Second Order Linear Partial Differential Equations

Part III

One-dimensional Heat Conduction Equation revisited; temperature distribution of a bar with insulated ends; nonhomogeneous boundary conditions; temperature distribution of a bar with ends kept at arbitrary temperatures; steady-state solution

Previously, we have learned that the general solution of a partial differential equation is dependent of boundary conditions. The same equation will have different general solutions under different sets of boundary conditions. We shall witness this fact, by examining additional examples of heat conduction problems with new sets of boundary conditions.

Keep in mind that, throughout this section, we will be solving the same partial differential equation, the homogeneous one-dimensional heat conduction equation:

$$\alpha^2 u_{xx} = u_t$$

where $u(x, t)$ is the temperature distribution function of a thin bar, which has length L, and the positive constant α^2 is the thermo diffusivity constant of the bar. The equation will now be paired up with new sets of boundary conditions.

Bar with both ends insulated

Now let us consider the situation where, instead of them being kept at constant 0 degree temperature, the two ends of the bar are also sealed with perfect insulation so that no heat could escape to the outside environment (recall that the side of the bar is always perfectly insulated in the one-dimensional assumption), or vice versa. The new boundary conditions are $u_x(0, t) = 0$ and $u_x(L, t) = 0$*, reflecting the fact that there will be no heat transferring, spatially, across the points $x = 0$ and $x = L$. (Hence, this is a *Neumann* type problem.) The heat conduction problem becomes the initial-boundary value problem below.

(Heat conduction eq.)	$\alpha^2 u_{xx} = u_t$,	$0 < x < L, \quad t > 0,$
(Boundary conditions)	$u_x(0, t) = 0$, and	$u_x(L, t) = 0,$
(Initial condition)	$u(x, 0) = f(x).$	

The first step is the separation of variables. The equation is the same as before. Therefore, it will separate into the exact same two ordinary differential equations as in the first heat conduction problem seen earlier. The new boundary conditions separate into

$$u_x(0, t) = 0 \;\rightarrow\; X'(0)T(t) = 0 \;\rightarrow\; X'(0) = 0 \;\text{ or }\; T(t) = 0$$
$$u_x(L, t) = 0 \;\rightarrow\; X'(L)T(t) = 0 \;\rightarrow\; X'(L) = 0 \;\text{ or }\; T(t) = 0$$

As before, we cannot choose $T(t) = 0$. Else we could only get the trivial solution $u(x,t) = 0$, rather than the general solution. Hence, the new boundary conditions should be $X'(0) = 0$ and $X'(L) = 0$.

Again, we end up with a system of two simultaneous ordinary differential equations. Plus a set of two boundary conditions that goes with the spatial independent variable x:

* The conditions say that the instantaneous rate of change with respect to x, the spatial variable (i.e., the rate of point-to-point heat transfer), is zero at each end. They do not suggest that the temperature is constant (that is, there is no change in temperature through time, which would require $u_t = 0$) at each end.

$$X'' + \lambda X = 0, \qquad X'(0) = 0 \quad \text{and} \quad X'(L) = 0,$$

$$T' + \alpha^2 \lambda T = 0.$$

The second step is to solve the eigenvalue problem

$$X'' + \lambda X = 0, \qquad X'(0) = 0 \quad \text{and} \quad X'(L) = 0.$$

The result is summarized below.

Case 1: If $\lambda < 0$: No such λ exists.

Case 2: If $\lambda = 0$: Zero is an eigenvalue. Its eigenfunction is the constant function $X_0 = 1$ (or any other nonzero constant).

Case 3: If $\lambda > 0$: The positive eigenvalues λ are

$$\lambda = \frac{n^2 \pi^2}{L^2}, \qquad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions that satisfy the said boundary conditions are

$$X_n = \cos \frac{n \pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

The third step is to substitute the positive eigenvalues found above into the equation of t and solve:

$$T' + \alpha^2 \frac{n^2 \pi^2}{L^2} T = 0.$$

Notice that this is exactly the same equation as in the first (both ends kept at 0 degree) heat conduction problem, due to the fact that both problems have the same set of eigenvalues (but with different eigenfunctions).

As a result, the solutions of the second equation are just the ones we have gotten the last time

$$T_n(t) = C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2}, \qquad n = 1, 2, 3, \ldots$$

There is this extra eigenvalue of $\lambda = 0$ that also needs to be accounted for. It has as an eigenfunction the constant $X_0(x) = 1$. Put $\lambda = 0$ into the second equation and we get $T' = 0$, which has only constant solutions $T_0(t) = C_0$. Thus, we get the (arbitrary) constant function $u_0(x, t) = X_0(x)T_0(t) = C_0$ as a solution. Therefore, the solutions of the one-dimensional heat conduction equation, with the boundary conditions $u_x(0, t) = 0$ and $u_x(L, t) = 0$, are in the form

$$u_0(x,t) = C_0,$$

$$u_n(x,t) = X_n(t)\, T_n(t) = C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2} \cos\frac{n\pi x}{L},$$

$n = 1, 2, 3, \ldots$

The general solution is their linear combination. Hence, for a bar with both ends insulated, the heat conduction problem has general solution:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2} \cos\frac{n\pi x}{L}.$$

Now set $t = 0$ and equate it with the initial condition $u(x, 0) = f(x)$:

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{L} = f(x).$$

We see that the requirement is that the initial temperature distribution $f(x)$ must be a Fourier cosine series. That is, it needs to be an even periodic function of period $2L$. If $f(x)$ is not already an even periodic function, then we will need to expand it into one and use the resulting even periodic extension of $f(x)$ in its place in the above equation. Once this is done, the coefficients C's in the particular solution are just the corresponding Fourier cosine coefficients of the initial condition $f(x)$. (Except for the constant term, where the relation $C_0 = a_0 / 2$ holds, instead.)

The explicit formula for C_n is, therefore,

$$C_n = a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots$$

$$C_0 = a_0 / 2$$

Example: Solve the heat conduction problem

$$3\,u_{xx} = u_t \quad , \qquad 0 < x < 8, \quad t > 0,$$
$$u_x(0, t) = 0, \text{ and } \quad u_x(8, t) = 0,$$
$$u(x, 0) = 9 - 3\cos(\pi x/4) - 6\cos(2\pi x).$$

First note that $\alpha^2 = 3$ and $L = 8$, and the fact that the boundary conditions indicating this is a bar with both ends perfectly insulated. Substitute them into the formula we have just derived to obtain the general solution for this problem:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n\, e^{-3n^2\pi^2 t/64} \cos\frac{n\pi x}{8}\ .$$

Check the initial condition $f(\mathrm{x})$, and we see that it is already in the require form of a Fourier cosine series of period 16. Therefore, there is no need to find its even periodic extension. Instead, we just need to extract the correct Fourier cosine coefficients from $f(\mathrm{x})$:

$$C_0 = a_0/2 = 9,$$
$$C_2 = a_2 = -3,$$
$$C_{16} = a_{16} = -6,$$
$$C_n = a_n = 0, \text{ for all other } n,\ n \neq 0, 2, \text{ or } 16.$$

Note that C_0 is actually $a_0/2$, due to the way we write the constant term of the Fourier series. But that shouldn't present any more difficulty. Since when you see a Fourier series, its constant term is already expressed in the form $a_0/2$. Therefore, you could just copy it down directly to be the C_0 term without thinking.

Finally, the particular solution is

$$u(x,t) = 9 - 3e^{-3(2^2)\pi^2 t/64}\cos(\frac{\pi x}{4}) - 6e^{-3(16^2)\pi^2 t/64}\cos(2\pi x)$$

Bar with two ends kept at arbitrary temperatures:
An example of nonhomogeneous boundary conditions

In both of the heat conduction initial-boundary value problems we have seen, the boundary conditions are homogeneous − they are all zeros. Now let us look at an example of heat conduction problem with simple nonhomogeneous boundary conditions. The general set-up is the same as the first example (where the both ends of the bar were kept at constant 0 degree, but were not insulated), except now the ends are kept at arbitrary (but constant) temperatures of T_1 degrees at the left end, and T_2 degrees at the right end. The initial condition, as usual, is arbitrary. The heat conduction problem is therefore given by the initial-boundary value problem:

$$\alpha^2 u_{xx} = u_t \ , \qquad 0 < x < L, \quad t > 0,$$

$$u(0, t) = T_1, \text{ and} \qquad u(L, t) = T_2,$$

$$u(x, 0) = f(x).$$

The boundary conditions is now nonhomogeneous (unless T_1 and T_2 are both 0, then the problem becomes identical to the earlier example), because at least one of the boundary values are nonzero.

The nonhomogeneous boundary conditions are rather easy to work with, more so than we might have reasonably expected. First, let us be introduced to the concept of the *steady-state solution*. It is the part of the solution $u(x, t)$ that is independent of the time variable t. Therefore, it is a function of the spatial variable alone. We can thusly rewrite the solution $u(x, t)$ as a sum of 2 parts, a time-independent part and a time-dependent part:

$$u(x, t) = v(x) + w(x, t).$$

Where $v(x)$ is the steady-state solution, which is independent of t, and $w(x, t)$ is called the *transient solution*, which does vary with t.

The Steady-State Solution

The steady-state solution, $v(x)$, of a heat conduction problem is the part of the temperature distribution function that is independent of time t. It represents the equilibrium temperature distribution. To find it, we note the fact that it is a function of x alone, yet it has to satisfy the heat conduction equation. Since $v_{xx} = v''$ and $v_t = 0$, substituting them into the heat conduction equation we get

$$\alpha^2 v_{xx} = 0.$$

Divide both sides by α^2 and integrate twice with respect to x, we find that $v(x)$ must be in the form of a degree 1 polynomial:

$$\boxed{v(x) = Ax + B.}$$

Then, rewrite the boundary conditions in terms of v: $u(0, t) = v(0) = T_1$, and $u(L, t) = v(L) = T_2$. Apply those 2 conditions to find that:

$$v(0) = T_1 = A(0) + B = B \qquad \rightarrow \qquad B = T_1$$

$$v(L) = T_2 = AL + B = AL + T_1 \qquad \rightarrow \qquad A = (T_2 - T_1)/L$$

Therefore,

$$v(x) = \frac{T_2 - T_1}{L}\,x + T_1.$$

Thing to remember: The steady-state solution is a time-independent function. It is obtained by setting the partial derivative(s) with respect to t in the heat equation (or, later on, the wave equation) to constant zero, and then solving the equation for a function that depends only on the spatial variable x.

Comment: Another way to understand the behavior of $v(x)$ is to think from the perspective of separation of variables. You could think of the steady-state solution as, during the separation of variables, the solution you would have obtained if $T(t) = 1$, the constant function 1. Therefore, the solution is independent of time, or *time-invariant*. Hence, $u(x, t) = X(x)T(t) = X(x) = v(x)$. We can, in addition, readily see the substitutions required for rewriting the boundary conditions prior to solving for the steady-state solution:

$$u(0, t) = X(0) = v(0) = T_1, \text{ and } u(L, t) = X(L) = v(L) = T_2.$$

That is, just rename the function u as v, ignore the time variable t, and put whatever x-coordinate specified directly into $v(x)$.

The solution of bar with two ends kept at arbitrary temperatures

Once the steady-state solution has been found, we can set it aside for the time being and proceed to find the transient part of solution, $w(x, t)$. First we will need to rewrite the given initial-boundary value problem slightly. Keep in mind that the initial and boundary conditions as originally given were meant for the temperature distribution function $u(x, t) = v(x) + w(x, t)$. Since we have already found $v(x)$, we shall now subtract out the contribution of $v(x)$ from the initial and boundary values. The results will be the conditions that the transient solution $w(x, t)$ alone must satisfy.

Change in the boundary conditions:

$$u(0, t) = T_1 = v(0) + w(0, t) \quad \rightarrow \quad w(0, t) = T_1 - v(0) = 0$$
$$u(L, t) = T_2 = v(L) + w(L, t) \quad \rightarrow \quad w(L, t) = T_2 - v(L) = 0$$

Note: Recall that $u(0, t) = v(0) = T_1$, and $u(L, t) = v(L) = T_2$.

Change in the initial condition:

$$u(x, 0) = f(x) = v(x) + w(x, 0) \quad \rightarrow \quad w(x, 0) = f(x) - v(x)$$

Consequently, the transient solution is a function of both x and t that must satisfy the new initial-boundary value problem:

$$\alpha^2 w_{xx} = w_t, \qquad 0 < x < L, \quad t > 0,$$

$$w(0, t) = 0, \text{ and} \qquad w(L, t) = 0,$$

$$w(x, 0) = f(x) - v(x).$$

Surprise! Notice that the new problem just described is precisely the same initial-boundary value problem associated with the heat conduction of a bar with both ends kept at 0 degree. Therefore, the transient solution $w(x, t)$ of the current problem is just the general solution of the previous heat conduction problem (with homogeneous boundary conditions), that of a bar with 2 ends kept constantly at 0 degree:

$$w(x,t)=\sum_{n=1}^{\infty} C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L}.$$

Where the coefficients C_n are equal to the corresponding Fourier sine coefficients b_n of the (newly rewritten) initial condition $w(x, 0) = f(x) - v(x)$. (Or those of $w(x, 0)$'s odd periodic extension, of period $2L$, if it is not already an odd periodic function of the correct period.) Explicitly, they are given by

$$C_n = b_n = \frac{2}{L}\int_0^L \left(f(x) - v(x)\right)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

Finally, combining the steady-state and transient solutions together, the general solution of the temperature distribution of a bar whose ends are kept at T_1 degrees at the left, and T_2 degrees at the right, becomes

$$u(x,t) = v(x) + w(x,t)$$
$$= \left(\frac{T_2 - T_1}{L}x + T_1\right) + \sum_{n=1}^{\infty} C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L}.$$

Example: Solve the heat conduction problem

$$8\,u_{xx} = u_t \quad, \qquad 0 < x < 5, \qquad t > 0,$$
$$u(0, t) = 10, \qquad u(5, t) = 90,$$
$$u(x, 0) = 16x + 10 + 2\sin(\pi x) - 4\sin(2\pi x) + \sin(6\pi x).$$

First we note that $\alpha^2 = 8$ and $L = 5$. Since $T_1 = 10$ and $T_2 = 90$, the steady-state solution is $v(x) = (90 - 10)x/5 + 10 = 16x + 10$. We then subtract $v(0) = 10$ from $u(0, t)$, $v(5) = 90$ from $u(5, t)$, and $v(x) = 16x + 10$ from $u(x, 0)$ to obtain a new set of initial-boundary values that the transient solution $w(x,t)$ alone must satisfy:

$$w(0, t) = 0, \text{ and } \qquad w(5, t) = 0,$$
$$w(x, 0) = 2\sin(\pi x) - 4\sin(2\pi x) + \sin(6\pi x).$$

Base on $\alpha^2 = 8$ and $L = 5$, we write down the general solution:

$$w(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-8n^2\pi^2 t/25} \sin\frac{n\pi x}{5}$$

The new initial condition, $f(x) - v(x)$, is already an odd periodic function of the period $T = 2L = 10$. Therefore, just extract the correct Fourier sine coefficients from it:

$C_5 = b_5 = 2$,
$C_{10} = b_{10} = -4$,
$C_{30} = b_{30} = 1$,
$C_n = b_n = 0$, for all other n, $n \neq 5, 10$, or 30.

Add together the steady-state and transient solutions, we have

$$u(x,t) = v(x) + w(x,t) = 16x + 10 + 2e^{-8(5^2)\pi^2 t/25} \sin(\pi x)$$
$$-4e^{-8(10^2)\pi^2 t/25} \sin(2\pi x) + e^{-8(30^2)\pi^2 t/25} \sin(6\pi x)$$

Back to the Steady-State Solution

A thing to remember: the steady-state solution of the one-dimensional homogeneous heat conduction equation is always in the form $v(x) = Ax + B$. Since it is independent of t, the effects of boundary conditions on $v(x)$ are also simplified as: $u(x_0, t) = v(x_0)$, and $u_x(x_0, t) = v'(x_0)$.

Nonhomogeneous heat conduction equations (that is, the equations themselves contain forcing function terms; not to be confused with the homogeneous equation with accompanying nonhomogeneous boundary conditions that we have just seen), however, could have different forms of $v(x)$.

Fact: The steady-state temperature distribution satisfies the property:

$$\lim_{t \to \infty} u(x,t) = v(x).$$

This relation is true only for the solution of heat conduction equation (modeling diffusion-like processes that are thermodynamically irreversible). Physically speaking, $v(x)$ describes the eventual state of maximum entropy as dictated by the *second law of Thermodynamics*.

Caution: The above relation is not true, in general, for solutions of the wave equation in the next section. This difference is due to the fact that the wave equation models wave-like motions which are thermodynamically reversible processes.

Further examples of steady-state solutions of the heat conduction equation:

Find $v(x)$, given each set of boundary conditions below.

1. $u(0, t) = 50, \quad u_x(6, t) = 0$

We are looking for a function of the form $v(x) = Ax + B$ that satisfies the given boundary conditions. Its derivative is then $v'(x) = A$. The two boundary conditions can be rewritten to be $u(0, t) = v(0) = 50$, and $u_x(6, t) = v'(6) = 0$. Hence,

$v(0) = 50 = A(0) + B = B \quad \rightarrow \quad B = 50$

$v'(6) = 0 = A \quad \rightarrow \quad A = 0$

Therefore, $v(x) = 0x + 50 = 50$.

2. $u(0, t) - 4u_x(0, t) = 0, \quad u_x(10, t) = 25$

The two boundary conditions can be rewritten to be $v(0) - 4\,v'(0) = 0$, and $v'(10) = 25$.

Hence,

$v(0) - 4\,v'(0) = 0 = (A(0) + B) - 4A = -4A + B$

$v'(10) = 25 = A \quad \rightarrow \quad A = 25$

Substitute $A = 25$ into the first equation: $0 = -4A + B = -100 + B$
$\rightarrow \quad B = 100$

Therefore, $v(x) = 25x + 100$.

3. $u(0, t) = 35, \qquad u(4, t) + 3u_x(4, t) = 0$

Rewriting the boundary conditions:
$v(0) = 35$, and $v(4) + 3\,v'(4) = 0$.

Hence,

$v(0) = 35 = A(0) + B = B \qquad \rightarrow \qquad B = 35$

$v(4) + 3\,v'(4) = 0 = (A(4) + B) + 3A = 7A + B$

$0 = 7A + 35$

$A = -5$

Therefore, $v(x) = -5x + 35$

Comment: Notice how in each example we have seen, the steady-state solution is uniquely determined by the boundary conditions alone. In general this is true – that the steady-state solution is *almost* always independent of the initial condition. The lone exception is the insulated-ends problem (with boundary conditions $u_x(0, t) = 0 = u_x(L, t)$). In this special case, the boundary conditions only tell us that the steady-state solution should be a constant, which turns out to be the constant term of the general solution. As we have seen, that constant is indeed dependent on the initial condition – it is just the constant term of the initial condition, as the latter is expanded into a Fourier cosine series of period $2L$. In all other cases the boundary conditions alone determine the steady-state solution (therefore, the limiting temperature) of a problem.

<u>Summary</u>: Solving Second Order Linear Partial Differential Equations

The Method of Separation of Variables:

0. (If the boundary conditions are nonhomogeneous) Solve for the steady-state solution, $v(x)$, which is a function of x only that satisfies both the PDE and the boundary conditions. Afterwards rewrite the problem's boundary and initial conditions to subtract out the contribution from the steady-state solution. Therefore, the problem is now transformed into one with homogeneous boundary conditions.

1. Separate the PDE into ODEs of one independent variable each. Rewrite the boundary conditions so they associate with only one of the variables.

2. One of the ODEs is a part of a two-point boundary value problem. Solve this problem for its eigenvalues and eigenfunctions.

3. Solve the other ordinary differential equation.

4. Multiply the results from steps (2) and (3), and sum up all the products to find the general solution respect to the given homogeneous boundary conditions. Add to it the steady-state solution (from step 0, if applicable) to find the overall general solution.

5. Expand the initial condition into a suitable (e.g. a sine or a cosine series, <u>depending on whichever type the eigenfunctions are</u>) Fourier series. Then compare it against $u(x, 0)$ to find the coefficients for the particular solution.

Summary of Heat Conduction Problems

Here is a list of heat conduction problems and their solutions. All solutions obey the homogeneous one-dimensional heat conduction equation

$$\alpha^2 u_{xx} = u_t .$$

They only differ in boundary conditions (which are given below for each problem). The initial condition is always arbitrary, $u(x, 0) = f(x)$.

1. **Bar with both ends kept at 0 degree** (boundary conditions: $u(0, t) = 0$, $u(L, t) = 0$)

$$u(x,t) = \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L} .$$

Expand $f(x)$ to be a Fourier sine series, then $C_n = b_n$.
Steady-state solution is $v(x) = 0$.

2. **Bar with both ends perfectly insulated** (boundary conditions: $u_x(0, t) = 0$, $u_x(L, t) = 0$)

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 n^2 \pi^2 t / L^2} \cos \frac{n\pi x}{L} .$$

Expand $f(x)$ to be a Fourier cosine series, then $C_0 = a_0 / 2$, and $C_n = a_n$, $n = 1, 2, 3, \ldots$
Steady-state solution is $v(x) = C_0$.

3. **Bar with T_1 degrees at the left end, and T_2 degrees at the right end** (boundary conditions: $u(0, t) = T_1$, $u(L, t) = T_2$)

$$u(x,t) = \left(\frac{T_2 - T_1}{L} x + T_1 \right) + \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L} .$$

Expand $(f(x) - v(x))$ to be a Fourier sine series, then $C_n = b_n$.

Steady-state solution is $v(x) = \frac{T_2 - T_1}{L} x + T_1$.

Exercises E-3.1:

1 – 7 Find the steady-state solution $v(x)$ of the heat conduction equation, given each set of boundary conditions below.

1. $u(0,t) = 200, \quad u(10,t) = 100$
2. $u(0,t) = 100, \quad u_x(10,t) = 50$
3. $u_x(0,t) = 8, \quad u(10,t) = 100$
4. $u_x(0,t) = 30, \quad u_x(10,t) = 10$
5. $u(0,t) + u_x(0,t) = 10, \quad u(10,t) = 100$
6. $u(0,t) + u_x(0,t) = 0, \quad u(10,t) - u_x(10,t) = 200$
7. $u(0,t) - 10u_x(0,t) = 30, \quad u(10,t) - 5u_x(10,t) = 0$

8. Solve the heat conduction problem of the given initial conditions.

$$2\,u_{xx} = u_t \quad , \qquad 0 < x < 6, \qquad t > 0,$$
$$u_x(0,\,t) = 0, \quad \text{and} \quad u_x(6,\,t) = 0,$$

(a) $u(x,\,0) = \pi + 3\cos(\pi x) - 4\cos(3\pi x/2) - \cos(3\pi x),$
(b) $u(x,\,0) = 4,$
(c) $u(x,\,0) = x^2,$
(d) $u(x,\,0) = 0.$

9. For each particular solution found in #8, find $\lim_{t\to\infty} u(x,t)$.

10. Solve the heat conduction problem of the given initial conditions.

$$2\,u_{xx} = u_t \quad , \qquad 0 < x < 6, \qquad t > 0,$$
$$u(0,\,t) = 40, \quad \text{and} \quad u(6,\,t) = 10,$$

(a) $u(x,\,0) = -5x + 40 + 5\sin(2\pi x) - 2\sin(5\pi x/2),$
(b) $u(x,\,0) = 0.$

11. For each particular solution found in #10, find $\lim_{t\to\infty} u(4,t)$.

12. Consider the heat conduction problem

$$9u_{xx} = u_t \quad , \qquad 0 < x < 4, \quad t > 0,$$
$$u(0, t) = 32, \text{ and } u(4, t) = 32,$$
$$u(x, 0) = f(x).$$

(a) Find its general solution.
(b) Write an explicit formula to determine the coefficients.
(c) Base on (a), find $\lim_{t \to \infty} u(x,t)$.

13. Consider the heat conduction equation below, subject to each of the 4 sets of boundary conditions.

$$9u_{xx} = u_t \quad , \qquad 0 < x < 10, \quad t > 0,$$

(i) $u(0, t) = 0$, and $u(10, t) = 0$,
(ii) $u_x(0, t) = 0$, and $u_x(10, t) = 0$,
(iii) $u(0, t) = 0$, and $u(10, t) = 100$
(iv) $u(0, t) = 100$, and $u(10, t) = 50$.

Given the common initial condition $u(x, 0) = 300$, determine the temperature at the midpoint of the bar (at $x = 5$) after a very long time has elapsed. Which set of boundary conditions will give the highest temperature at that point?

Answers E-3.1:

1. $v(x) = -10x + 200$ 2. $v(x) = 50x + 100$ 3. $v(x) = 8x + 20$
4. $v(x)$ does not exist. 5. $v(x) = 10x$ 6. $v(x) = 25x - 25$
7. $v(x) = -2x + 10$
8. (a) $u(x,t) = \pi + 3e^{-2\pi^2 t}\cos(\pi x) - 4e^{-9\pi^2 t/2}\cos(3\pi x/2) - e^{-18\pi^2 t}\cos(3\pi x)$,
   (b) $u(x,t) = 4$,
   (d) $u(x,t) = 0$.
9. (a) $\lim_{t\to\infty} u(x,t) = \pi$; (b) $\lim_{t\to\infty} u(x,t) = 4$; (c) $\lim_{t\to\infty} u(x,t) = 25/3$;
   (d) $\lim_{t\to\infty} u(x,t) = 0$.
10. (a) $u(x,t) = -5x + 40 + 5e^{-8\pi^2 t}\sin(2\pi x) - 2e^{-25\pi^2 t/2}\sin(5\pi x/2)$.
11. (a) and (b) $\lim_{t\to\infty} u(4,t) = 20$.
12. (a) $u(x,t) = 32 + \sum_{n=1}^{\infty} C_n e^{-9n^2\pi^2 t/16} \sin\frac{n\pi x}{4}$
    (b) $C_n = \frac{1}{2}\int_0^4 \left(f(x) - 32\right)\sin\frac{n\pi x}{4}\,dx$, $n = 1, 2, 3, \ldots$
    (c) $\lim_{t\to\infty} u(x,t) = 32$
13. Boundary conditions (ii) will give the highest temperature.
At $x = 5$, the temperature is $\lim_{t\to\infty} u(5,t) = v(5) = 300$.