.

(f) (2 points) Consider now the following nonhomogeneous equation, whose corresponding homogeneous equation is given above:

\[
y'' + p(t)y' + q(t)y = g(t).
\]

Suppose \( Y = t - e^{-t} \cos(5t) \) is a known solution of this nonhomogeneous equation. True or false: \( y_6(t) = -2t^2 - 4 + e^{-t} \cos(5t) \) is another solution of this nonhomogeneous equation.
12. (14 points) Consider the second order nonhomogeneous linear equation

\[ y'' + 4y' + 5y = 10t + 3 - 2e^{-2t}. \]

(a) (3 points) Find \( y_c(t) \), the solution of its corresponding homogeneous equation.

(b) (6 points) Find a particular function \( Y(t) \) that satisfies the equation.

(c) (1 point) Write down the general solution of the equation.

(d) (4 points) What is the form of particular solution \( Y \) that you would use to solve the following equation using the Method of Undetermined Coefficients? DO NOT ATTEMPT TO SOLVE THE COEFFICIENTS.

\[ y'' + 4y' + 5y = e^{3t}t^2 - 6te^{-2t}\sin(t). \]
13. (12 points) Consider a mass-spring system described by the equation

\[ 2u'' + \gamma u' + ku = 0, \quad \gamma \geq 0, \quad k > 0. \]

Answer the following questions. Be sure to justify your answers. Full credit will not be given without supporting work.

(a) (2 points) Suppose the spring was stretched 5 meters by the mass to its equilibrium position. Find the value of \( k \). You may use \( g = 10 \) as the gravitational constant.

(b) (2 points) When \( \gamma = 0 \) and \( k = 16 \), what is the natural period of the system?

(c) (2 points) Suppose \( k = 200 \). For what value(s) of \( \gamma \) would the system be critically damped?

(d) (2 points) Suppose \( \gamma = 16 \) and \( k = 12 \). Will any nonzero solution of the equation cross the equilibrium position more than once?

(e) (2 points) Suppose \( \gamma = 8 \) and \( k = 10 \). Find the quasi frequency of the system.

(f) (2 points) Consider the displacement of the system in part (e) above, determine \( \lim_{t \to \infty} u(t) \).