Math 231 Midterm Prep

1. Let $\mathbf{v} = \langle 1, 1, 2 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$.

a. Find the unit vector in the same direction as $\mathbf{v}$.

$|\mathbf{v}| = \sqrt{1+1+4} = \sqrt{6}$. Unit vector is $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \left\langle \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}} \right\rangle$.

b. Find the dot product $\mathbf{v} \cdot \mathbf{w}$.

$\mathbf{v} \cdot \mathbf{w} = (1)(-2) + (1)(3) + (2)(1) = -2 + 3 + 2 = 3.$

c. Find the cross product $\mathbf{v} \times \mathbf{w}$.

Using determinants:

$$\begin{aligned}
\mathbf{v} \times \mathbf{w} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -2 & 3 & 1 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \\
&= (1-6)\,\mathbf{i} - (1+4)\,\mathbf{j} + (3+2)\,\mathbf{k} \\
&= -5\,\mathbf{i} - 5\,\mathbf{j} + 5\,\mathbf{k} \qquad \text{or} \qquad \langle -5, -5, 5 \rangle
\end{aligned}$$

You can use polynomial expansion or the huge formula if you prefer.

d. Find the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} = \frac{3}{14}\langle -2, 3, 1 \rangle = \left\langle \frac{-3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

2. What is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ in terms of $|\mathbf{a}|$, $|\mathbf{b}|$, and θ (the angle between the vectors)?

$|\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin\theta$. This is also $|\mathbf{a} \times \mathbf{b}|$.

3. **a.** Given two vectors, how can you tell if they are parallel?

One vector is a scalar multiple of the other.

b. Given two lines, how can you tell if they are parallel?

The vectors for the directions of the line are parallel.

c. Given two planes, how can you tell if they are parallel?

Their normal vectors are parallel.

d. Given a vector and a line, how can you tell if they are parallel?

The vector giving the direction of the line is parallel to the given vector.

e. Given a vector and a plane, how can you tell if they are parallel?

The plane's normal vector is **orthogonal** to the given vector.

f. Given a line and a plane, how can you tell if they are parallel?

The plane's normal vector is **orthogonal** to the vector giving the direction of the line.

4. **a.** Given two vectors, how can you tell if they are orthogonal?

Their dot product is zero.

b. Given two lines, how can you tell if they are perpendicular?

The vectors for the directions of the line are orthogonal.

c. Given two planes, how can you tell if they are orthogonal?

Their normal vectors are orthogonal.

d. Given a vector and a line, how can you tell if they are perpendicular?

The vector giving the direction of the line is orthogonal to the given vector.

e. Given a vector and a plane, how can you tell if they are normal?

The plane's normal vector is **parallel** to the given vector.

f. Given a line and a plane, how can you tell if they are normal?

The plane's normal vector is **parallel** to the vector giving the direction of the line.

5. Consider the lines with parametric equations

$$\begin{aligned} \ell_1 &: \quad x = t \quad , \; y = 2t - 2, \; z = t + 10 \\ \ell_2 &: \quad x = 1 - s, \; y = -2s \quad , \; z = 7 - 3s \end{aligned}$$

a. Find the point of intersection of ℓ_1 and ℓ_2.

The intersection occurs when

$$t = 1 - s, \qquad 2t - 2 = -2s, \qquad t + 10 = 7 - 3s.$$

Substituting $t = 1 - s$ into the third equation gives

$$\begin{aligned} t + 10 &= 7 - 3s \\ (1 - s) + 10 &= 7 - 3s \\ 11 - s &= 7 - 3s \\ 4 &= -2s \\ s &= -2 \end{aligned}$$

When $s = -2,\ t = 1 - s = 1 - (-2) = 3.$ This is the point

$$\begin{aligned} \mathbf{r}_1(3) &= \langle\, 3, 2(3) - 2, 3 + 10 \,\rangle = \langle\, 3, 4, 13 \,\rangle \\ \mathbf{r}_2(-2) &= \langle\, 1 - (-2), -2(-2), 7 - 3(-2) \,\rangle = \langle\, 3, 4, 13 \,\rangle \end{aligned}$$

(Calculating both $\mathbf{r}_1(3)$ and $\mathbf{r}_2(-2)$ is just to double-check that they are in fact the same point.)

b. Find a vector normal to the plane containing ℓ_1 and ℓ_2.

The line ℓ_1 is parallel to $\mathbf{v}_1 = \langle 1, 2, 1 \rangle$, and the line ℓ_2 is parallel to $\mathbf{v}_2 = \langle -1, -2, -3 \rangle$. A vector normal to the plane will be orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Use

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -4, 2, 0 \rangle .$$

Any vector parallel to this, e.g., $\langle -2, 1, 0 \rangle$ or $\langle 2, -1, 0 \rangle$, is also correct.

c. Write an equation for the plane containing ℓ_1 and ℓ_2 in "standard form" (that is, $ax + by + cz + d = 0$).

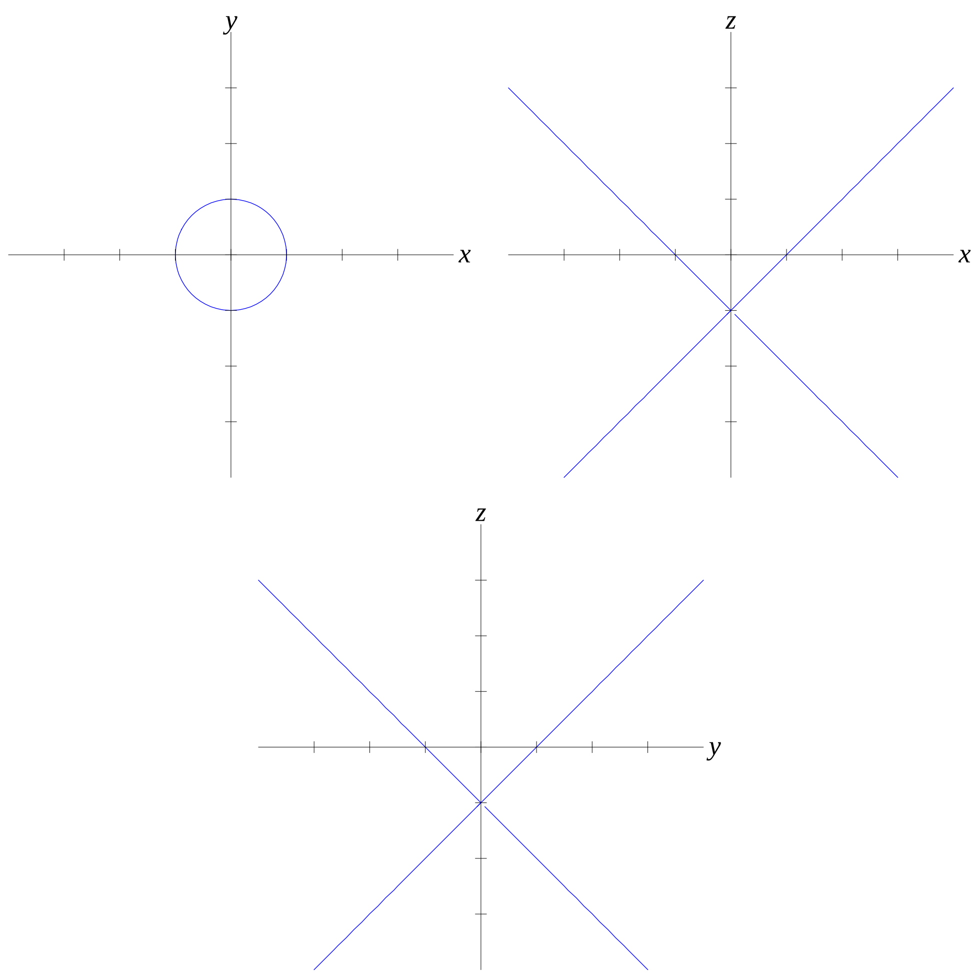
The plane through the point $(3, 4, 13)$ normal to $\langle -4, 2, 0 \rangle$ is

$$\begin{aligned} -4(x-3) + 2(y-4) + 0(z-13) &= 0 \\ -4x + 12 + 2y - 8 + 0 &= 0 \\ -4x + 2y + 4 &= 0 \end{aligned}$$

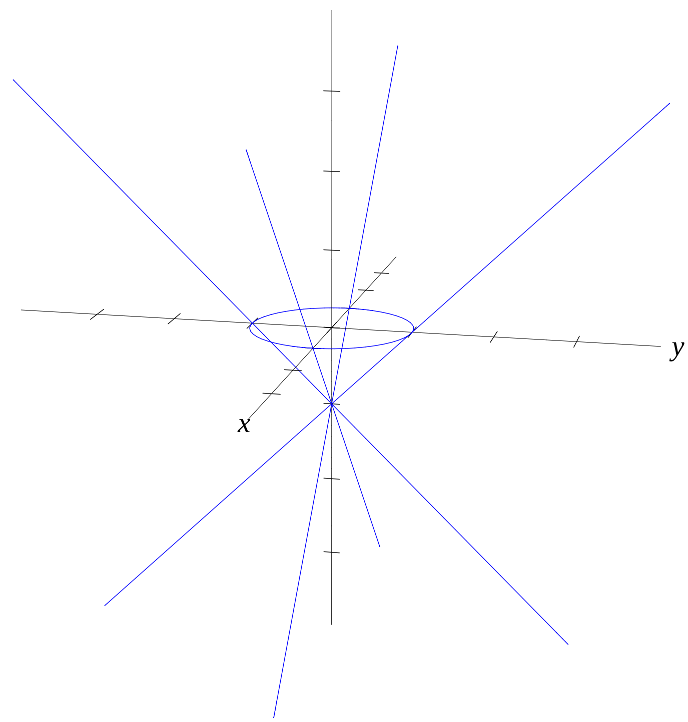
Any multiple of this equation, such as $-2x + y + 2 = 0$ or $2x - y - 2 = 0$, is also correct.

6. a. Sketch and identify the surface $x^2 + y^2 - (z+1)^2 = 0$.

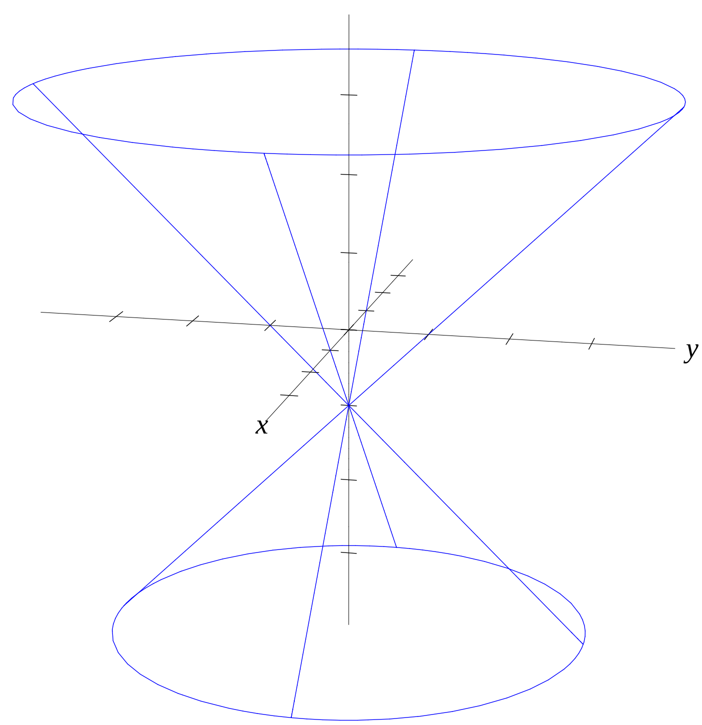
To understand this surface, draw its cross-sections in the coordinate planes:



Drawing all of these at once gives

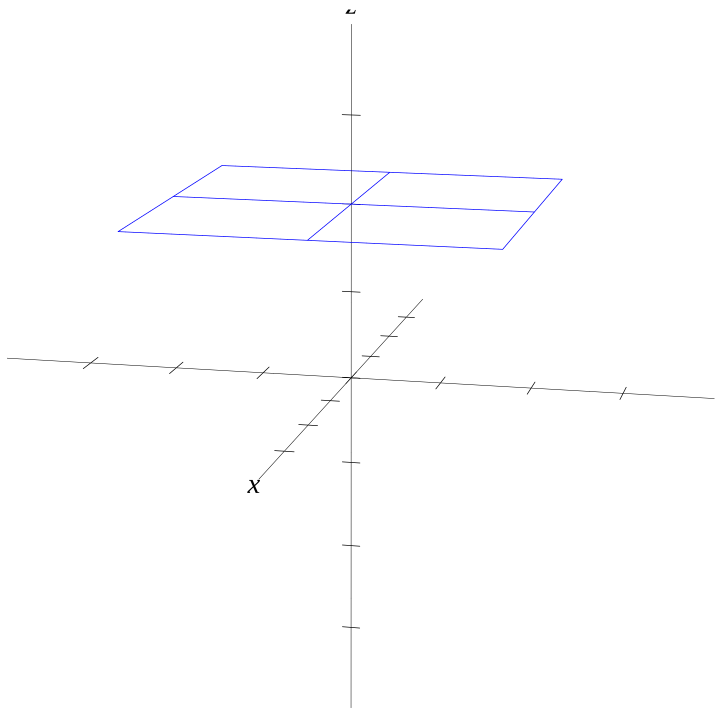


This is a double-cone. Your drawing should look more like this:



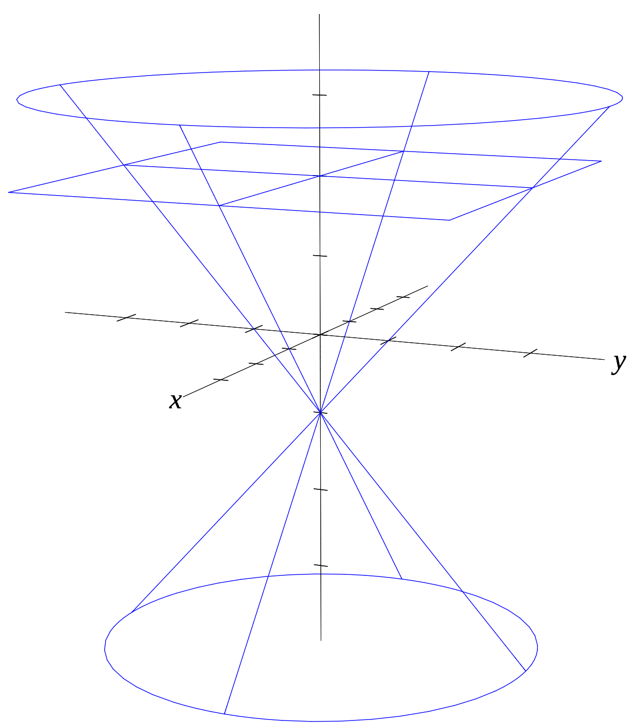
b. Sketch and identify the surface $z = 2$.

This is a plane parallel to the xy-plane at a height of 2.

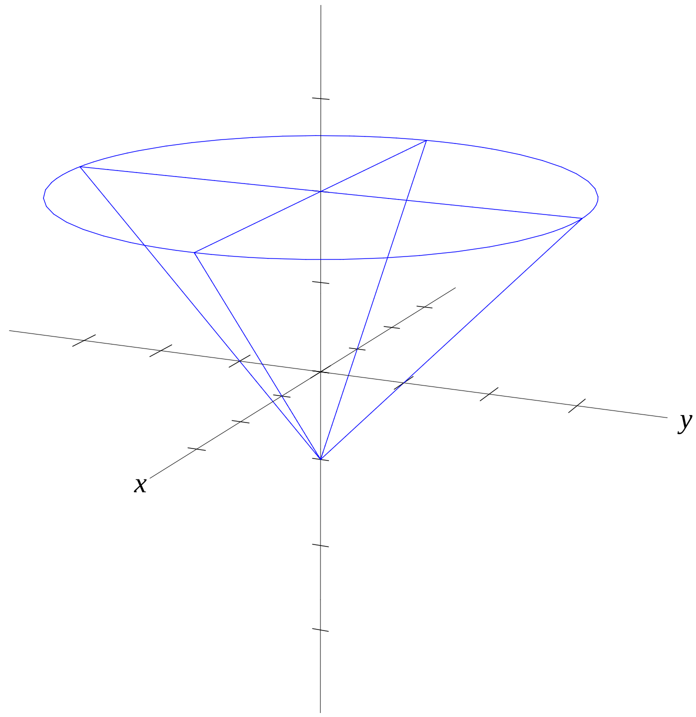


c. Sketch the **region** in space bounded by $x^2 + y^2 - (z+1)^2 = 0$ and $z = 2$.

Just drawing the cone and the plane on the same axes gives

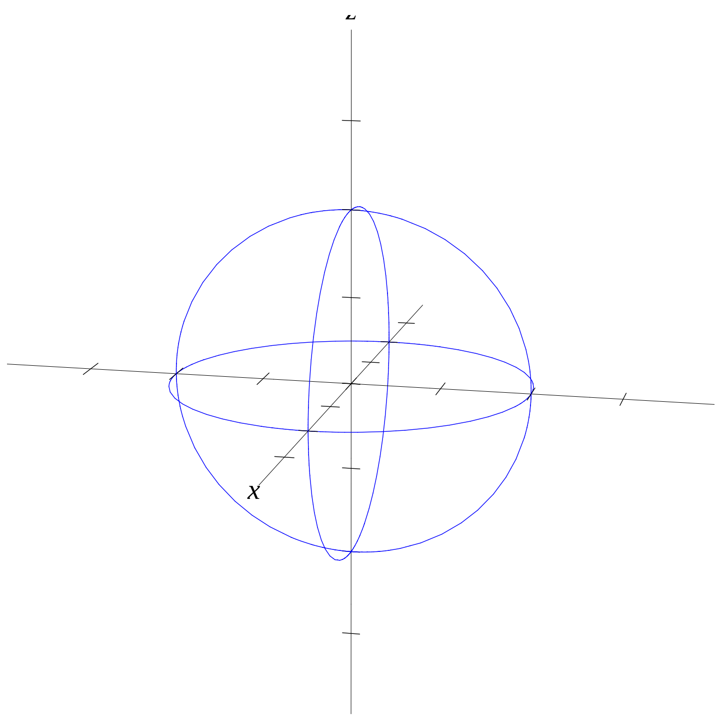


The region bounded by the cone and the plane is a "chopped off cone:"



7. **a.** Sketch and identify the surface $x^2 + y^2 + z^2 = 4$.

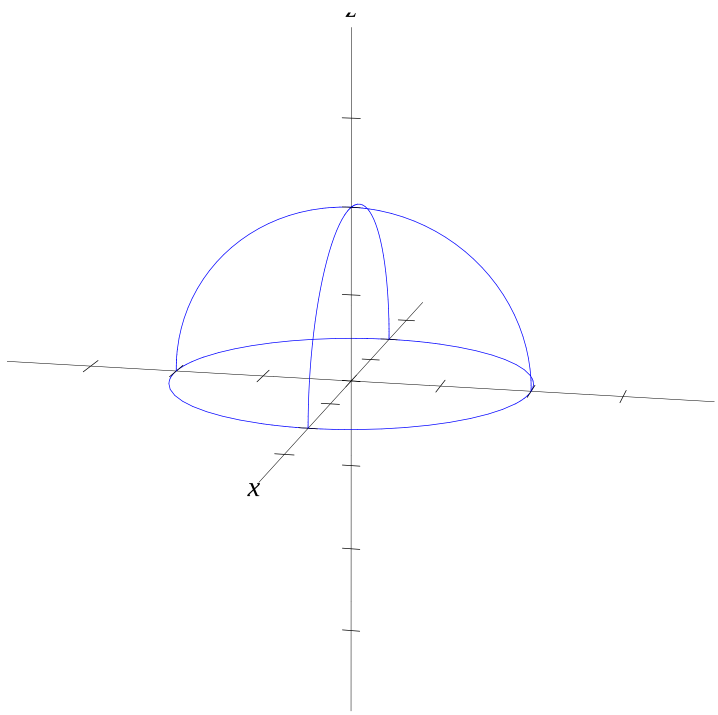
This is a sphere of radius 2.



b. Sketch and identify the surface $z = \sqrt{4 - x^2 - y^2}$.

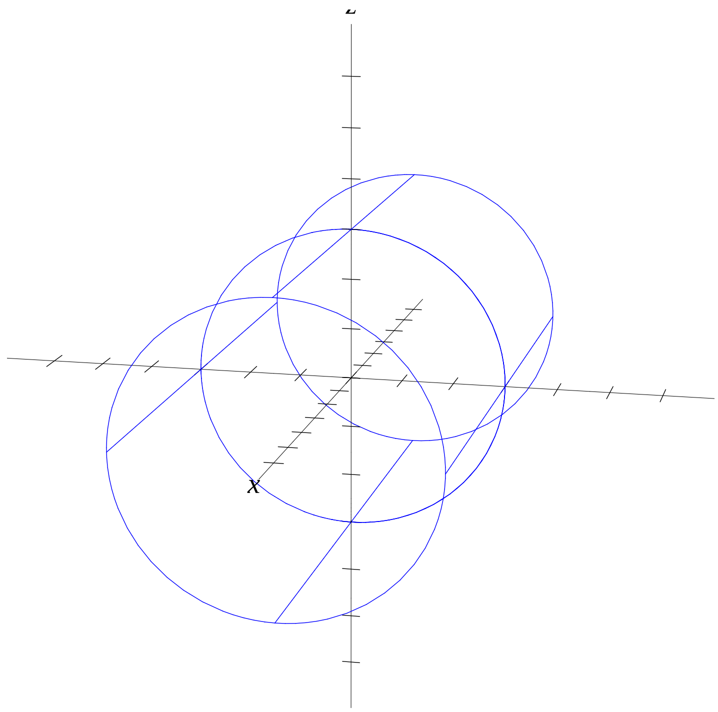
$$\begin{aligned} z &= \sqrt{4 - x^2 - y^2} \\ z^2 &= 4 - x^2 - y^2 \\ x^2 + y^2 + z^2 &= 4 \end{aligned}$$

This is the same equation! However, since z is now written as a square root, z must be positive (or 0). That means this a actually a hemisphere:

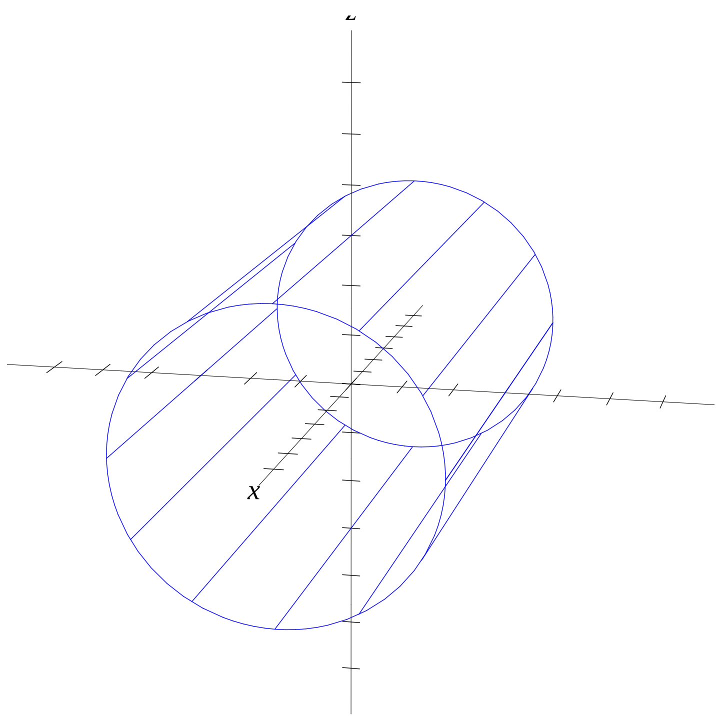


8. **a.** Sketch and identify the surface $y^2 + z^2 = 9$.

In the yz-plane, this would be a circle. In 3D space, this equation describes a cylinder.

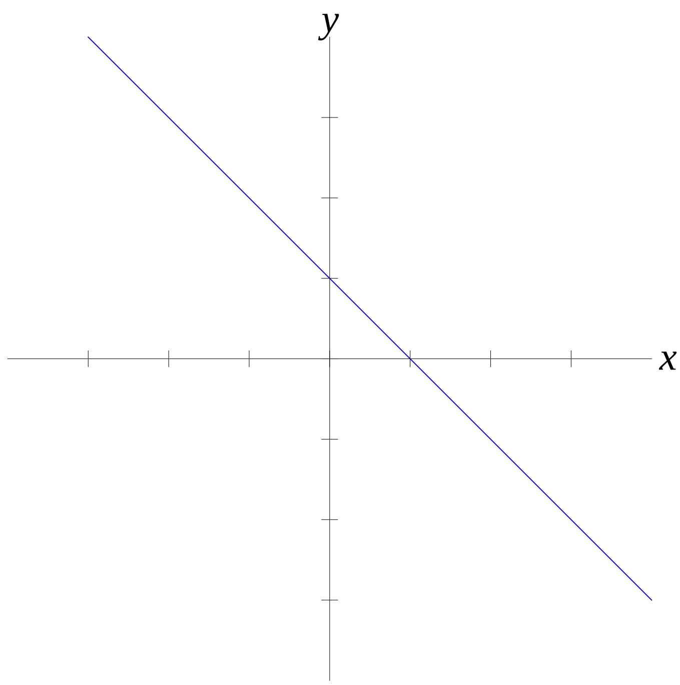


It might be easier to see with more lines (and without the circle in the center):

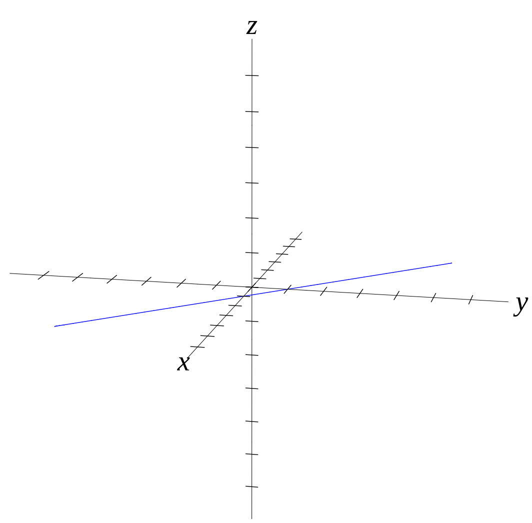


b. Sketch and identify the surface $x + y = 1$.

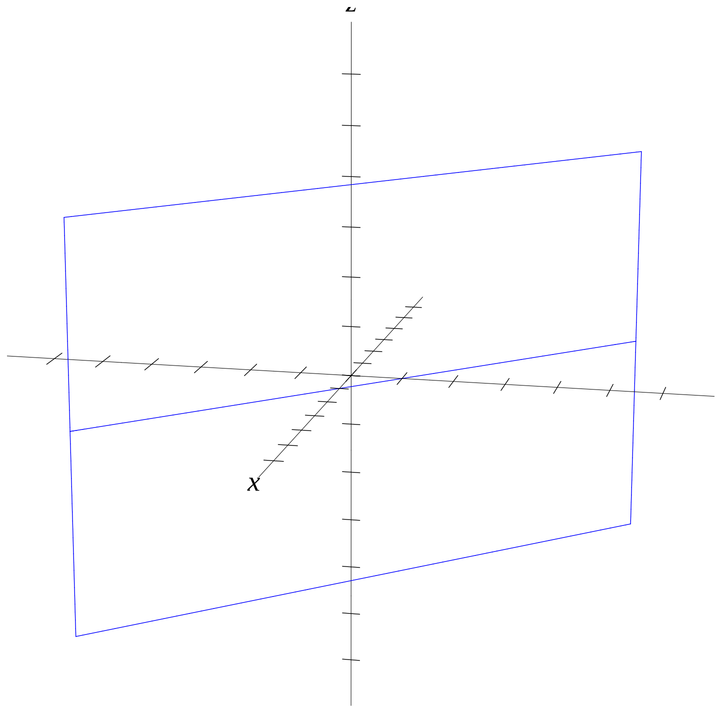
This is a plane. Since it is not parallel to a coordinate plane, it is a little harder to draw. However, since z is absent from this equation, we can also think of it as a cylinder. In the xy-plane, the equation $x + y = 1$ is a line:



Here is the same line drawn in 3D space:

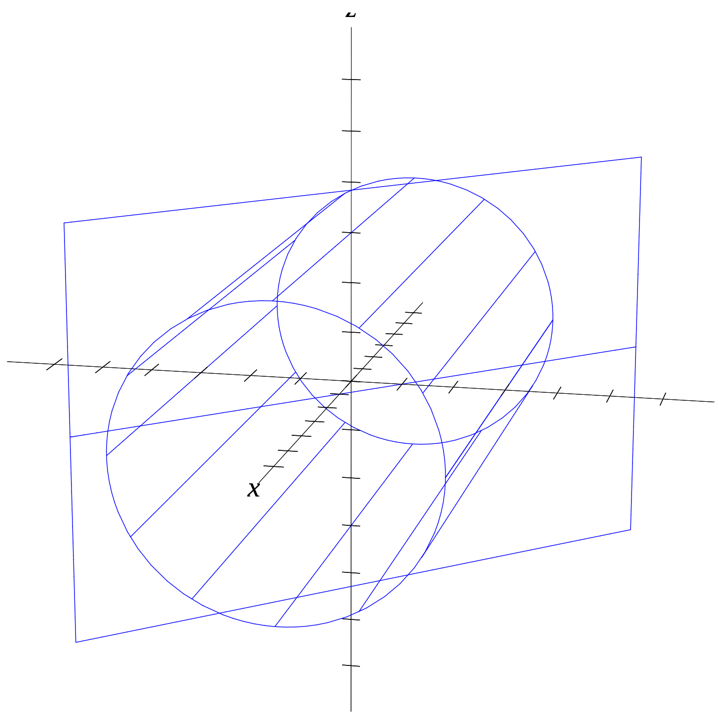


Since z is not in the equation $x + y = 1$, the z-coordinate for these points can be anything. Thus we can move this line vertically up and down to get a plane:

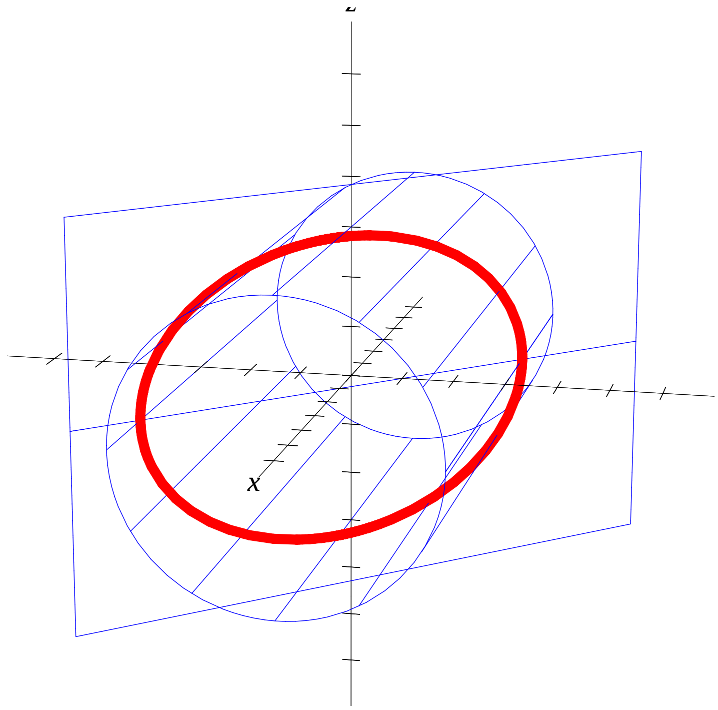


c. Sketch both $y^2 + z^2 = 9$ and $x + y = 1$ and mark their curve of intersection.

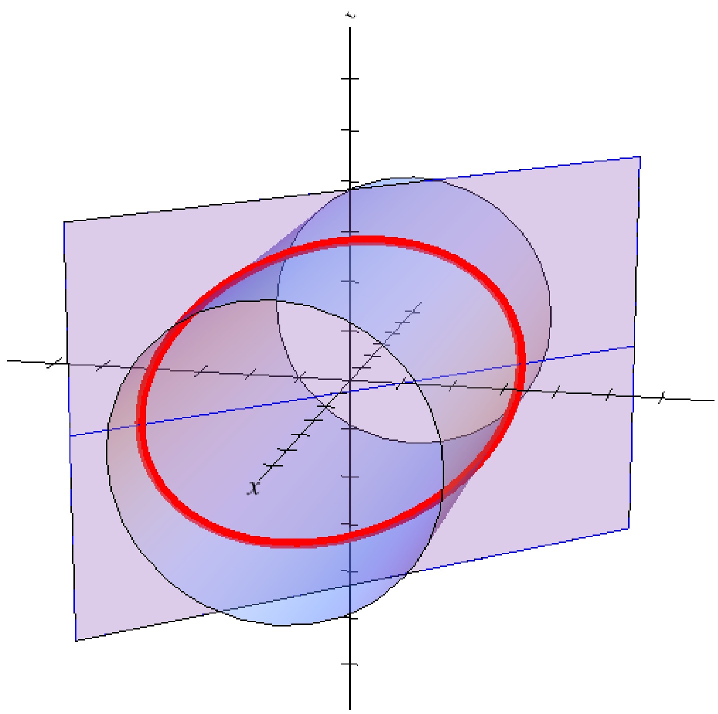
Drawing both the cylinder and the plane looks like this:



The curve of intersection is the following:



The drawing above is what a hand drawing should look like. Below is a drawing with shading done by a computer:



d. Give a vector function that traces out the intersection of $y^2 + z^2 = 9$ and $x + y = 1$.

We are trying to fill in the blanks in the vector function

$$\mathbf{r}(t) = \langle \underbrace{\qquad\quad}_{x}, \underbrace{\qquad\quad}_{y}, \underbrace{\qquad\quad}_{z} \rangle$$

To get $y^2 + z^2$ to equal 9, use

$$\mathbf{r}(t) = \langle \underline{\qquad\qquad}, 3\cos t, 3\sin t \rangle .$$

To get $x + y$ to equal 1, you need $x = 1 - y$:

$$\mathbf{r}(t) = \langle 1 - 3\cos t, 3\cos t, 3\sin t \rangle .$$

That's it.

9. Consider the ellipse $x^2 + \dfrac{y^2}{4} = 1$ in the xy-plane.

a. Give a vector function in 2D that traces out the ellipse **counter-clockwise**.

This ellipse has a "horizontal radius" of 1 and a "vertical radius" of 2.

$$\mathbf{r}(t) = \langle \cos t, 2\sin t \rangle .$$

b. Give a vector function in 2D that traces out the ellipse **clockwise**.

$$\mathbf{r}(t) = \langle \sin t, 2\cos t \rangle .$$

10. Consider the helix $\mathbf{r}(t) = \langle 12\cos t, 5t, 12\sin t \rangle$.

a. Calculate the unit tangent vector $\mathbf{T}$ at the the point $(-12, 5\pi, 0)$.

The point $(-12, 5\pi, 0)$ is when $t = \pi$.

$$\begin{aligned}
\mathbf{r}' &= \langle 12\sin t, 5, -12\cos t \rangle \\
|\mathbf{r}'| &= \sqrt{12^2\sin^2 t + 5^2 + 12^2\cos^2 t} \\
&= \sqrt{12^2 + 5^2} = \sqrt{169} = 13 \\
\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} &= \left\langle \frac{12}{13}\sin t, \frac{5}{13}, \frac{-12}{13}\cos t \right\rangle \\
\mathbf{T}(\pi) &= \left\langle 0, \frac{5}{13}, \frac{12}{13} \right\rangle
\end{aligned}$$

b. Calculate the unit normal vector $\mathbf{N}$ at the the point $(-12, 5\pi, 0)$.

The point $(-12, 5\pi, 0)$ is when $t = \pi$.

$$\begin{aligned} \mathbf{T} &= \langle\, 12\sin t, 5, -12\cos t\,\rangle \\ \mathbf{T}' &= \langle\, 12\cos t, 0, 12\sin t\,\rangle \\ |\,\mathbf{T}'\,| &= \sqrt{12^2 + 0^2} = 12 \\ \mathbf{N} = \frac{\mathbf{T}'}{|\,\mathbf{T}'\,|} &= \langle\, \cos t, 0, \sin t\,\rangle \\ \mathbf{N}(\pi) &= \langle\, -1, 0, 0\,\rangle \end{aligned}$$

c. Calculate the unit binormal vector $\mathbf{B}$ at the the point $(-12, 5\pi, 0)$.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left\langle\, 0, \frac{5}{13}, \frac{12}{13}\,\right\rangle \times \langle\, -1, 0, 0\,\rangle = \left\langle\, 0, \frac{-12}{13}, \frac{5}{13}\,\right\rangle$$

d. Calculate the curvature at the point $(-12, 5\pi, 0)$.

$$\kappa(\pi) = \frac{|\,\mathbf{T}'(\pi)\,|}{|\,\mathbf{r}'(\pi)\,|} = \frac{12}{13}$$

e. Calculate the distance travelled along the curve from $(12, 0, 0)$ to $(-12, 5\pi, 0)$.

The point $(12, 0, 0)$ is when $t = 0$. Thus the distance travelled is the arclength from $t = 0$ to $t = \pi$.

$$L = \int_0^\pi |\,\mathbf{r}'(t)\,|\,\mathrm{d}t = \int_0^\pi 13\,\mathrm{d}t = \Big[13t\Big]_{t=0}^{t=\pi} = 13\pi.$$

f. Reparametrize $\mathbf{r}(t)$ by arclength starting from $(12, 0, 0)$.

The arclength function is

$$s(t) = \int_0^t |\,\mathbf{r}'(u)\,|\,\mathrm{d}u = \int_0^t 13\,\mathrm{d}u = 13t.$$

Solving for $s = 13t$ gives $t = \frac{s}{13}$. Plugging this into $\mathbf{r}(t)$ gives

$$\begin{aligned} \mathbf{r}(t) &= \langle\, 12\cos t, 5t, 12\sin t\,\rangle \\ \mathbf{r}(s) &= \left\langle\, 12\cos\frac{s}{13}, \frac{5s}{13}, 12\sin\frac{s}{13}\,\right\rangle \end{aligned}$$

11. A particle moves along the path $\mathbf{r}(t) = \langle \sqrt{t+1}, t-5, t^2 \rangle$. At what point in space is the velocity of this particle parallel to vector $\langle 1, 4, 24 \rangle$?

The velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{1}{2\sqrt{t+1}}, 1, 2t \right\rangle.$$

For the vector $\langle 1, 4, 24 \rangle$, the z-component is 6 times the y-component. For this to happen for $\mathbf{v}(t)$, we would need $2t$ to be 6 times 1. That is $2t = 6$, so $t = 3$. To double-check,

$$\mathbf{v}(3) = \left\langle \frac{1}{2\sqrt{3+1}}, 1, 2 \cdot 3 \right\rangle = \left\langle \frac{1}{4}, 1, 6 \right\rangle.$$

This is indeed parallel to $\langle 1, 4, 24 \rangle$. The point in space at which this occurs is

$$\mathbf{r}(3) = \left\langle \sqrt{3+1}, 3-5, 3^2 \right\rangle = \langle 2, -2, 9 \rangle.$$

12. Find the position vector $\mathbf{r}(t)$ for a particle that starts at the origin and has velocity vector $\mathbf{v}(t) = \langle t, 4, 1-2t \rangle$.

Find anti-derivatives (aka indefinite integrals) for t, 4, and $1 - 2t$.

$$\mathbf{r}(t) = \left\langle \tfrac{1}{2}t^2, 4t, t - t^2 \right\rangle + \mathbf{C}.$$

Right now, $\mathbf{r}(0) = \langle 0, 0, 0 \rangle + \mathbf{C}$. Since $\mathbf{r}(0)$ should be $\langle 0, 0, 0 \rangle$, in fact $\mathbf{C} = \langle 0, 0, 0 \rangle$. Thus

$$\mathbf{r}(t) = \left\langle \tfrac{1}{2}t^2, 4t, t - t^2 \right\rangle.$$

13. A particle starts at the origin and has velocity $\mathbf{v}(t) = \langle\, t, 3t^2, 2t - 1\,\rangle$. Which of the following is a point through which the point travels?

(a) $(2, 8, 2)$

(b) $(1, 3, 1)$

(c) $(1, 0, 2)$

(d) $(2, 6, 1)$

(e) $(2, 3, 2)$

Find anti-derivatives (aka indefinite integrals) for t, $3t^2$, and $2t - 1$.

$$\mathbf{r}(t) = \left\langle\, \tfrac{1}{2}t^2, t^3, t^2 - t \,\right\rangle + \mathbf{C}.$$

Right now, $\mathbf{r}(0) = \langle\, 0, 0, 0\,\rangle + \mathbf{C}$. Since $\mathbf{r}(0)$ should be $\langle\, 0, 0, 0\,\rangle$, in fact $\mathbf{C} = \langle\, 0, 0, 0\,\rangle$. Thus

$$\mathbf{r}(t) = \left\langle\, \tfrac{1}{2}t^2, t^3, t^2 - t \,\right\rangle.$$

Only answer (a) is a value of $\mathbf{r}(t)$:

$$\mathbf{r}(2) = \left\langle\, \tfrac{1}{2} \cdot 2^2, 2^3, 2^2 - 2 \,\right\rangle = \langle\, 2, 8, 2\,\rangle.$$

14. Which of the following statements is NOT true?

(a) At any point on a curve, the curvature κ is a scalar that is greater than or equal to 0.

(b) Two vectors are orthogonal if and only if their dot product is zero.

(c) The equation $x^2 - y + z^2 = 1$ describes an elliptic paraboloid.

(d) The equation $ax + by + cz = 0$ describes a plane that passes through the origin.

(e) If two lines are not parallel, they intersect at exactly one point.

All of these are true except (e). In 3D space, two lines could be skew.

15. Suppose $|\mathbf{a}| = 6$, $|\mathbf{b}| = \sqrt{3}$, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/6$. Which of the following is the value of $\mathbf{a} \cdot \mathbf{b}$?

(a) -1

(b) $3\sqrt{3}$

(c) 0

(d) 9

(e) Not enough information.

(f) None of the above.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta = 6 \cdot \sqrt{3} \cdot \cos\frac{\pi}{6} = 6 \cdot \sqrt{3} \cdot \frac{\sqrt{3}}{2} = \frac{6 \cdot 3}{2} = 9.$$

This is answer (d).

16. Which of the following planes is normal to the line $x = t + 1,\ y = 3t - 1,\ z = 2t - 3$?

(a) $x - 3y - 2z = 0$

(b) $x - y - 3z + 2 = 0$

(c) $2x + 6y + 4z - 1 = 0$

(d) $2x + 6y + z = 0$

(e) $2x + y - 3z + 3 = 0$

The normal vector of the given plane is $\langle 1, 3, 2 \rangle$, so we need a plane whose normal vector is parallel to $\langle 1, 3, 2 \rangle$. The only plane that does this is (c). The normal vector of $2x + 6y + 4z - 1 = 0$ is $\langle 2, 6, 4 \rangle$, which is $2 \cdot \langle 1, 3, 2 \rangle$, so these normals are parallel.

17. What is the vector projection of $\mathbf{v} = \langle 5, -1/2, 0 \rangle$ onto $\mathbf{w} = \langle 8, -1, 4 \rangle$?

(a) $\langle 4, -1/2, 2 \rangle$

(b) $\langle 16, -2, 4 \rangle$

(c) $\langle 10, -1, 0 \rangle$

(d) $\langle -8, 1, -4 \rangle$

(e) $\frac{1}{101} \langle 810, 81, 0 \rangle$

(f) None of the above.

$$\begin{aligned} \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} &= \frac{40 + \frac{1}{2}}{64+1+16} \langle 8, -1, 4 \rangle \\ &= \frac{40.5}{81} \langle 8, -1, 4 \rangle \\ &= \tfrac{1}{2} \langle 8, -1, 4 \rangle \\ &= \left\langle 4, \tfrac{-1}{2}, 2 \right\rangle \end{aligned}$$

This is answer (a).

18. Which of the following pairs of vectors are orthogonal?

(a) $\langle 23, 4, 1 \rangle$	and	$\langle 0, 1, -3 \rangle$
(b) $\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$	and	$19\,\mathbf{i} + \mathbf{j} + 7\,\mathbf{k}$
(c) $\left\langle \frac{1}{2}, 3, -1 \right\rangle$	and	$\langle -18, 4, 1 \rangle$
(d) $-6\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$	and	$\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$
(e) $\langle 3, -5, 2 \rangle$	and	$\langle -1, 3, 2 \rangle$
(f) $2\,\mathbf{i} - \mathbf{j} + 12\,\mathbf{k}$	and	$-4\,\mathbf{i} + 3\,\mathbf{j}$

The only pair whose dot product is zero is (b):

$$(\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}) \cdot (19\,\mathbf{i} + \mathbf{j} + 7\,\mathbf{k}) = 1(19) + 2(1) + (-3)(7) = 19 + 2 - 21 = 0.$$