

The Predator-Prey Equations

An application of the nonlinear system of differential equations in mathematical biology / ecology: to model the *predator-prey relationship* of a simple eco-system.

Suppose in a *closed* eco-system (i.e. no migration is allowed into or out of the system) there are only 2 types of animals: the predator and the prey. They form a simple food-chain where the predator species hunts the prey species, while the prey grazes vegetation. The size of the 2 populations can be described by a simple system of 2 nonlinear first order differential equations (a.k.a. the *Lotka-Volterra equations*, which originated in the study of fish populations of the Mediterranean during and immediately after WW I).

Let $x(t)$ denotes the population of the prey species, and $y(t)$ denotes the population of the predator species. Then

$$\begin{aligned}x' &= ax - \alpha xy \\y' &= -cy + \gamma xy\end{aligned}$$

a , c , α , and γ are positive constants.

Note that in the absence of the predators (when $y = 0$), the prey population would grow exponentially. If the preys are absence (when $x = 0$), the predator population would decay exponentially to zero due to starvation.

This system has two critical points. One is the origin, and the other is in the first quadrant.

$$\begin{aligned} 0 = x' = ax - \alpha xy = x(a - \alpha y) &\rightarrow x = 0 \text{ or } a - \alpha y = 0 \\ 0 = y' = -cy + \gamma xy = y(-c + \gamma x) &\rightarrow y = 0 \text{ or } -c + \gamma x = 0 \end{aligned}$$

Therefore, the critical points are $(0, 0)$ and $(\frac{c}{\gamma}, \frac{a}{\alpha})$.

The Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{bmatrix}$$

At $(0, 0)$, the linearized system has coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

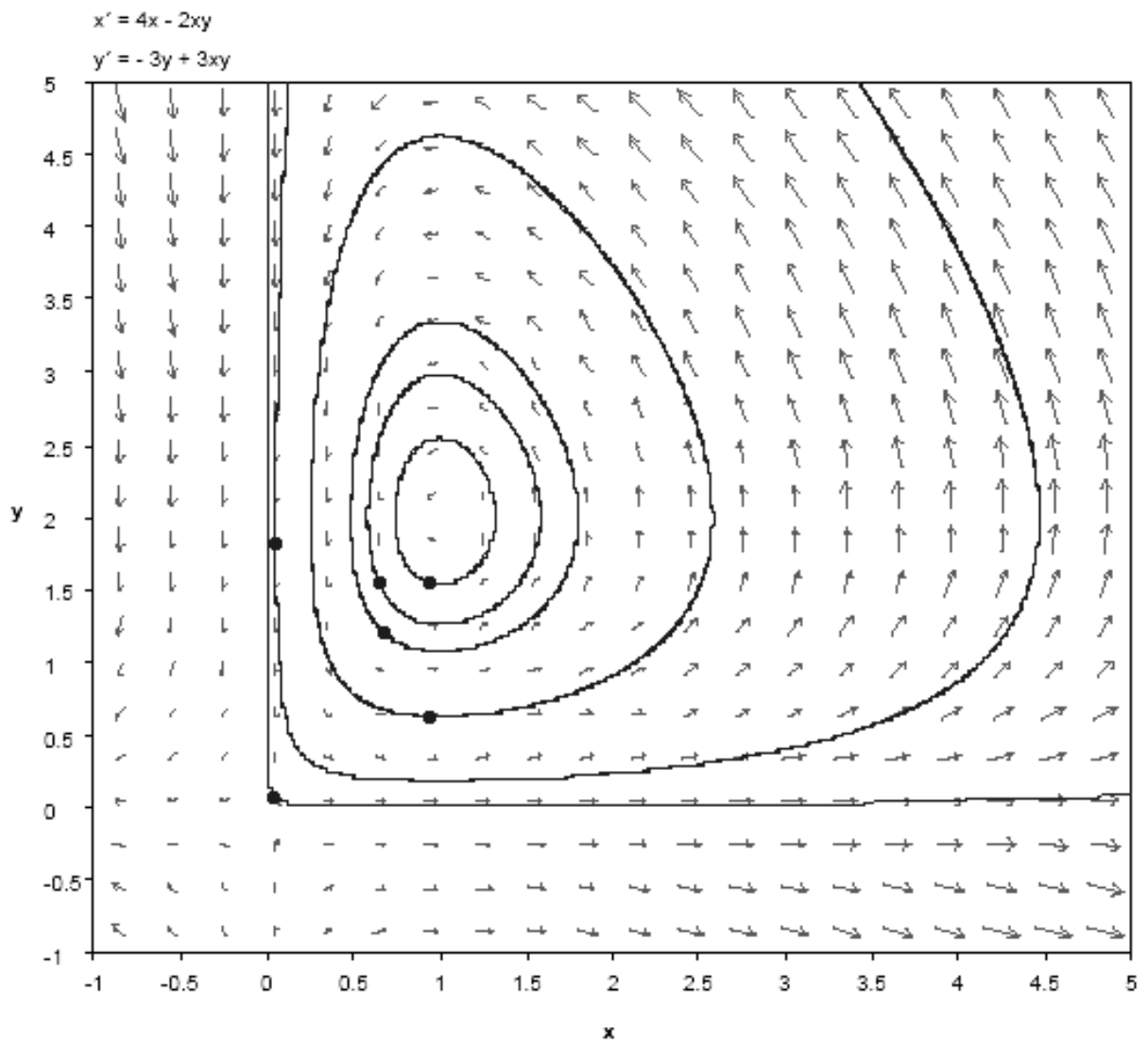
The eigenvalues are a and $-c$. Hence, it is an unstable saddle point.

At $(\frac{c}{\gamma}, \frac{a}{\alpha})$, the linearized system has coefficient matrix

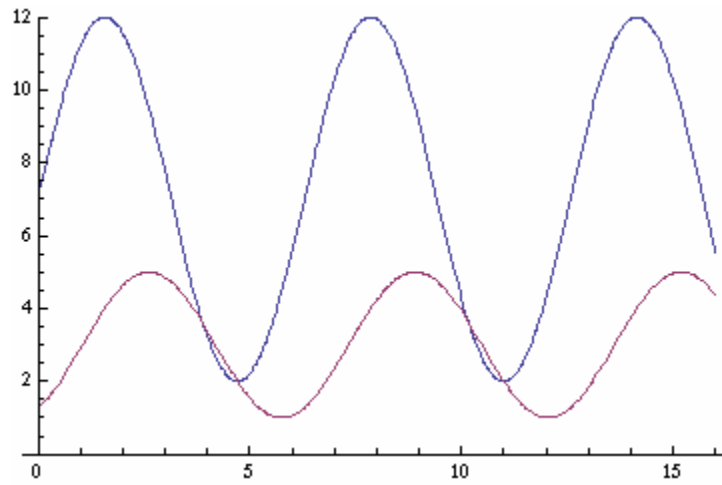
$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{a\gamma}{\alpha} & 0 \end{bmatrix}$$

The eigenvalues are $\pm\sqrt{ac}i$. It is a stable center. (Previously, we have learned that the purely imaginary eigenvalues case in a nonlinear system is ambiguous, with several possible behaviors. But in this example it really is a center. See its phase portrait on the next page.)

The phase portrait of one such system of Lotka-Volterra equations is shown here:



When $x(t)$ and $y(t)$ are plotted individually versus t , we see that the periodic variation of the predator population $y(t)$ lags slightly behind the prey population $x(t)$. An example is shown below.



Note: The higher curve is that of $x(t)$, the lower curve that of $y(t)$. Notice that y lags x slightly.

Variations of the basic Lotka-Volterra equations

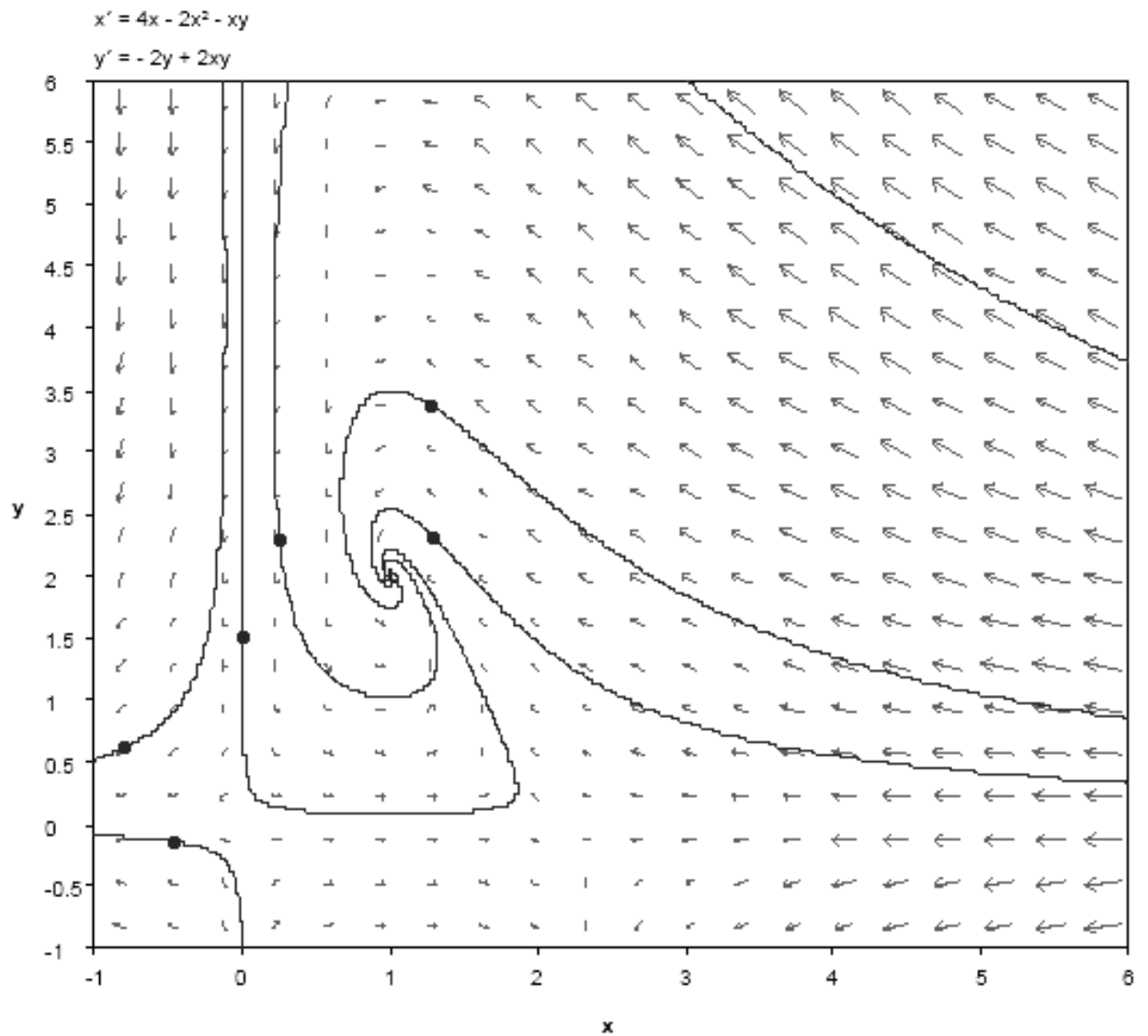
One obvious shortcoming of the basic predator-prey system is that the population of the prey species would grow unbounded, exponentially, in the absence of predators. There is an easy solution to this unrealistic behavior. We'll just replace the exponential growth term in the first equation by the two-term logistic growth expression:

$$\begin{aligned}x' &= (ax - rx^2) - \alpha xy \\y' &= -cy + \gamma xy\end{aligned}$$

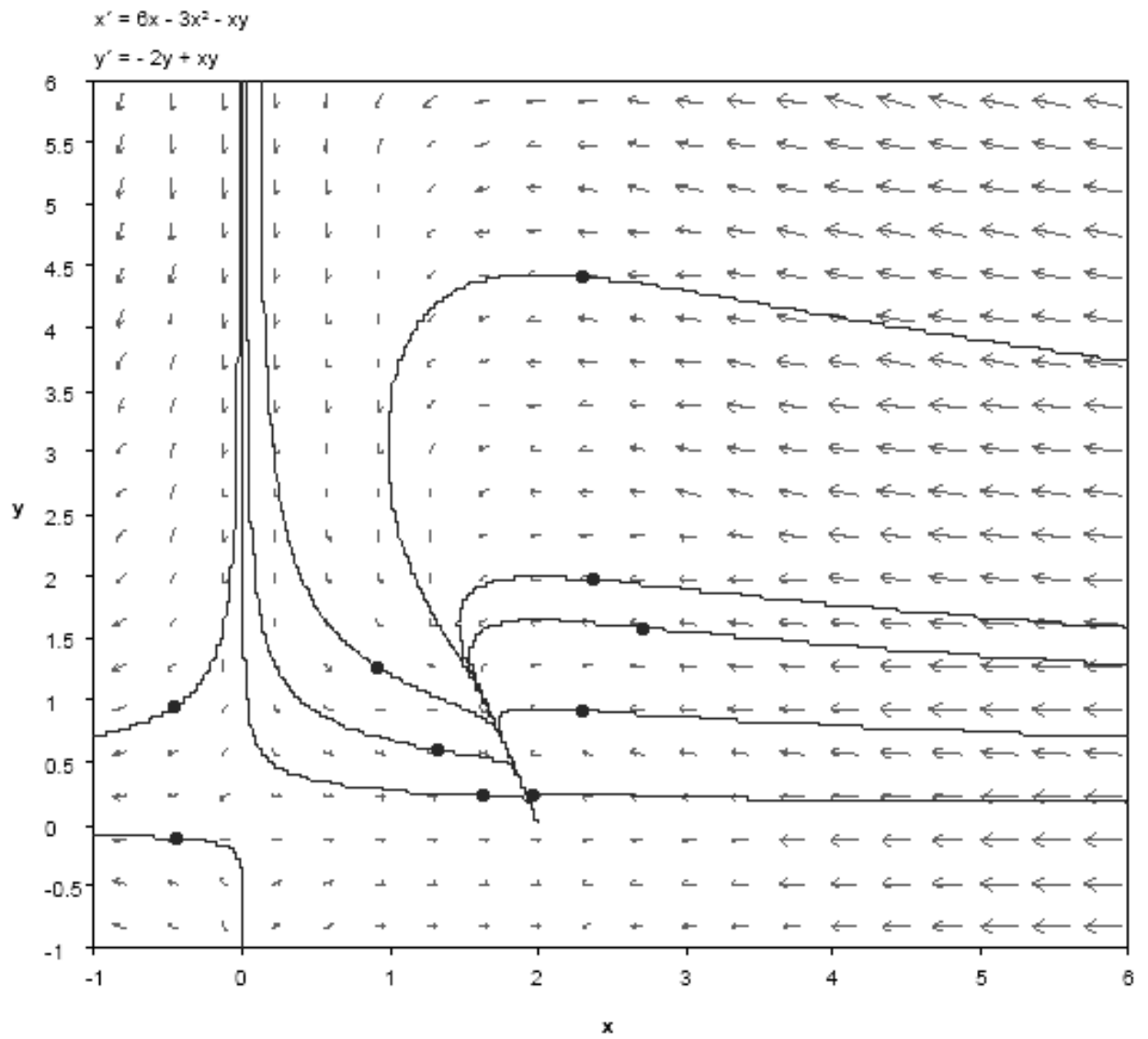
Therefore, in the absence of predators, the first equation becomes the *logistic equation*. The prey population would instead stabilize at the environmental carrying capacity given by the logistic equation.

In this new system there will be 3 critical points: at the origin, in the first quadrant, and the third on the positive x -axis. This third critical point is at the environmental carrying capacity of prey x , while $y = 0$. That is, the additional critical point is at $(k, 0)$, where k is the environmental carrying capacity for the prey species seen earlier in the course. It is the limiting population of the prey species in the absence of the predator species. The critical point in the first quadrant could be an asymptotically stable node or spiral point.

Example (an asymptotically stable spiral point in the first quadrant):



Example (an asymptotically stable node in the first quadrant):



If the predator species has an alternate food source (omnivores such as bears, for example, could possibly subsist on plants alone), then it needs not to die out due to starvation even if the preys are totally absence. In this case we could replace the $-cy$ term in the second equation by logistic-growth terms as well:

$$\begin{aligned}x' &= (ax - rx^2) - \alpha xy \\y' &= (by - cy^2) + \gamma xy\end{aligned}$$

More complex food-chains can be similarly constructed as systems of more than 2 equations. For example, suppose there is an enclosed eco-system containing 3 species. Species x is a grass-grazer whose population in isolation would obey the logistic equation, and that it is preyed upon by species y who, in turn, is the sole food source of species z . Then their respective population might be modeled by the 3-equation system:

$$\begin{aligned}x' &= (ax - rx^2) - \alpha xy \\y' &= -cy + \gamma xy - \delta yz \\z' &= -dz + \lambda yz\end{aligned}$$

What do the equations mean?

- (1) At the bottom of the food chain, population x grows logistically in the absence of its natural predator (y , in this case); while it decreases due to hunting by y .
- (2) Population y starves in the absence of its sole food source (x , in this example); it grows by hunting and eating x ; it is, in turn, hunted by z and therefore it would decrease due to interaction with z .
- (3) Population z sits atop the food chain. It starves in the absence of its sole food source (y , in this example); it grows by hunting and eating y . Populations x and z do not interact directly.