Second Order Linear Nonhomogeneous Differential Equations; Method of Undetermined Coefficients

We will now turn our attention to nonhomogeneous second order linear equations, equations with the standard form

$$y'' + p(t)y' + q(t)y = g(t), \qquad g(t) \neq 0. \qquad (*)$$

Each such nonhomogeneous equation has a *corresponding homogeneous equation*:

$$y'' + p(t)y' + q(t)y = 0. \qquad (**)$$

Note that the two equations have the same left-hand side, (**) is just the homogeneous version of (*), with $g(t) = 0$.

We will focus our attention to the simpler topic of nonhomogeneous second order linear equations with constant coefficients:

$$a y'' + b y' + c y = g(t).$$

Where a, b, and c are constants, $a \neq 0$; and $g(t) \neq 0$. It has a corresponding homogeneous equation $a y'' + b y' + c y = 0$.

Solution of the nonhomogeneous linear equations

It can be verify easily that the difference $y = Y_1 - Y_2$, of any two solutions of the nonhomogeneous equation (*), is always a solution of its corresponding homogeneous equation (**). Therefore, every solution of (*) can be obtained from a single solution of (*), by adding to it all possible solutions of its corresponding homogeneous equation (**). As a result:

Theroem: The general solution of the second order nonhomogeneous linear equation

$$y'' + p(t)y' + q(t)y = g(t)$$

can be expressed in the form

$$y = y_c + Y$$

where Y is *any* specific function that satisfies the nonhomogeneous equation, and $y_c = C_1 y_1 + C_2 y_2$ is a general solution of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

(That is, y_1 and y_2 are a pair of fundamental solutions of the corresponding homogeneous equation; C_1 and C_2 are arbitrary constants.)

The term $y_c = C_1 y_1 + C_2 y_2$ is called the *complementary solution* (or the *homogeneous solution*) of the nonhomogeneous equation. The term Y is called the *particular solution* (or the *nonhomogeneous solution*) of the same equation.

Comment: It should be noted that the "complementary solution" is never actually a solution of the given nonhomogeneous equation! It is merely taken from the corresponding homogeneous equation as a component that, when coupled with a particular solution, gives us the general solution of a nonhomogeneous linear equation. On the other hand, the particular solution is necessarily always a solution of the said nonhomogeneous equation. Indeed, in a slightly different context, it must be a "particular" solution of a certain initial value problem that contains the given equation and whatever initial conditions that would result in $C_1 = C_2 = 0$.

In the case of nonhomgeneous equations with constant coefficients, the complementary solution can be easily found from the roots of the characteristic polynomial. They are always one of the three forms:

$$y_c = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$y_c = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$$

$$y_c = C_1 e^{rt} + C_2 t e^{rt}$$

Therefore, the only task remaining is to find the particular solution Y, which is any one function that satisfies the given nonhomogeneous equation. That might sound like an easy task. But it is quite nontrivial.

There are two general approaches to find Y: the Methods of Undetermined Coefficients, and Variation of Parameters. We will only study the former in this class.

Method of Undetermined Coefficients

The *Method of Undetermined Coefficients* (sometimes referred to as the method of *Judicious Guessing*) is a systematic way (almost, but not quite, like using "educated guesses") to determine the general form/type of the particular solution $Y(t)$ based on the nonhomogeneous term $g(t)$ in the given equation. The basic idea is that many of the most familiar and commonly encountered functions have derivatives that vary little (in the form/type of function) from their parent functions: exponential, polynomials, sine and cosine. (Contrast them against log functions, whose derivatives, while simple and predictable, are rational functions; or tangent, whose higher derivatives quickly become a messy combinations of the powers of secant and tangent.) Consequently, when those functions appear in $g(t)$, we can predict the type of function that the solution Y would be. Write down the (best guess) form of Y, leaving the coefficient(s) undetermined. Then compute Y' and Y'', put them into the equation, and solve for the unknown coefficient(s). We shall see how this idea is put into practice in the following three simple examples.

Example: $y'' - 2y' - 3y = e^{2t}$

The corresponding homogeneous equation $y'' - 2y' - 3y = 0$ has characteristic equation $r^2 - 2r - 3 = (r + 1)(r - 3) = 0$. So the complementary solution is $y_c = C_1 e^{-t} + C_2 e^{3t}$.

The nonhomogeneous equation has $g(t) = e^{2t}$. It is an exponential function, which does not change form after differentiation: an exponential function's derivative will remain an exponential function with the same exponent (although its coefficient might change due to the effect of the Chain Rule). Therefore, we can very reasonably expect that $Y(t)$ is in the form $A e^{2t}$ for some unknown coefficient A. Our job is to find this as yet undetermined coefficient.

Let $Y = A e^{2t}$, then $Y' = 2A e^{2t}$, and $Y'' = 4A e^{2t}$. Substitute them back into the original differential equation:

$$(4A e^{2t}) - 2(2A e^{2t}) - 3(A e^{2t}) = e^{2t}$$

$$-3A e^{2t} = e^{2t}$$

$$A = -1/3$$

Hence, $Y(t) = \dfrac{-1}{3} e^{2t}$.

Therefore, $y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - \dfrac{1}{3} e^{2t}$.

Thing to remember: When an exponential function appears in $g(t)$, use an exponential function of the same exponent for Y.

Example: $$y'' - 2y' - 3y = 3t^2 + 4t - 5$$

The corresponding homogeneous equation is still $y'' - 2y' - 3y = 0$. Therefore, the complementary solution remains $y_c = C_1 e^{-t} + C_2 e^{3t}$.

Now $g(t) = 3t^2 + 4t - 5$. It is a degree 2 (i.e., quadratic) polynomial. Since polynomials, like exponential functions, do not change form after differentiation: the derivative of a polynomial is just another polynomial of one degree less (until it eventually reaches zero). We expect that $Y(t)$ will, therefore, be a polynomial of the same degree as that of $g(t)$. (Why will their degrees be the same?)

So, we will let Y be a generic quadratic polynomial: $Y = At^2 + Bt + C$. It follows $Y' = 2At + B$, and $Y'' = 2A$. Substitute them into the equation:

$$(2A) - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 + 4t - 5$$

$$-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) = 3t^2 + 4t - 5$$

The corresponding terms on both sides should have the same coefficients, therefore, equating the coefficients of like terms.

t^2 :	$3 = -3A$		$A = -1$
t :	$4 = -4A - 3B$	$\rightarrow$	$B = 0$
1 :	$-5 = 2A - 2B - 3C$		$C = 1$

Therefore, $Y = -t^2 + 1$, and $y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - t^2 + 1$.

Thing to remember: When a polynomial appears in $g(t)$, use a generic polynomial of the same degree for Y. That is $Y = A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0$. Note that if $g(t)$ is a (nonzero) constant, it is considered a polynomial of degree 0, and Y would therefore also be a generic polynomial of degree 0. That is, Y is an arbitrary nonzero constant: $Y = A_0$. Recall that the degree of a polynomial is the highest power that appears. Therefore, the rule can be stated a little differently to say that "look for the highest power of t in $g(t)$, then list it and all the lower powers (down to the constant term) in Y.

Example: $$y'' - 2y' - 3y = 5\cos(2t)$$

Again, the same corresponding homogeneous equation as the previous examples means that $y_c = C_1 e^{-t} + C_2 e^{3t}$ as before.

The nonhomogeneous term is $g(t) = 5\cos(2t)$. Cosine and sine functions do change form, slightly, when differentiated, but the pattern is simple, predictable, and repetitive: their respective forms just change to each other's. Consequently, we should choose the form $Y = A\cos(2t) + B\sin(2t)$. (Why do we choose to employ both cosine and sine?) Substitute Y, $Y' = -2A\sin(2t) + 2B\cos(2t)$, and $Y'' = -4A\cos(2t) - 4B\sin(2t)$ into the equation:

$(-4A\cos(2t) - 4B\sin(2t)) - 2(-2A\sin(2t) + 2B\cos(2t)) - 3(A\cos(2t) + B\sin(2t)) = 5\cos(2t)$

$(-4A - 4B - 3A)\cos(2t) + (-4B + 4A - 3B)\sin(2t) = 5\cos(2t)$

$(-7A - 4B)\cos(2t) + (4A - 7B)\sin(2t) = 5\cos(2t) + 0\sin(2t)$

Compare the coefficients:

$\cos(2t)$:	$5 = -7A - 4B$	$\rightarrow$	$A = -7/13$
$\sin(2t)$:	$0 = 4A - 7B$	$\rightarrow$	$B = -4/13$

Therefore, $Y = \frac{-7}{13}\cos(2t) - \frac{4}{13}\sin(2t)$, and

$$y = C_1 e^{-t} + C_2 e^{3t} - \frac{7}{13}\cos(2t) - \frac{4}{13}\sin(2t)$$

Thing to remember: When either cosine or sine appears in $g(t)$, <u>both</u> cosine and sine (of the same frequency) must appear in Y.

When $g(t)$ is a sum of several terms

When $g(t)$ is a sum of several functions: $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, we can break the equation into n parts and solve them separately. Given

$$y'' + p(t)\, y' + q(t)\, y = g_1(t) + g_2(t) + \ldots + g_n(t)$$

we change it into

$$y'' + p(t)\, y' + q(t)\, y = g_1(t)$$

$$y'' + p(t)\, y' + q(t)\, y = g_2(t)$$

$$\vdots$$

$$y'' + p(t)\, y' + q(t)\, y = g_n(t).$$

Solve them individually to find respective particular solutions $Y_1, Y_2, \ldots, Y_n$. Then add up them to get $Y = Y_1 + Y_2 + \ldots + Y_n$.

Comment: The above is a consequence of the general version of the *Superposition Principle*[*]:

General Principle of Superposition: If y_1 is a solution of the equation

$$y'' + p(t)\, y' + q(t)\, y = g_1(t),$$

and y_2 is a solution of the equation

$$y'' + p(t)\, y' + q(t)\, y = g_2(t).$$

Then, for any pair of constants C_1 and C_2, the function $y = C_1\, y_1 + C_2\, y_2$ is a solution of the equation

$$y'' + p(t)\, y' + q(t)\, y = C_1\, g_1(t) + C_2\, g_2(t).$$

[*] Note that when $g_1(t) = g_2(t) = 0$, the above becomes the homogeneous linear equation version of the Superposition Principle seen in an earlier section.

Example: $$y'' - 2y' - 3y = e^{2t} + 3t^2 + 4t - 5 + 5\cos(2t)$$

Solve each of the sub-parts:

$$y'' - 2y' - 3y = e^{2t} \quad \rightarrow \quad Y_1(t) = \frac{-1}{3}e^{2t}$$

$$y'' - 2y' - 3y = 3t^2 + 4t - 5 \quad \rightarrow \quad Y_2(t) = -t^2 + 1$$

$$y'' - 2y' - 3y = 5\cos(2t) \quad \rightarrow \quad Y_3(t) = \frac{-7}{13}\cos(2t) - \frac{4}{13}\sin(2t)$$

Then add up the partial results:

$$Y(t) = \frac{-1}{3}e^{2t} - t^2 + 1 - \frac{7}{13}\cos(2t) - \frac{4}{13}\sin(2t).$$

The general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{3t} - \frac{1}{3}e^{2t} - t^2 + 1 - \frac{7}{13}\cos(2t) - \frac{4}{13}\sin(2t).$$

If initial conditions are present, only apply the initial values after the general solution is found to find the particular solution. While it might be tempting to solve for the coefficients C_1 and C_2 as soon as they appear (they would appear with the complementary solution y_c, at the very beginning), we nevertheless could not have found them without knowing Y. Since the initial values consist of contribution from both parts y_c and Y. Therefore, we must wait until we have found the general solution in its entirety before applying the initial values to find C_1 and C_2.

Example: $y'' - 2y' - 3y = 3t^2 + 4t - 5, \qquad y(0) = 9, \quad y'(0) = -4$

First find the general solution: $y = C_1 e^{-t} + C_2 e^{3t} - t^2 + 1$.
Then use the initial conditions to find that $C_1 = 7$ and $C_2 = 1$.

The particular solution is: $y = 7e^{-t} + e^{3t} - t^2 + 1$.

A (possible) glitch?

There is a complication that occurs under a certain circumstance…

Example: $$y'' - 2y' - 3y = 5e^{3t}$$

The old news is that $y_c = C_1 e^{-t} + C_2 e^{3t}$. Since $g(t) = 5e^{3t}$, we should be able to use the form $Y = A e^{3t}$, just like in the first example, right? But if we substitute Y, $Y' = 3A e^{3t}$, and $Y'' = 9A e^{3t}$ into the differential equation and simplify, we would get the equation

$$0 = 5e^{3t}.$$

That means there is no solution for A. Our method (that has worked well thus far) seems to have failed. The same outcome (an inability to find A) also happens when $g(t)$ is a multiple of e^{-t}. But, for any other exponent our choice of the form for Y works. What is so special about these two particular exponential functions, e^{3t} and e^{-t}, that causes our method to misfire? (Hint: What is the complementary solution of the nonhomgeneous equation?)

The answer is that those two functions are exactly the terms in y_c. Being a part of the complementary solution (the solution of the corresponding homogeneous equation) means that any constant multiple of either functions will ALWAYS results in zero on the right-hand side of the equation. Therefore, it is impossible to match the given $g(t)$.

The cure: The remedy is surprisingly simple: <u>multiply our usual choice by *t*</u>. In the above example, we should instead use the form $Y = A t e^{3t}$.

In general, whenever your initial choice of the form of Y has any term in common with the complementary solution, then you must alter it by multiplying your initial choice of Y by t, <u>as many times as necessary but no more than necessary</u>.

Example: $$y'' - 6y' + 9y = e^{3t}$$

The complementary solution is $y_c = C_1 e^{3t} + C_2 t e^{3t}$. $g(t) = e^{3t}$, therefore, the initial choice would be $Y = A e^{3t}$. But wait, that is the same as the first term of y_c, so multiply Y by t to get $Y = A t e^{3t}$. However, the new Y is now in common with the second term of y_c. Multiply it by t again to get $Y = A t^2 e^{3t}$. That is the final, correct choice of the general form of Y to use. (*Exercise*: Verify that neither $Y = A e^{3t}$, nor $Y = A t e^{3t}$ would yield an answer to this problem.)

Once we have established that $Y = A t^2 e^{3t}$, then $Y' = 2At e^{3t} + 3A t^2 e^{3t}$, and $Y'' = 2A e^{3t} + 12At e^{3t} + 9A t^2 e^{3t}$. Substitute them back into the original equation:

$$(2A e^{3t} + 12At e^{3t} + 9A t^2 e^{3t}) - 6(2At e^{3t} + 3A t^2 e^{3t}) + 9(A t^2 e^{3t}) = e^{3t}$$

$$2A e^{3t} + (12 - 12)At e^{3t} + (9 - 18 + 9)A t^2 e^{3t} = e^{3t}$$

$$2A e^{3t} = e^{3t}$$

$$A = 1/2$$

Hence, $Y(t) = \frac{1}{2} t^2 e^{3t}$.

Therefore, $y = C_1 e^{3t} + C_2 t e^{3t} + \frac{1}{2} t^2 e^{3t}$. Our "cure" has worked!

Since a second order linear equation's complementary solution only has two parts, there could be at most two shared terms with Y. Consequently we would only need to, at most, apply the cure twice (effectively multiplying by t^2) as the worst case scenario.

The lesson here is that <u>you should always find the complementary solution first</u>, since the correct choice of the form of Y depends on y_c. Therefore, you need to have y_c handy before you write down the form of Y. Before you finalize your choice, always compare it against y_c. And if there is anything those two have in common, multiplying your choice of form of Y by t. (However, you should do this <u>ONLY</u> when there actually exists something in common; you should never apply this cure unless you know for sure that a common term exists between Y and y_c, else you will not be able to find the correct answer!) Repeat until there is no shared term.

When $g(t)$ is a product of several functions

If $g(t)$ is a product of two or more simple functions, e.g. $g(t) = t^2 e^{5t}\cos(3t)$, then our basic choice (before multiplying by t, if necessary) should be a product consist of the corresponding choices of the individual components of $g(t)$. One thing to keep in mind: <u>that there should be only as many undetermined coefficients in Y as there are distinct terms</u> (after expanding the expression and simplifying algebraically).

Example: $$y'' - 2y' - 3y = t^3 e^{5t}\cos(3t)$$

We have $g(t) = t^3 e^{5t}\cos(3t)$. It is a product of a degree 3 polynomial[†], an exponential function, and a cosine. Out choice of the form of Y therefore must be a product of their corresponding choices: a generic degree 3 polynomial, an exponential function, and both cosine and sine. Try

Correct form: $Y = (At^3 + Bt^2 + Ct + D)\, e^{5t}\cos(3t) + (Et^3 + Ft^2 + Gt + H)\, e^{5t}\sin(3t)$

Wrong form: $Y = (At^3 + Bt^2 + Ct + D)\, E\, e^{5t}(F\cos(3t) + G\sin(3t))$

Note in the correct form above, each of the eight distinct terms has its own unique undetermined coefficient. Here is another thing to remember: that those coefficients should all be <u>independent</u> of each others, each uniquely associated with only one term.

In short, when $g(t)$ is a product of basic functions, $Y(t)$ is chosen based on:

i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
iii. Each distinct term must have its own coefficient, not shared with any other term.

[†] A power such as t^n is really just an n-th degree polynomial with only one (the n-th term's) nonzero coefficient.

Another way (longer, but less prone to mistakes) to come up with the correct form is to do the following.

Start with the basic forms of the corresponding functions that are to appear in the product, without assigning any coefficient. In the above example, they are $(t^3 + t^2 + t + 1)$, e^{5t}, and $\cos(3t) + \sin(3t)$.

Multiply them together to get all the distinct terms in the product:

$$(t^3 + t^2 + t + 1)\, e^{5t}(\cos(3t) + \sin(3t))$$
$$= t^3 e^{5t} \cos(3t) + t^2 e^{5t} \cos(3t) + t\, e^{5t} \cos(3t) + e^{5t} \cos(3t)$$
$$+ t^3 e^{5t} \sin(3t) + t^2 e^{5t} \sin(3t) + t\, e^{5t} \sin(3t) + e^{5t} \sin(3t)$$

Once we have expanded the product and identified the distinct terms in the product (8, in this example), then we insert the undetermined coefficients into the expression, one for each term:

$$Y = A\, t^3 e^{5t} \cos(3t) + B\, t^2 e^{5t} \cos(3t) + C\, t\, e^{5t} \cos(3t)$$
$$+ D\, e^{5t} \cos(3t) + E\, t^3 e^{5t} \sin(3t) + F\, t^2 e^{5t} \sin(3t) + G\, t\, e^{5t} \sin(3t)$$
$$+ H\, e^{5t} \sin(3t)$$

Which is the correct form of Y seen previously.

Therefore, whenever you have doubts as to what the correct form of Y for a product is, just first explicitly list all of terms you expect to see in the result. Then assign each term an undetermined coefficient.

Remember, however, the result obtained still needs to be compared against the complementary solution for shared term(s). If there is any term in common, then the entire complex of product that is the choice for Y must be multiplied by t. Repeat as necessary.

Example: $$y'' + 25y = 4t^3 \sin(5t) - 2e^{3t} \cos(5t)$$

The complementary solution is $y_c = C_1 \cos(5t) + C_2 \sin(5t)$. Let's break up $g(t)$ into 2 parts and work on them individually.

$g_1(t) = 4t^3 \sin(5t)$ is a product of a degree 3 polynomial and a sine function. Therefore, Y_1 should be a product of a generic degree 3 polynomial and both cosine and sine:

$$Y_1 = (At^3 + Bt^2 + Ct + D)\cos(5t) + (Et^3 + Ft^2 + Gt + H)\sin(5t)$$

The validity of the above choice of form can be verified by our second (longer) method. Note that the product of a degree 3 polynomial and both cosine and sine: $(t^3 + t^2 + t + 1)\times(\cos(5t) + \sin(5t))$ contains 8 distinct terms listed below.

$t^3\cos(5t)$	$t^2\cos(5t)$	$t\cos(5t)$	$\cos(5t)$
$t^3\sin(5t)$	$t^2\sin(5t)$	$t\sin(5t)$	$\sin(5t)$

Now insert 8 independent undetermined coefficients, one for each:

$$\begin{aligned} Y_1 = {} & At^3\cos(5t) + Bt^2\cos(5t) + Ct\cos(5t) + D\cos(5t) + \\ & Et^3\sin(5t) + Ft^2\sin(5t) + Gt\sin(5t) + H\sin(5t) \end{aligned}$$

However, there is still one important detail to check before we could put the above expression down for Y_1. Is there anything in the expression that is shared with $y_c = C_1 \cos(5t) + C_2 \sin(5t)$? As we can see, there are – both the fourth and the eighth terms. Therefore, we need to multiply everything in this entire expression by t. Hence,

$$\begin{aligned} Y_1 &= t(At^3 + Bt^2 + Ct + D)\cos(5t) + \\ &\qquad t(Et^3 + Ft^2 + Gt + H)\sin(5t) \\ &= (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + \\ &\qquad (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t). \end{aligned}$$

The second half of $g(t)$ is $g_2(t) = -2e^{3t}\cos(5t)$. It is a product of an exponential function and cosine. So our choice of form for Y_2 should be a product of an exponential function with both cosine and sine.

$$Y_2 = I e^{3t}\cos(5t) + J e^{3t}\sin(5t).$$

There is no conflict with the complementary solution – even though both $\cos(5t)$ and $\sin(5t)$ are present within both y_c and Y_2, they appear alone in y_c, but in products with e^{3t} in Y_2, making them parts of completely different functions. Hence this is the correct choice.

Finally, the complete choice of Y is the sum of Y_1 and Y_2.

$$Y = Y_1 + Y_2 = (At^4 + Bt^3 + Ct^2 + Dt)\cos(5t) + (Et^4 + Ft^3 + Gt^2 + Ht)\sin(5t) + I e^{3t}\cos(5t) + J e^{3t}\sin(5t).$$

Example: $$y'' - 8y' + 12y = t^2 e^{6t} - 7t\sin(2t) + 4$$

Complementary solution: $y_c = C_1 e^{2t} + C_2 e^{6t}$.

The form of particular solution is

$Y = (At^3 + Bt^2 + Ct)e^{6t} + (Dt + E)\cos(2t) + (Ft + G)\sin(2t) + H.$

Example: $$y'' + 10y' + 25y = t e^{-5t} - 7t^2 e^{2t}\cos(4t) + 3t^2 - 2$$

Complementary solution: $y_c = C_1 e^{-5t} + C_2\, t e^{-5t}$.

The form of particular solution is

$Y = (At^3 + Bt^2)e^{-5t} + (Ct^2 + Dt + E)e^{2t}\cos(4t) + (Ft^2 + Gt + H)e^{2t}\sin(4t)$
$+ It^2 + Jt + K.$

Example: Find a second order linear equation with constant coefficients whose general solution is

$$y = C_1 e^t + C_2 e^{-10t} + 4t^2.$$

The solution contains three parts, so it must come from a nonhomogeneous equation. The complementary part of the solution, $y_c = C_1 e^t + C_2 e^{-10t}$ suggests that $r = 1$ and $r = -10$ are the two roots of its characteristic equation. Hence, $r - 1$ and $r + 10$ are its two factors. Therefore, the characteristic equation is $(r - 1)(r + 10) = r^2 + 9r - 10$.

The corresponding homogeneous equation is, as a result,

$$y'' + 9y' - 10y = 0.$$

Hence, the nonhomogeneous equation is

$$y'' + 9y' - 10y = g(t).$$

The nonhomogeneous part $g(t)$ results in the particular solution $Y = 4t^2$. As well, $Y' = 8t$ and $Y'' = 8$. Therefore,

$$g(t) = Y'' + 9Y' - 10Y = 8 + 9(8t) - 10(4t^2) = 8 + 72\mathrm{t} - 40\, t^2.$$

The equation with the given general solution is, therefore,

$$y'' + 9y' - 10y = 8 + 72\mathrm{t} - 40\, t^2.$$

The 6 Rules-of-Thumb of the Method of Undetermined Coefficients

1. If an exponential function appears in $g(t)$, the starting choice for $Y(t)$ is an exponential function of the same exponent.

2. If a polynomial appears in $g(t)$, the starting choice for $Y(t)$ is a generic polynomial of the same degree.

3. If either cosine or sine appears in $g(t)$, the starting choice for $Y(t)$ needs to contain both cosine and sine of the same frequency.

4. If $g(t)$ is a sum of several functions, $g(t) = g_1(t) + g_2(t) + \ldots + g_n(t)$, separate it into n parts and solve them individually.

5. If $g(t)$ is a product of basic functions, the starting choice for $Y(t)$ is chosen based on:

   i. $Y(t)$ is a product of the corresponding choices of all the parts of $g(t)$.
   ii. There are as many coefficients as the number of distinct terms in $Y(t)$.
   iii. Each distinct term must have its own coefficient, not shared with any other term.

6. Before finalizing the choice of $Y(t)$, compare it against $y_c(t)$. If there is any shared term between the two, the present choice of $Y(t)$ needs to be multiplied by t. Repeat until there is no shared term.

Remember that, in order to use Rule 6 you always need to find the complementary solution first.

SUMMARY: Method of Undetermined Coefficients

Given $$a\,y'' + b\,y' + c\,y = g(t)$$

1. Find the complementary solution y_c.

2. Subdivide, if necessary, $g(t)$ into parts: $g(t) = g_1(t) + g_2(t) \ldots + g_k(t)$.

3. For each $g_i(t)$, choose the form of its corresponding particular solution $Y_i(t)$ according to:

$g_i(t)$	$Y_i(t)$
$P_n(t)$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$
$P_n(t)\, e^{at}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)\, e^{at}$
$P_n(t)\, e^{at} \cos \mu t$ and/or $P_n(t)\, e^{at} \sin \mu t$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0)\, e^{at} \cos \mu t$ + $t^s(B_n t^n + B_{n-1} t^{n-1} + \ldots + B_0)\, e^{at} \sin \mu t$

Where $s = 0$, 1, or 2, is the **minimum** number of times the choice must be multiplied by t so that it shares no common terms with y_c.

$P_n(t)$ denotes a n-th degree polynomial. If there is no power of t present, then $n = 0$ and $P_0(t) = C_0$ is just the constant coefficient. If no exponential term is present, then set the exponent $a = 0$.

4. $Y = Y_1 + Y_2 + \ldots + Y_k$.

5. The general solution is $y = y_c + Y$.

6. Finally, apply any initial conditions to determine the as yet unknown coefficients C_1 and C_2 in y_c.

Exercises B-2.1:

1 − 10 Find the general solution of each nonhomogeneous equation.

1. $y'' + 4y = 8$

2. $y'' + 4y = 8t^2 - 20t + 8$

3. $y'' + 4y = 5\sin 3t - 5\cos 3t$

4. $y'' + 4y = 24e^{-2t}$

5. $y'' + 4y = 8\cos 2t$

6. $y'' + 2y' = 2te^{-t}$

7. $y'' + 2y' = 6e^{-2t}$

8. $y'' + 2y' = 12t^2$

9. $y'' - 6y' - 7y = 13\cos 2t + 34\sin 2t$

10. $y'' - 6y' - 7y = 8e^{-t} - 7t - 6$

11 – 15 Solve each initial value problem.

11. $y'' - 6y' - 7y = -9e^{-2t}$, $\quad y(0) = -2, \quad y'(0) = -13$

12. $y'' - 6y' - 7y = 6e^{2t}$, $\quad y(0) = 5, \quad y'(0) = -3$

13. $y'' - 4y' + 4y = 2e^{2t} - 12\cos 3t - 5\sin 3t$, $\quad y(0) = -2, \quad y'(0) = 4$

14. $y'' - 2y' - 8y = 8t^2 + 20t + 2$, $\quad y(0) = 0, \quad y'(0) = -8$

15. $y'' - 2y' + 4y = 8t - 12\sin 2t$, $\quad y(0) = -2, \quad y'(0) = 8$

16 – 18 Determine the most suitable form of $Y(t)$ that should be used when solving each equation.

16. $y'' - 9y = 6t^4e^{3t} - 2e^{-3t}\sin 9t$

17. $y'' - 4y' + 4y = 5t^3 - 2 - t^2e^{2t} + 4e^{2t}\cos t$

18. $y'' + 4y' + 20y = t^2 e^{-2t} \sin 4t - 3\cos 4t - t e^{-2t}$

19. Find a second order linear equation with constant coefficients whose general solution is $y = C_1 \cos 4t + C_2 \sin 4t - e^t \sin 2t$.

20. Find a second order linear equation with constant coefficients whose general solution is $y = C_1 e^{-2t} + C_2 t e^{-2t} + t^3 - 3t$.

21. Suppose $y_1 = 2t \sin 3t$ is a solution of the equation

$$y'' + 2y' + 2y = g_1(t),$$

and $y_2 = \cos 6t - e^{-t} \cos t$ is a solution of the equation

$$y'' + 2y' + 2y = g_2(t).$$

What is the general solution of

$$y'' + 2y' + 2y = 5g_1(t) - 2g_2(t)?$$

22. Suppose the equation $y'' - 4y' - 5y = g(t)$ has $y = 3t^3$ as a solution.
(a) Which one(s) of the following functions is/are also a solution(s)?

(i) $y = e^{-t} + 3t^3$, (ii) $y = \pi e^{5t} + 3t^3$, (iii) $y = 2e^{5t} + 4e^{-t}$,
(iv) $y = e^{-t} + e^{5t} + 6t^3$, (v) $y = 3e^{5t} - 4e^{-t} + 3t^3$

(b) What is the general solution of the equation? (c) Find $g(t)$. (d) Given that $y(0) = 3$ and $y'(0) = 3$, solve the initial value problem.

Answers B-2.1:

1. $y = C_1 \cos 2t + C_2 \sin 2t + 2$
2. $y = C_1 \cos 2t + C_2 \sin 2t + 2t^2 - 5t + 1$
3. $y = C_1 \cos 2t + C_2 \sin 2t + \cos 3t - \sin 3t$
4. $y = C_1 \cos 2t + C_2 \sin 2t + 3e^{-2t}$
5. $y = C_1 \cos 2t + C_2 \sin 2t + 2t \sin 2t$
6. $y = C_1 e^{-2t} + C_2 - 2t e^{-t}$
7. $y = C_1 e^{-2t} + C_2 - 3t e^{-2t}$
8. $y = C_1 e^{-2t} + C_2 + 2t^3 - 3t^2 + 3t$
9. $y = C_1 e^{-t} + C_2 e^{7t} + \cos 2t - 2\sin 2t$
10. $y = C_1 e^{-t} + C_2 e^{7t} + t - t e^{-t}$
11. $y = e^{-t} - 2e^{7t} - e^{-2t}$
12. $y = 5e^{-t} + \dfrac{2}{5}e^{7t} - \dfrac{2}{5}e^{2t}$
13. $y = -2e^{2t} + 5t e^{2t} + t^2 e^{2t} + \sin 3t$
14. $y = e^{-2t} - e^{4t} - t^2 - 2t$
15. $y = 2\sqrt{3}\, e^t \sin(\sqrt{3}\, t) + 2t + 1 - 3\cos 2t$
16. $Y = (At^5 + Bt^4 + Ct^3 + Dt^2 + Et)e^{3t} + Fe^{-3t}\cos 9t + Ge^{-3t}\sin 9t$
17. $Y = At^3 + Bt^2 + Ct + D + (Et^4 + Ft^3 + Gt^2)e^{2t} + He^{2t}\cos t + Ie^{2t}\sin t$
18. $Y = (At^3 + Bt^2 + Ct)e^{-2t}\cos 4t + (Dt^3 + Et^2 + Ft)e^{-2t}\sin 4t + G\cos 4t$
    $+ H\sin 4t + (It + J)e^{-2t}$
19. $y'' + 16y = -4e^t \cos 2t - 13e^t \sin 2t$
20. $y'' + 4y' + 4y = 4t^3 + 12t^2 - 6t - 12$
21. $y = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + 10t \sin 3t - 2\cos 6t$
22. (a) i, ii, v (b) $y = C_1 e^{-t} + C_2 e^{5t} + 3t^3$
    (c) $g(t) = 18t - 36t^2 - 15t^3$ (d) $y = e^{5t} + 2e^{-t} + 3t^3$